Math 112 Review Sheet

This note contains the most important definitions, theorems, problem solving techniques and concepts that you need to know for the final exam. You should only consider this note as a *survey* of the material covered in class. Do not ignore your notes, problem sets or your textbook. This review sheet should help you study the most important concepts faster.

Definition: A **function** is a rule that assigns to each element x in a set D exactly one element called $f(x)$, in a set E. The set D is called the **domain** of the function f. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain D.

Definition: If f is a function with domain D, then its **graph** is the set of all points (x,y) in the $xy-$plane where $y=f(x)$.

Note: All functions that we consider in this course are real valued functions whose domains are a subset of real numbers. This means the domains and ranges are always subsets of real numbers. To find the domain of a function $f(x)$ you need to consider all restrictions that we can have on x. For example all denominators should be non-zero, all numbers under the square root should be non-negative, etc.

The Vertical Line Test: A curve on the $xy-$plane is the graph of a function if no vertical line crosses this curve at more than one point.

Definition: A function $f(x)$ is called **even** if $f(-x)=f(x)$ for any x in the domain of f. The graph of an even function is symmetric about the $y-$axis. $f(x)$ is called **odd** if $f(-x)=-f(x)$ for any x in the domain of f. The graph of an odd function is symmetric about the origin.

Definition: A function f is called **increasing** on an interval I if $f(x_1)<f(x_2)$ for any two numbers $x_1<x_2$ in I. f is called **decreasing** on I if $f(x_1)>f(x_2)$ for any two numbers $x_1<x_2$ in I.

Definition: A function f is called **linear** if its graph is a line, i.e. $f(x)=mx+b$ for two constants m and b.

Definition: A function f is called a **polynomial** if there is a non-negative integer n and real numbers $a_0, a_1, ..., a_{n-1}, a_n$ such that $f(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$ for any real number x. Constants $a_0, a_1,..., a_{n-1}, a_n$ are called **coefficients.** If $a_n \neq 0$, a_n is called the leading coefficient of f and n is called **degree** of f and is denoted by $\deg f$.

Definition: A **power function** is a function of the form $f(x)=x^a$ where a is a constant.

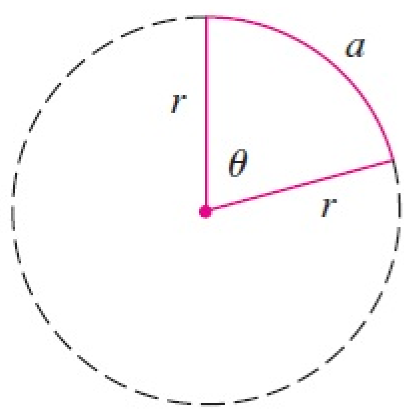
Definition: A **rational function** is a function of the form $f(x) = p(x)/q(x)$ where $p(x)$ and $q(x)$ are polynomials.

Definition: A function f is an **exponential function** if $f(x) = a^x$ where a is a positive constant.

Remark: It is important to distinguish exponential and power functions. In exponential functions the base is a constant, but in power functions the exponent is a constant.

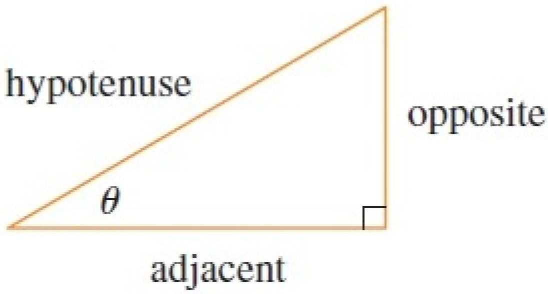
Angles are measured in **degrees** or in **radians**. The angle of a complete revolution is 360^o and is 2π radians. If an angle is D degrees and R radians then the following formula relates the two: $D/360 = R/2\pi$.

The **arc length** a of a sector of a circle of radius r with central angle θ is evaluated by $a = r\theta$.



The **standard position** of an angle occurs when we place its vertex at the origin and its initial side on the positive $x-$axis. **Positive** angles are angles measured counter-clockwise. **Negative** angles are the ones measured clockwise.

For an acute angle θ we can find its trigonometric values using a right triangle as follows:

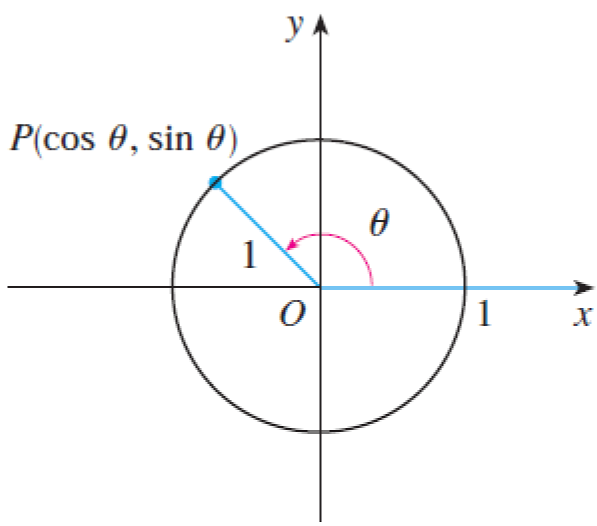


$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

To find trigonometric functions of angles that are not acute, we need to use the unit circle, which is a circle centered at the origin whose radius is one. Considering standard positioning of an angle θ the x and y coordinates of the point of intersection of the terminal side of the angle with the unit circle are $\cos\theta$ and $\sin\theta$ respectively, as shown in the picture below:



Important Trigonometric Identities:

The following trigonometric identities are important and you need to memorize them.

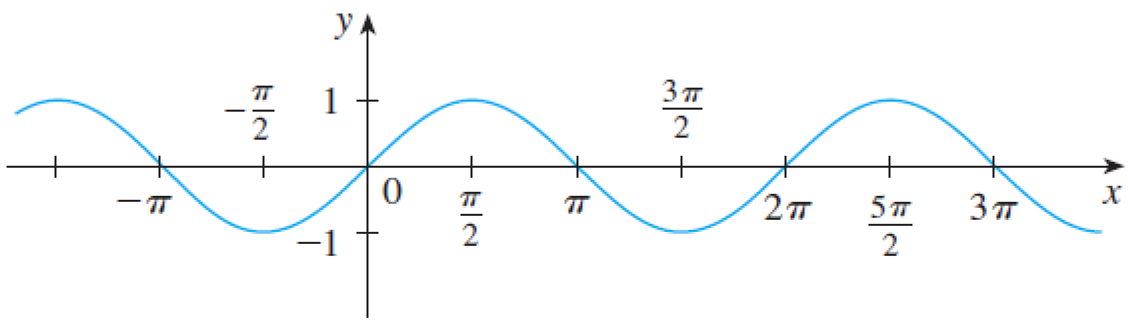
$\cos^2\theta + \sin^2\theta = 1$,
$\sec^2\theta = 1 + \tan^2\theta$, $\csc^2\theta = 1 + \cot^2\theta$,
$\sec\theta = 1/\cos\theta$, $\csc\theta = 1/\sin\theta$,
$\tan\theta = \sin\theta/\cos\theta$, $\cot\theta = \cos\theta/\sin\theta$,

$\sin(2\pi + \theta) = \sin\theta, \ \cos(2\pi + \theta) = \cos\theta$

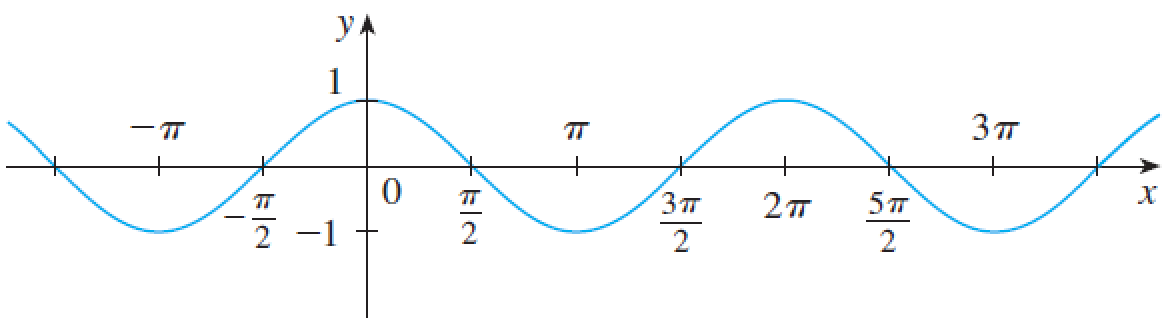
The functions $\sin x$, $\tan x$ and $\cot x$ are odd functions and $\cos x$ is an even function.

Ranges of functions $\sin x$ and $\cos x$ are $[-1, 1]$ and ranges of functions $\tan x$ and $\cot x$ are the set of all real numbers.

Graphs of functions $\sin x$ and $\cos x$ are shown below:

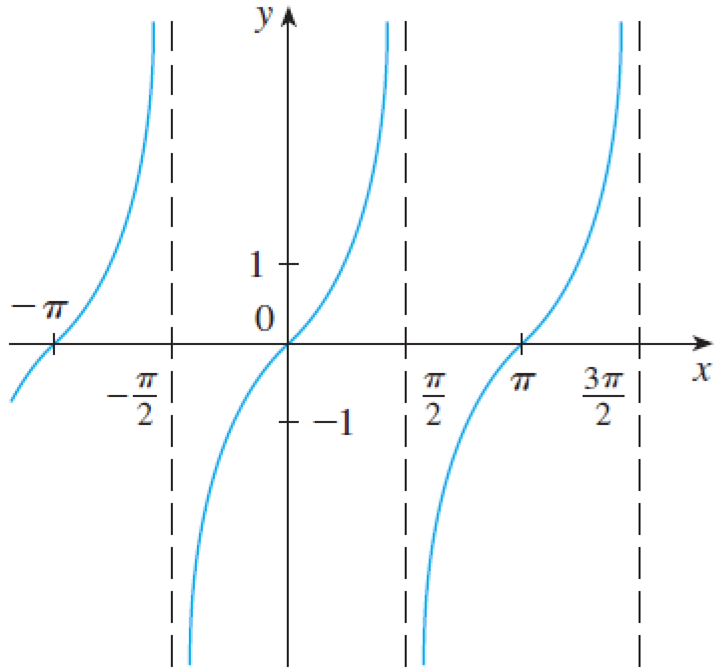


(a) $f(x) = \sin x$

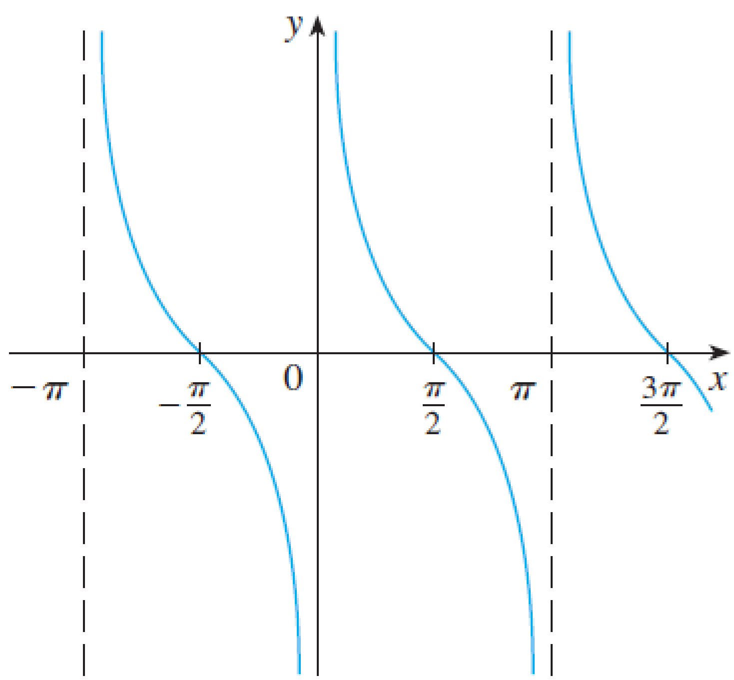


(b) $g(x) = \cos x$

Graphs of $\tan x$ and $\cot x$ are as follows



(a) $y = \tan x$



(b) $y = \cot x$

The following table contains trigonometric values of some certain acute angles that you need to know.

x	$\sin x$	$\cos x$	$\tan x$	$\cot x$
0	0	1	0	undefined
$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$\sqrt{3}/3$
$\pi/2$	1	0	undefined	0

Suppose $c > 0$ is a constant. To obtain the graph of

- $f(x) + c$ move the graph of $f(x)$, a distance of c upward.
- $f(x) - c$ move the graph of $f(x)$, a distance of c downward.
- $f(x + c)$ move the graph of $f(x)$, a distance of c to the left.
- $f(x - c)$ move the graph of $f(x)$, a distance of c to the right.

Suppose $c > 1$ is a constant. To obtain the graph of

- $cf(x)$ stretch the graph of $f(x)$ vertically with a factor of c.
- $f(x)/c$ shrink the graph of $f(x)$ vertically with a factor of c.
- $f(cx)$ shrink the graph of $f(x)$ horizontally with a factor of c.
- $f(x/c)$ stretch the graph of $f(x)$ horizontally with a factor of c.
- $-f(x)$ reflect the graph of $f(x)$ about the $x-$axis.
- $f(-x)$ reflect the graph of $f(x)$ about the $y-$axis.

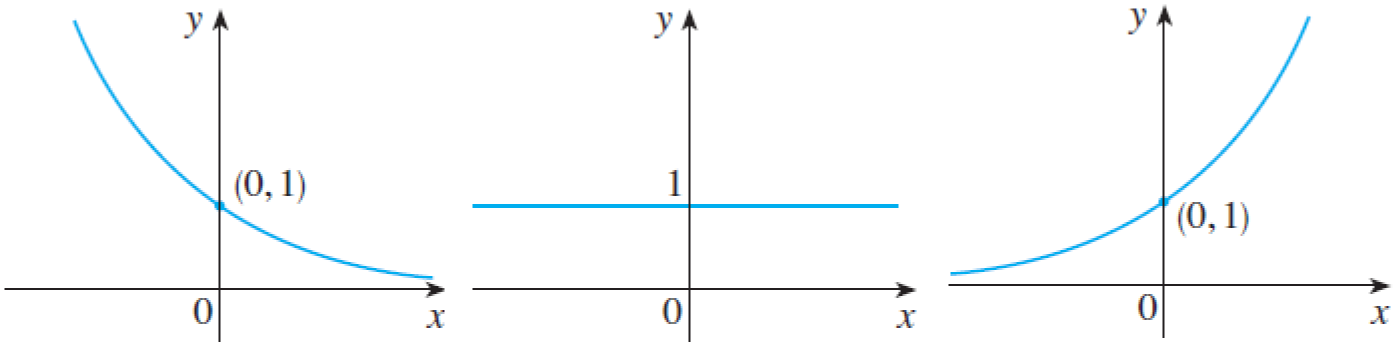
Definition: For two functions f and g, the **composite function** $f \circ g$ is defined by $f \circ g(x) = f(g(x))$.

Note that $f \circ g$ and $g \circ f$ are not the same.

Laws of Exponents: For all positive numbers a and b and all real numbers x and y we have:

- $(ab)^x = a^x b^x$
- $a^x a^y = a^{x+y}$
- $a^x / a^y = a^{x-y}$
- $(a^x)^y = a^{xy}$

Let a be a positive constant. Graphs of exponential functions a^x for different values of a can be seen in the following figure:



(a) $y = a^x,\ 0 < a < 1$ (b) $y = 1^x$ (c) $y = a^x,\ a > 1$

Definition: The number e is a number such that the slope of tangent line to the graph of e^x at $(0,1)$ is exactly 1. This number is approximately $e \approx 2.7$

Definition: A function $f(x)$ is **one-to-one** if for any two different values of $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.

Note: To check if a function is one-to-one or not, start with the equation $f(x_1) = f(x_2)$. Simplify and see if you can show $x_1 = x_2$.

- If you show $x_1 = x_2$ then f is one-to-one.
- If you can't, look for two *different* numbers x_1 and x_2 with $f(x_1) = f(x_2)$. If you find such x_1 and x_2, then f is not one-to-one.

Horizontal Line Test: A function is one-to-one if no horizontal line crosses its graph more than once.

Definition: For a one-to-one function f, its **inverse** f^{-1} is defined by: For any x in the range of f, $f^{-1}(x) = y$ if and only if $f(y) = x$.

Domain of f = Range of f^{-1}
Range of f =Domain of f^{-1}

Note: The functions f^{-1} and $1/f$ are <u>NOT</u> the same functions. For example for $f(x) = 2x$, $f^{-1}(x) = x/2$ but $1/f(x) = \frac{1}{2x}$.

Cancellation Equations: For a one-to-one function f and any x in the domain of f and any y in the range of f, we have:

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(y)) = y.$$

Note: To find the inverse function of a one-to-one function $f(x)$ you need to

1. Write $y = f(x)$ and solve this equation for x.
2. Interchange x and y. The resulting function y as an expression of x would be the inverse function $f^{-1}(x)$.

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Definition: For any positive constant $a \neq 1$ the inverse function of the exponential function a^x is called the logarithmic function $\log_a^x$. In other words:

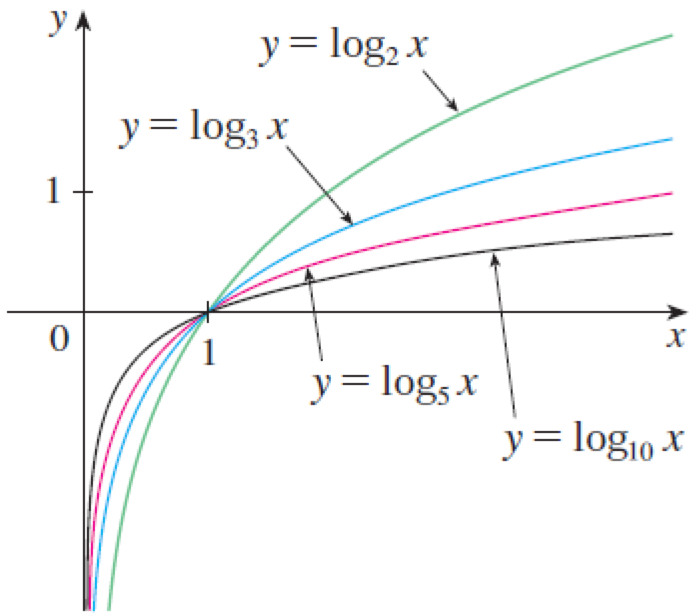
$$\log_a^x = y \text{ if and only if } a^y = x$$

Using the cancellation formulas we get:

- $a^{\log_a^x} = x$
- $\log_a^{a^x} = x$

The range of logarithmic functions are the set of all real numbers. Their domains are the set of all positive numbers.

The following figure shows how graphs of logarithmic functions change when the base changes:



Laws of Logarithms: For any positive number $a \neq 1$, any real number r, and any positive numbers x and y, we have:

- $\log_a^x + \log_a^y = \log_a^{xy}$
- $\log_a^x - \log_a^y = \log_a^{x/y}$
- $\log_a^{x^r} = r\log_a^x$
- $\log_a^1 = 0$

Natural Logarithms: If the base of a logarithmic function is the number e then this logarithmic function is called the natural logarithmic function and is denoted by $\ln x$.

Change of Base Formula: For all positive numbers x and $a \neq 1$ we have: $\log_a^x = \ln x / \ln a$.

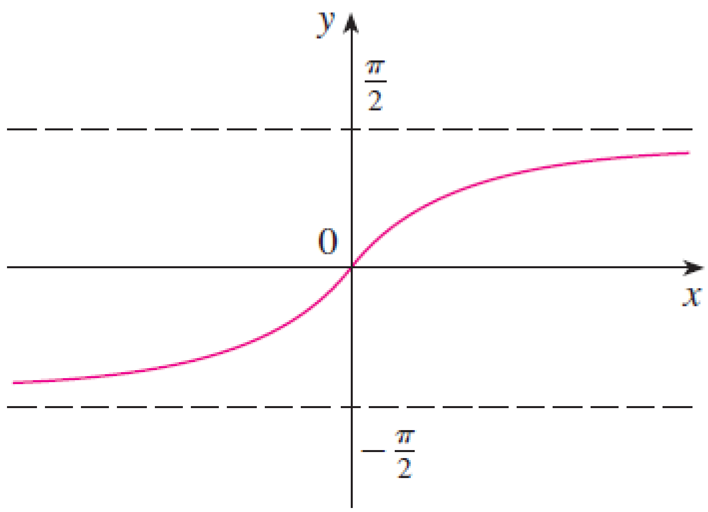
The **inverse sine function** is a function whose domain is $[-1, 1]$ and whose range is $[-\pi/2, \pi/2]$ and is defined by: For any number x, $-1 \leq x \leq 1$, $\sin^{-1} x$ is an angle θ with $-\pi/2 \leq \theta \leq \pi/2$ whose sine is x, i.e. $\sin^{-1} x = \theta$, when $\sin\theta = x$ and $-\pi/2 \leq \theta \leq \pi/2$.

The **inverse cosine function** is defined as: $\cos^{-1} x = \theta$ where $\cos\theta = x$ and $0 \leq \theta \leq \pi$.
The **inverse tangent function** is defined as: $\tan^{-1} x = \theta$ where $\tan\theta = x$ and $-\pi/2 < \theta < \pi/2$.
The **inverse cotangent function** is defined as: $\cot^{-1} x = \theta$ where $\cot\theta = x$ and $0 < \theta < \pi$.

Note:

- Domain of $\sin^{-1}x$ = Domain of $\cos^{-1}x$ = $[-1,1]$
- Domain of $\tan^{-1}x$ =Domain of $\cot^{-1}x$ = $(-\infty,\infty)$
- Range of $\sin^{-1}x = [-\pi/2,\pi/2]$
- Range of $\tan^{-1}x = (-\pi/2,\pi/2)$
- Range of $\cos^{-1}x = [0,\pi]$
- Range of $\cot^{-1}x = (0,\pi)$

Graphs of these inverse trigonometric functions are obtained by reflecting the graphs of trigonometric functions about the line $y = x$. Graph of $\tan^{-1}x$ is specially important.



$y = \tan^{-1}x$

Sometimes $\sin^{-1}x$ is written as $\arcsin x$. Similarly we may write $\cos^{-1}x$ and $\tan^{-1}x$ as $\arccos x$ and $\arctan x$.

Note: $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ and $\cot^{-1}x$ are NOT the same as $1/\sin x$, $1/\cos x$, $1/\tan x$ and $1/\cot x$.

Chapter 2

Definition: Assume $f(x)$ is defined for numbers close to a but possibly not at a. We say $\lim_{x\to a} f(x) = L$ if we can make $f(x)$ arbitrarily close to L when x is sufficiently close to a, but not equal to a.

Definition: Assume $f(x)$ is defined for numbers close to a and less than a .We say $\lim_{x\to a^-} f(x) = L$ if we can make $f(x)$ arbitrarily close to L when x is sufficiently close to a and less than a. Similarly we can define $\lim_{x\to a^+} f(x) = L.$

Theorem: $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L.$

Note: If a function is piecewise defined and you want to find its limit at one of the "border numbers" you need to use the above theorem and evaluate the right-hand and left-hand limits. If these limits are the same then the function has a limit. If they are different then the limit does not exist.

Definition: Let f be a function defined on numbers close to a except possibly at a, then $\lim_{x\to a} f(x) = \infty$ if $f(x)$ can be made arbitrarily large when x is sufficiently close to a but not equal to a.

Definition: Let f be a function defined on numbers close to a except possibly at a, then $\lim_{x\to a} f(x) = -\infty$ if $f(x)$ can be made arbitrarily large negative when x is sufficiently close to a but not equal to a.

Definition: The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if one of the following is true:

$$\lim_{x\to a} f(x) = \infty, \quad \lim_{x\to a^-} f(x) = \infty, \quad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty, \quad \lim_{x\to a^-} f(x) = -\infty, \quad \lim_{x\to a^+} f(x) = -\infty$$

Note: To find vertical asymptotes of a rational function $f(x)$, generally you need to look at the roots of the denominator of $f(x)$. After finding these roots, make sure the limit of $f(x)$ at these numbers equals $\pm\infty$.

Limit Laws: Let n be a positive integer and a and c two real constants and f and g two functions that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then we have the following:

- $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
- $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$

- $\lim_{x\to a}[f(x)/g(x)] = \lim_{x\to a} f(x)/\lim_{x\to a} g(x)$, provided $\lim_{x\to a} g(x) \neq 0$
- $\lim_{x\to a} f(x)^n = [\lim_{x\to a} f(x)]^n$
- $\lim_{x\to a} c = c$
- $\lim_{x\to a} x^n = a^n$
- $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, provided $\lim_{x\to a} f(x) > 0$ when n is even.
- $\lim_{x\to a} f(x) = f(a)$ if $f(x)$ is a polynomial, a rational function, a root function, a trigonometric function, an inverse trigonometric function, an exponential function or a logarithmic function. [This is called the Direct Substitution Property.]

Theorem: If $f(x) \le g(x)$ for any x close to a, except possibly at a. Then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$ provided both limits exist.

Squeeze Theorem: Assume $g(x) \le f(x) \le h(x)$ for all x close to a but not equal to a. In addition assume $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$. Then $\lim_{x\to a} f(x) = L$.

Squeeze Theorem is mostly used when dealing with limits involving functions that are "complicated" but can be bounded between two numbers.

Definition: A function f is **continuous** at a if $\lim_{x\to a} f(x) = f(a)$.

To Check continuity of f at a, we need to make sure all of the following statements are true:

1. f is defined at a, i.e. a is in the domain of f.
2. $\lim_{x\to a} f(x)$ exists.
3. $\lim_{x\to a} f(x) = f(a)$.

Definition: A function f is **continuous from the right** at a if $\lim_{x\to a^+} f(x) = f(a)$. It is **continuous from the left** at a if $\lim_{x\to a^-} f(x) = f(a)$.

Definition: f is continuous on an interval if it is continuous at any number in that interval. (At the endpoints of the interval by continuous we mean "continuous from the right or left".)

Theorem: Let c and a be two real numbers. If f and g are continuous at a then the following functions are all continuous at a:

- $f(x) \pm g(x)$

- $cf(x)$
- $f(x)g(x)$
- $f(x)/g(x)$ if $g(a) \neq 0$.

Theorem: Any function of one of the following types is continuous on its domain: polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions and logarithmic functions.

Theorem:

1. If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$.
2. If f is continuous at $g(a)$ and g is continuous at a, then the composite function $f \circ g$ is continuous at a.

The Intermediate Value Theorem: Suppose $f(x)$ is a continuous function on a closed interval $[a,b]$. If N is a number between $f(a)$ and $f(b)$, then there is a number c in $[a,b]$ such that $f(c) = N$.

Note: You can use the above theorem to show an equation $f(x) = 0$ has a root. To do that you need to show $f(x)$ is continuous, $f(x)$ is positive at some point and negative at another point.

Definition: Let L be a real number. For a function f defined on an interval (a,∞), we say $\lim_{x\to\infty} f(x) = L$ if $f(x)$ can be made arbitrarily close to L when x is sufficiently large.

Definition: For a function f defined on an interval $(-\infty,a)$, we say $\lim_{x\to-\infty} f(x) = L$ if $f(x)$ can be made arbitrarily close to L when x is sufficiently large negative.

Definition: For a function f defined on an interval $(-\infty,a)$, we say $\lim_{x\to-\infty} f(x) = \infty$ if $f(x)$ can be made arbitrarily large when x is sufficiently large negative. Similarly we can define the notions $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = -\infty$.

Definition: The horizontal line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

Theorem: For any positive number r, we have:

- $\lim_{x\to\infty} 1/x^r = 0$ and $\lim_{x\to\infty} x^r = \infty$.
- $\lim_{x\to-\infty} 1/x^r = 0$ provided x^r is defined for every x.

Note: For two polynomials $p(x)$ and $q(x)$, to evaluate $\lim_{x\to\pm\infty} p(x)/q(x)$ divide both numerator and denominator by the highest power of x that occurs in the denominator and then use the above theorem.

Theorem: If $a>1$ and $0<b<1$ are two constants, then

- $\lim_{x\to\infty} a^x = \infty$
- $\lim_{x\to-\infty} a^x = 0$
- $\lim_{x\to\infty} b^x = 0$
- $\lim_{x\to-\infty} b^x = \infty$

Definition: The **derivative** of a function f at a number a is given by

$$f'(a) = \lim_{x\to a}(f(x)-f(a))/(x-a) = \lim_{h\to 0}(f(a+h)-f(a))/h$$ [You may use either limits.]

if this limit exists as real number. When this limit exists, we say f is **differentiable** at a. We say f is differentiable on an interval if it is differentiable at any point in that interval.

Definition: The **tangent line** to the graph of $f(x)$ at the point $P(a, f(a))$ is the line through P whose slope is given by $m = f'(a)$.

Definition: Assume a particle moves along the x – axis and its position function at time t is given by $f(t)$, then the **velocity** of this particle at time $t=a$ (denoted by $v(a)$) is $f'(a)$. The **acceleration** of this particle at time $t=a$ is given by $v'(a)$.

Definition: The derivative $f'(a)$ is the rate of change of $y = f(x)$ with respect to x when $x = a$.

Theorem: If a function f is differentiable at a then it is continuous at a.

A function $f(x)$ fails to be differentiable at a if one of the following occurs:

- $f(x)$ is not continuous at a.
- The graph of $f(x)$ has a corner at $(a, f(a))$, i.e. the graph of $f(x)$ changes direction abruptly at $(a, f(a))$.
- The graph of $f(x)$ has a vertical tangent line when $x = a$.

Definition: The derivative of the function $f'(x)$ is a new function denoted by $f''(x)$ and called the second derivative of f. The third derivative of f is the derivative of $f''(x)$ denoted by $f'''(x)$. The n – th derivative of $f(x)$ is denoted by $f^{(n)}(x)$.

Notations: These notations are all used for the derivatives of a function $y = f(x)$:

$$y' = f'(x) = \frac{df}{dx} = \frac{dy}{dx} = D_x f(x)$$

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n} = \frac{d^n f}{dx^n}$$

Chapter 3

Rules of Differentiations: Let c be a constant, and f and g be two functions, then:

1. $\frac{d}{dx}(c) = 0$ [derivative of a constant function is zero].
2. **The Power Rule**: $\frac{d}{dx}(x^c) = cx^{c-1}$ provided x^c is defined.
3. **The Constant Multiple Rule**: $\frac{d}{dx}(cf(x)) = cf'(x)$.
4. **The Sum Rule**: $[f(x) + g(x)]' = f'(x) + g'(x)$.
5. **The Difference Rule**: $[f(x) - g(x)]' = f'(x) - g'(x)$.
6. $(e^x)' = e^x$.
7. **The Product Rule**: $(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x)$.
8. **The Quotient Rule**: $(f(x)/g(x))' = (f'(x)g(x) - f(x)g'(x))/g^2(x)$.
9. **Derivatives of Trigonometric Functions**:
   - $(\sin x)' = \cos x$
   - $(\cos x)' = -\sin x$
   - $(\tan x)' = \sec^2 x$
   - $(\cot x)' = -\csc^2 x$
10. **Derivative of Exponential Functions**: $(c^x)' = c^x \ln c$, where $c > 0$.

Note: Make sure to distinguish between derivative of a power function $[(x^c)' = cx^{c-1}]$ and derivative of an exponential function $[(c^x)' = c^x \ln c]$.

The Chain Rule: If g is differentiable at x and f is differentiable at $u = g(x)$, then the composite function $F = f \circ g$ is differentiable at x and $F'(x) = f'(g(x)) \cdot g'(x)$. If we set $y = f(u)$, we can write that as $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, which is easier to remember.

Implicit Differentiation: To find the derivative of a function y in terms of x given by an implicit equation $F(x,y) = 0$ we differentiate both sides of this equality using The Chain Rule. Then we solve the resulting equation for y' to evaluate y' in terms of x and y.

Derivatives of Inverse Trigonometric Functions:

- $(\sin^{-1}x)' = 1/\sqrt{1-x^2}$
- $(\cos^{-1}x)' =- 1/\sqrt{1-x^2}$
- $(\tan^{-1}x)' = 1/(1+x^2)$
- $(\cot^{-1}x)' =- 1/(1+x^2)$

Derivative of Logarithmic Functions: For a positive constant a,

- $(\log_a^x)' = 1/(x\ln a)$
- $(\ln x)' = 1/x$

Logarithmic Differentiation: When dealing with functions involving several products, quotients and powers we can often simplify differentiating by taking logarithms. To do that we follow these steps:

- Take the natural logarithm of both sides of $y = f(x)$.
- Use the laws of logarithms to simplify this equation.
- Differentiate both sides using the Chain Rule.
- Solve the resulting equation for y'.

Note: Logarithmic differentiation is often used when differentiating a function involving expressions of x as exponents, e.g. x^x, $(\ln x)^{x^2}$.

Theorem: If k is a constant and $y(x)$ is a function of x such that $\frac{dy}{dx} = ky$, then $y(x) = y(0)e^{kx}$.

When dealing with word problems it is always important to follow these steps:

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
6. Use the Chain Rule to differentiate both sides of the equation with respect to .
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Note: The most common error when solving related rate problems, is to substitute the given numerical value too early. Make sure to do the substitution after differentiation is done.

Assume $f(x)$ is differentiable at a. Then $f(x)$ can be approximated by

$$f(x) \approx f(a) + f'(a)(x-a).$$

This approximation is called the **linear approximation** or the **tangent line approximation** of f at a.

Definition: The **differential** dy of a function $y = f(x)$ is $dy = f'(x)dx$.

Chapter 4

Definition: Let c be in the domain D of a function $f(x)$, then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D.
- The maximum and minimum value are called **extreme values** of f.

Definition: Let c be in the domain D of a function f such that f is defined for numbers close to c. Then $f(c)$ is a

- **local minimum** value if $f(c) \leq f(x)$ for any x near c.
- **local maximum** value if $f(c) \geq f(x)$ for any x near c.

The Extreme Value Theorem: A continuous function f on a closed interval $[a,b]$ attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ in this interval.

Definition: A number c in the domain of f is called a **critical number** if either $f'(c) = 0$ or $f'(c)$ does not exist.

Fermat's Theorem: If f has a local minimum or a local maximum at c, then c is a critical number for f.

Note: The converse of the Fermat's Theorem is not true, i.e. if $f'(c) = 0$, c is not necessarily a local maximum or a local minimum for f.

The Closed Interval Method: To find extreme values of a continuous function f on a closed interval $[a,b]$,

1. Find all critical numbers of f on (a,b).
2. Evaluate f at these critical numbers.
3. Evaluate $f(a)$ and $f(b)$.
4. The largest value from Steps 2 and 3 is the absolute maximum value and the smallest value from Steps 2 and 3 is the absolute minimum value of f on $[a, b]$.

Note: When using the Closed Interval Method, make sure the function is continuous and the interval is closed.

The Rolle's Theorem: Let f be a function such that,

1. f is continuous on the closed interval $[a,b]$.
2. f is differentiable on the open interval (a,b).
3. $f(a) = f(b)$.

Then there is a number $a < c < b$ such that $f'(c) = 0$.

The Mean Value Theorem: Let f be a function such that,

1. f is continuous on the closed interval $[a,b]$.
2. f is differentiable on the open interval (a,b).

Then there is a number c in (a,b) such that $f'(c) = (f(b) - f(a))/(b - a)$.

Note: Make sure to check <u>all</u> conditions are satisfied when using the Rolle's or the Mean Value Theorems.

Note: You may often use the Rolle's Theorem to show a certain equation $f(x) = 0$ has *at most* one root. To do that assume in contrary it has two roots x_1 and x_2, i.e. $f(x_1) = f(x_2) = 0$. Then use the Rolle's Theorem to deduce $f'(c) = 0$ for some c between x_1 and x_2 and show it is not possible. Combine this method with the IVT when you want to show an equation has *exactly* one root.

Increasing/Decreasing Test: Let f be a differentiable function on an interval I.

- If $f'(x) > 0$ for any x in I, then $f(x)$ is increasing.
- If $f'(x) < 0$ for any x in I, then $f(x)$ is decreasing.

Note: To find local maximum and minimum values of a function $f(x)$,

1. Find all critical numbers of $f(x)$.
2. Draw a table with three rows.
    a. In the first row put all critical numbers of $f(x)$.
    b. In the second row determine the sign of $f'(x)$ between these critical numbers.
    c. Using the I/D Test, in the third row determine whether $f(x)$ is increasing or decreasing between these critical numbers.
3. If c is a critical number and f is decreasing near and to the left of c and f is increasing near and to the right of c, then f has a local minimum at c.
4. If c is a critical number and f is decreasing near and to the right of c and f is increasing near and to the left of c, then f has a local maximum at c.

Note: To understand and remember without having to memorize steps 3 and 4, visualize how the graph will look like.

Definition: For a differentiable function on an interval I,

- If the graph of a function f lies above all of its tangent lines on I, then it is called **concave upward** on I.
- If the graph of a function f lies below all of its tangent lines on I, then it is called **concave downward** on I.

Concavity Test: Let f be a function such that $f''(x)$ exists for all x in an interval I.

- If $f''(x) > 0$ for all x in I, then f is concave upward on I.
- If $f''(x) < 0$ for all x in I, then f is concave downward on I.

Definition: A point $P(a, f(a))$ on the graph of a function f is called an **inflection point,** if f is continuous at a and the curve changes its concavity from concave upward to concave downward or vice-versa.

The Second Derivative Test: Suppose $f'(c) = 0$ and $f''(x)$ is continuous near c.

- If $f''(c) > 0$, then f has a local minimum at c.
- If $f''(c) < 0$, then f has a local miaximum at c.

Note: To determine concavity of a function $f(x)$,

1. Evaluate its second derivative $f''(x)$.
2. Find all roots of $f''(x) = 0$.
3. Draw a table with three rows.
4. In the first row put all values of x that satisfy $f''(x) = 0$, i.e. all roots of $f''(x) = 0$.
5. In the second row determine the sign of $f''(x)$ between these roots.
6. Use the concavity test to determine the concavity of $f(x)$ and collect this data in the third row.

L'Hospital's Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I containing a (except possibly at a). Suppose we have one of the following cases:

1. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ (In which case we say $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$.)
2. $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$ (In which case we say $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of the form $\frac{\infty}{\infty}$.)

Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists as a real number or is $\pm\infty$.

Note: One common error is to use the l'Hospital's Rule without verifying the limit is in form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Make sure to avoid this common error. This error is especially common when you need to use l'Hospital's Rule several time. You can use this rule several times if needed but each time make sure to check the conditions are satisfied.

Note: You can use l'Hospitals' Rule for one-sided limits and limits at infinity, i.e. when $x \to \pm\infty$ or $x \to a^{\pm}$.

Indeterminate Forms: To evaluate limits of indeterminate forms most of the time we turn the limit into an indeterminate quotient and then apply the l'Hospital's Rule.

- **Indeterminate Quotients**: To evaluate $\lim_{x\to a} \frac{f(x)}{g(x)}$ of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ you need to use the l'Hospital's Rule.
- **Indeterminate Products**: To evaluate $\lim_{x\to a} f(x)g(x)$ of the form $0 \times \infty$, (i.e. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$) you need to write fg as $\frac{f}{1/g}$ or $\frac{g}{1/f}$ to get an indeterminate quotient. The use the l'Hospital's Rule. Note that sometimes one of $\frac{f}{1/g}$ or $\frac{g}{1/f}$ is easier to work with than the other.
- **Indeterminate Differences**: Limits of type $\infty - \infty$ are called indeterminate differences, i.e. we want to evaluate $\lim_{x\to a} f(x) - g(x)$ where $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$. To evaluate such limits we use our knowledge of algebra to simplify the expression $f(x) - g(x)$ and turn it into a quotient and then use the l'Hospital's Rule.
- **Indeterminate Powers**: When evaluating a limit of the form $\lim_{x\to a} f(x)^{g(x)}$ we might get one of the following three indeterminate forms:

    - $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, indeterminate form of type 0^0.
    - $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$, indeterminate form of type 1^∞.
    - $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$, indeterminate form of type ∞^0.

    To evaluate such limits, take $y = f(x)^{g(x)}$. Take the $\ln$ from both sides and use laws of logarithms to get $g(x)\ln f(x)$. This way you end up with an indeterminate product. Evaluate this new limit using the method for evaluating limits of indeterminate products. This would give you $\lim_{x\to a} \ln f(x)^{g(x)} = L$. Using this you get $\lim_{x\to a} f(x)^{g(x)} = e^L$.

Note: The following forms of limits are <u>not</u> indeterminate:

- 0^∞ is not indeterminate. This is perhaps the most common form of *determinate* limits that a lot of students mistaken for an indeterminate form. This in fact is equal to zero, because if you raise a small number to a large exponent the number gets even smaller, so $0^\infty = 0$.

- $\infty^{\infty} = \infty$.
- $\infty \cdot \infty = \infty$
- $\infty + \infty = \infty$

To Sketch a Curve $y = f(x)$,

1. Find the domain of $f(x)$.
2. Find x and y intercepts. The y – intercept is $f(0)$ and the x – intercept may be evaluated by solving $f(x) = 0$. The x – intercept may be impossible to find as sometimes solving $f(x) = 0$ is impossible. Plot these intercepts.
3. Check whether f is odd or even. If it is either odd or even, you only need to graph $f(x)$ for positive (or negative) values of x.
   a. If f is odd reflect the graph obtained for positive values of x about the origin to get the complete graph of f.
   b. If f is even reflect he graph obtained for positive values of x about the y – axis to get the complete graph of f.

   Check whether f is periodic. If it is periodic with period p ,(i.e. $f(x+p) = f(x)$ for all x in the domain of f) then only graph f on the interval $[0,p]$ and repeat the same graph on intervals $[p,\ 2p],\ [2p,\ 3p],\ [3p,\ 4p],\ \cdots$ and $\cdots,\ [-3p,\ -2p],\ [-2p,\ -p],\ [-p,\ 0]$.
4. Find all horizontal and vertical asymptotes of f and draw them. as dashed lines
5. Use the I/D Test to find intervals where f is increasing or decreasing.
6. Find all local maximum and minimum values. Plot them.
7. Find all inflection points and concavity intervals of f. Plot the inflection points.
8. Sketch the curve: The curve should pass through the points that you have plotted in the previous steps. Start by looking at step 4 and see what is the value of $\lim_{x \to -\infty} f(x)$ to see how the graph should look like for large negative values of x. Then move to smaller negative numbers and make the curve pass through the points that you have plotted. Use the information you found to make the graph increasing or decreasing. Use the information about its concavity to draw the curve more accurately.

Optimization Problems: These are problems asking to find the maximum or minimum of a quantity given some information. Follow these steps to solve such problems:

1. Understand the Problem: The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?
2. Draw a Diagram: In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.
3. Introduce Notation: Assign a symbol to the quantity that is to be maximized or minimized (let’s call it Q for now). Also select symbols for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for

example, A for area, h for height, t for time.

4. Express Q in terms of some of the other symbols from Step 3.
5. If Q has been expressed as a function of more than one variable in Step 4, us the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus Q will be expressed as a function of one variable x, say, $Q = f(x)$. Write the domain of this function.
6. Use the methods of Sections 4.1 and 4.3 to find the absolute maximum or minimum value of f. In particular, if the domain of f is a closed interval, then the Closed Interval Method can be used.

Newton's Method: To approximate a root of an equation $f(x) = 0$, we start with an initial number x_1 which is our "best guess" for this root. Then evaluate a "better" estimate for this root by $x_2 = x_1 - f(x_1)/f'(x_1)$. Then use this new estimate to evaluate $x_3 = x_2 - f(x_2)/f'(x_2)$. Repeating this procedure we get better and better approximates of this root. In general you find x_{n+1} using the formula $x_{n+1} = x_n - f(x_n)/f'(x_n)$. If this method fails to give us good approximations we need to start with a better x_1.

Definition: A function $F(x)$ is called an **anti-derivative** of $f(x)$ on an interval I, if $f(x) = F'(x)$.

Theorem:

- If $F(x)$ and $G(x)$ are two anti-derivatives of a function $f(x)$ on an interval I, then there is a constant C such that $F(x) = G(x) + C$ for any x in I.
- If $F(x)$ is an anti-derivative of $f(x)$ on an interval, then all anti-derivatives of $f(x)$ are of the form $F(x) + C$ where C is a constant.

Note: To find an anti-derivative of a function $f(x)$ you often need to remember formulas of derivatives and try to think backward. You need to ask yourself "what function has a derivative equal to $f(x)$?"

Chapter 5

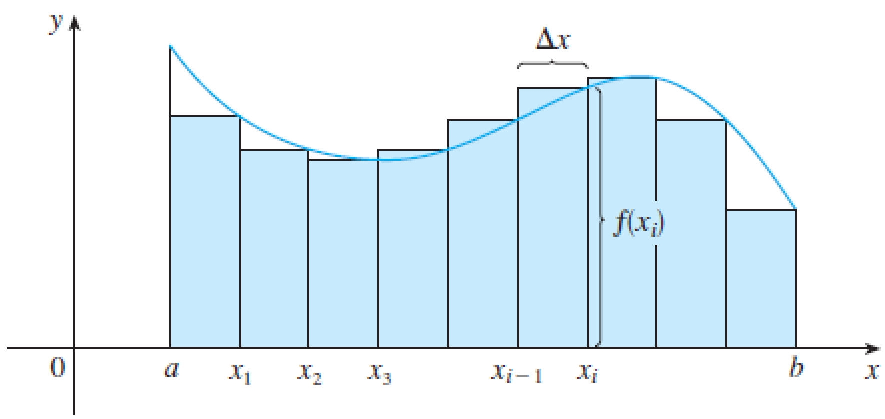
Definition: The area A of the region that lies under the graph of a continuous function f, above the $x-$axis and between vertical lines $x = a$ and $x = b$ is the limit of the sum of the areas of approximating rectangles as shown below:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x + f(x_n)\Delta x]$$

and

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty}[f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-2})\Delta x + f(x_{n-1})\Delta x]$$

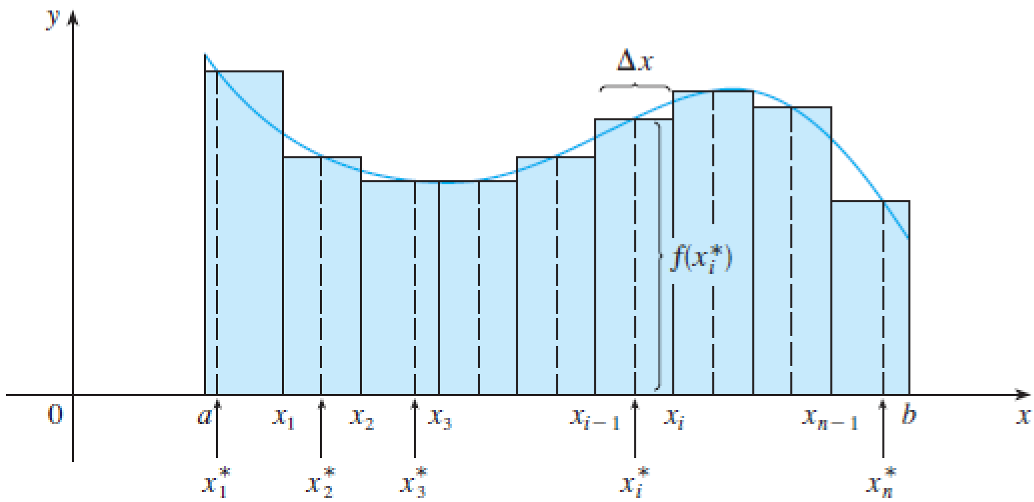
$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$



The first formula involving R_n is obtained by forming rectangles based on the right endpoints of each sub-interval. The second formula involves L_n which is obtained by using the left endpoints of sub-intervals.

Instead of taking right or left endpoints of these subintervals one can take any point (called a sample point) in these subintervals to get the following formula for the area A:

$$A = \lim_{n\to\infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$



. **Notation**: Instead of writing $x_1 + x_2 + \cdots + x_{n-1} + x_n$ we can write it in a more compact form as $\Sigma_{i=1}^{n} x_i$. For example the above formula would be shortened as $A = \lim_{n\to\infty} \Sigma_{i=1}^{n} f(x_i^*)\Delta x$.

Note: The sum $\Sigma_{i=1}^{n} f(x_i^*)\Delta x$ is called a **Riemann Sum**.

Definition: Let f be a function defined over $[a,b]$. Divide this interval into n subintervals of equal length $\Delta x = (b-a)/n$. Let $x_0 = a,\ x_1,\ \cdots,\ x_{n-1},\ x_n = b$ be the endpoints of these subintervals. and let x_i^* be a sample point in the i – th interval $[x_{i-1}, x_i]$. The **definite integral** of $f(x)$ from a to b is defined to be,

$$\int_a^b f(x)\, dx \;= \lim_{n\to\infty} \Sigma_{i=1}^{n} f(x_i^*)\Delta x$$

provided this limit exists and is the same for all possible choices of sample points.

Note: If the above limit exists and is the same for all choices of sample points, we say f is **integrable** on $[a,b]$. The function $f(x)$ is called the **integrand.** The numbers a and b are called **limits of integration**. The number a is called the **lower limit** and b is called the **upper limit**.

Theorem: If f is continuous on $[a,b]$ or has a finite number of discontinuities on this interval, then it is integrable on $[a,b]$.

The Midpoint Rule: If we pick the midpoints as our sample points of subintervals we get the following approximation which is called the Midpoint Rule:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \Sigma_{i=1}^{n} f(\overline{x_i})\Delta x \text{ where } \Delta x = (b-a)/n \text{ and } \overline{x_i} = \frac{x_{i-1}+x_i}{2}.$$

The following approximation is called a **midpoint approximation** for $\int_a^b f(x)\,dx$,

$$\int_a^b f(x)\,dx \approx \Sigma_{i=1}^{n} f(\overline{x_i})\Delta x.$$

Properties of Definite Integrals: For any continuous functions f and g and constants a, b and c, we have:

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$
- $\int_a^a f(x)\,dx = 0.$
- $\int_a^b c\,dx = c(b-a).$
- $\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx = \int_a^b f(x) \pm g(x)\,dx.$
- $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$
- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$
- If $a < b$ and $f(x) \le g(x)$ for all x in $[a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$

Note: In general integral of product (or ratio) of two functions is <u>not</u> equal to the product (or ratio) of their integrals.

$$\int_a^b f(x)\cdot g(x)\,dx \ne \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx \text{ and } \int_a^b f(x)/g(x)\,dx \ne \int_a^b f(x)\,dx / \int_a^b g(x)\,dx$$

The Fundamental Theorem of Calculus: Let f be a continuous function on $[a,b]$ and define a function $g(x)$ by $g(x) = \int_a^x f(t)\,dt$, then:

1. g is differentiable on (a,b) and continuous on $[a,b]$ and $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$ where $F(x)$ is any anti-derivative of $f(x)$.

Note: Part 1 of FToC is often used to take derivatives of functions defined as integrals of other functions. When using FToC, Part 1 you need to make sure the lower limit is a constant and the upper limit is the same variable that you are using for differentiation.

- If both limits of integration are non-constants you need to use properties of integrals to make one of them constant. (i.e. use $\int_a^b + \int_b^c = \int_a^c$.)
- If the upper limit is a constant instead of the lower limit switch the limits of integration by using another property of integrals, i.e. $\int_a^b = -\int_b^a$.
- When the lower limit is a constant and the upper limit is a function of x, say $u = h(x)$ to differentiate $\int_a^{h(x)} f(t)\,dt$ with respect to x, you need to use the Chain Rule as follows:

$$\tfrac{d}{dx}\left(\int_a^{h(x)} f(t)\,dt\right) = \tfrac{d}{dx}\left(\int_a^{u} f(t)\,dt\right) = \tfrac{d}{du}\left(\int_a^{u} f(t)\,dt\right) \cdot \tfrac{du}{dx} = f(u) \cdot u' = f(h(x)) \cdot h'(x)$$

Note: Given $g(x) = \int_{k(x)}^{h(x)} f(t)\,dt$ if you are asked to evaluate $g'(x)$ at a given number, e. g. $g'(1)$, do <u>not</u> evaluate $g(1)$ and then differentiate. You need to differentiate $g(x)$ *first*, then substitute $x = 1$. Evaluating $g(1)$ first and then differentiating will always give us zero, since $g(1)$ is just a constant and does not depend on x, but differentiating $g(x)$ and then substituting 1 for x most of the times gives us a non-zero number.