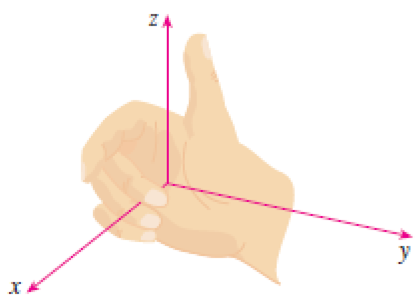
Math 120 Review Sheet

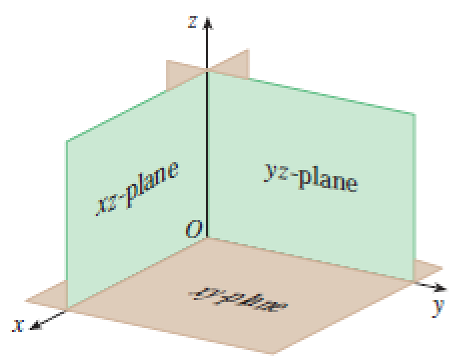
This note contains the most important definitions, theorems, problem solving techniques and concepts that you need to know for the final exam. You should only consider this note as a *survey* of the material covered in class. Do not ignore your notes, problem sets or your textbook. This review sheet should help you study the most important concepts faster.

Chapter 12

For locating points in the 3-dimensional space we need 3 numbers. To get these three numbers we need three axes called x, y and $z-$axis. We usually consider x and y axes to be horizontal and the $z-$axis to be vertical. The direction of the $z-$axis is determined by the **right-hand rule** shown in the picture below.



Right-hand rule



Coordinate planes

The three axes make three **coordinate planes** called xy, xz and yz planes shown in the above picture. These coordinate planes divide the 3-dimensional space into eight **octants**. The **first octant** is the octant determined by the positive axes. For any point P in the 3-D space the x **coordinate** of P is the directed distance of P to the $yz-$plane. Similarly one can define the $y-$coordinate and the $z-$coordinate of P. We assign to any point in the space a triple (a, b, c), where a, b, and c are the x, y and z coordinates of P, respectively. Similarly for any triple (a, b, c) of real numbers you can find a point P in the space. This coordinate system is called a **three dimensional rectangular (or Cartesian) coordinate system**.

Distance Formula: The distance between any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the 3-D space is calculated by $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$.

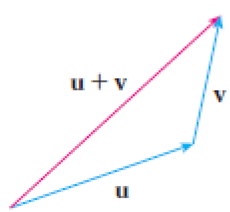
Equation of a Sphere: The equation of a sphere of radius r centered at (x_0, y_0, z_0) is given by

$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}=r \text{ or } (x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2.$$

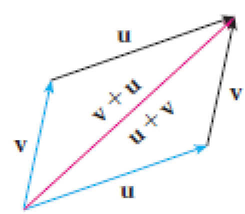
Often times we work with spheres centered at the origin. In which case equation of the sphere is given by, $x^2+y^2+z^2=r^2$.

Definition: A **vector** is a concept used for a quantity that has both a magnitude and a direction. Vectors are usually represented by a directed segment using an arrow. The arrow indicates the direction. The arrow shows the direction from the **initial point** or tail of this vector to its **terminal point** or tip. Two vectors with the same length and same direction are called **equal vectors.**

Definition: If $\vec{v}$ and $\vec{u}$ are two vectors positioned in a way that the initial point of $\vec{v}$ is the same as the terminal point of $\vec{u}$, then $\vec{u}+\vec{v}$ is the vector whose initial point is the initial point of $\vec{u}$ and its terminal point is the terminal point of $\vec{v}$. This definition is summarized in the picture below as the **Triangle Law**:



The Triangle Law



The Parallelogram Law

If the initial points of $\vec{u}$ and $\vec{v}$ are the same, then we can draw a vector equal to $\vec{v}$ from the terminal point of $\vec{u}$ and use the Triangle Law. This law -shown in the above picture- is called the **Parallelogram Law.**

Definition: By a **scalar** we mean a real number. For a scalar c and a vector $\vec{u}$, we can define the **scalar multiple** $c\vec{u}$ to be a vector whose length is equal to $|c|$ times the length of $\vec{u}$ and whose direction is the same as the direction of $\vec{u}$ if $c>0$ and opposite the direction of $\vec{u}$ if $c<0$. If $c=0$, the vector $c\vec{u}$ is the zero vector.

Definition: Two non-zero vectors are **parallel** if they are scalar multiples of one another. The vector $(-1)\vec{u}$ is called the **negative** of $\vec{u}$ and is denoted by $-\vec{u}$. The **difference** $\vec{u}-\vec{v}$ is defined to be the sum $\vec{u}+(-\vec{v})$.

Definition: If the initial point of a vector $\vec{u}$ is placed at the origin and its terminal point has coordinates (x, y, z) then these coordinates are called the components of $\vec{u}$ and we write

$\vec{u} = <x, y, z>$.

If coordinates of the initial point of vector $\vec{u}$ are (x_1, y_1, z_1) and coordinates of its terminal point are (x_2, y_2, z_2) then $\vec{u} = <x_2 - x_1, y_2 - y_1, z_2 - z_1>$.

The **length** or **magnitude** of a vector $\vec{u}$ is denoted by $|\vec{u}|$.

Given two vectors $\vec{u} = <x, y, z>$ and $\vec{v} = <a, b, c>$ and a scalar d, we have:

- $\vec{u} + \vec{v} = <x + a, y + b, z + c>$
- $\vec{u} - \vec{v} = <x - a, y - b, z - c>$
- $d\vec{u} = <dx, dy, dz>$
- $|\vec{u}| = \sqrt{x^2 + y^2 + z^2}$

For any positive integer n, we denote by V_n the set of all ordered n – tuples $\vec{u} = <x_1, x_2, \cdots, x_n>$ where $x_1, x_2, \cdots, x_n$ are real numbers. These real numbers are called components of $\vec{u}$.

Properties of Sum and Scalar Multiplication: For any three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in V_n and scalars c and d, we have the following:

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $\vec{u} + \vec{0} = \vec{u}$
- $\vec{u} + (-\vec{u}) = \vec{0}$
- $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
- $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
- $(cd)\vec{u} = c(d\vec{u})$
- $1 \cdot \vec{u} = \vec{u}$

Definition: The vectors $\vec{i} = <1, 0, 0>, \vec{j} = <0, 1, 0>$ and $\vec{k} = <0, 0, 1>$ are called the **standard basis vectors.**

Note: Any vector $\vec{u} = <x, y, z>$ can be written as $\vec{u} = x\vec{i} + y\vec{j} + z\vec{k}$.

Definition: A vector is called a **unit vector** if its length is 1.

Note: If $\vec{u} \neq \vec{0}$ then the vector $\vec{u}/|\vec{u}|$ is the unit vector that has the same direction as $\vec{u}$.

Definition: The **dot product** of two vectors $\vec{u} = <x_1, y_1, z_1>$ and $\vec{v} = <x_2, y_2, z_2>$ is the number $\vec{u} \cdot \vec{v} = x_1x_2 + y_1y_2 + z_1z_2$. This is sometimes called the **scalar product**, or the **inner product** of

$\vec{u}$ and $\vec{v}$.

Properties of the Dot Product: Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be three vectors and c be a scalar, then:

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $\vec{u} \cdot \vec{u} = |\vec{u}|^2$
- $\vec{u} \cdot \vec{0} = 0$
- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
- $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$

Theorem: Let θ be the angle between vectors $\vec{u}$ and $\vec{v}$, then $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$.

Note: Given components of two vectors you can use the above formula to find the cosine of the angle between these two vectors, which may be used to evaluate this angle.

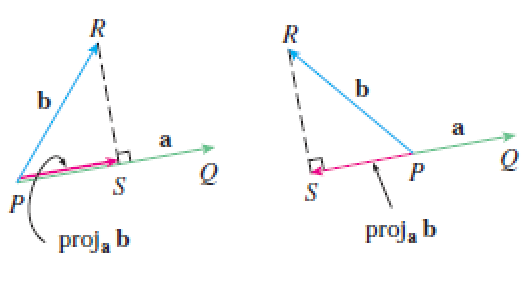
Definition: Two non-zero vectors are called **orthogonal** or **perpendicular** if the angle between them is $\pi/2$.

Note: Two non-zero vectors are orthogonal if and only if their dot product is zero.

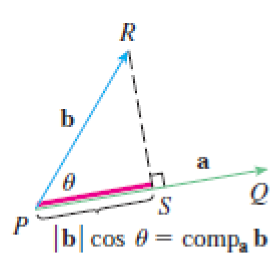
Definition: The **direction angles** of a non-zero vector $\vec{u} =< a, b, c >$ are the angles α, β and γ (in $[0, \pi]$) that $\vec{u}$ makes with the positive x, y and $z-$axes. The cosines of these angles are called the **direction cosines** of $\vec{u}$.

Note: Given the above notations we have: $\cos\alpha = a/|\vec{u}|$, $\cos\beta = b/|\vec{u}|$, $\cos\gamma = c/|\vec{u}|$.

The **vector projection** of a vector $\vec{b}$ onto a vector $\vec{a}$, shown in the picture below, is denoted by $proj_a\vec{b}$. The **scalar projection of** $\vec{b}$ **onto** $\vec{a}$ (or the **component of** $\vec{b}$ **along** $\vec{a}$) is defined to be the signed magnitude of this vector projection and is denoted by $comp_a\vec{b}$.



Vector projections



Scalar projection

These two quantities can be evaluated from the following formulas:

$$proj_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} \text{ and } comp_{\vec{a}}\vec{b} = \vec{a}\cdot\vec{b}/|\vec{a}|.$$

Definition: For any four real numbers a, b, c and d, define the **determinant of order 2** by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A **determinant of order 3** is defined in terms of determinants of order 2 as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Definition: The **cross product** $\vec{a}\times\vec{b}$ of two vectors $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$ is defined as the following determinant of order 3:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Note: Cross products are vectors but dot products are scalars. You cannot add vectors and scalars. So an expression like $\vec{u}\times\vec{v}+\vec{u}\cdot\vec{v}$ does not have any meaning.

The **scalar triple product** of $\vec{u}$, $\vec{v}$ and $\vec{w}$ is $\vec{u}\cdot(\vec{v}\times\vec{w})$.

Properties of Cross Products: Let $\vec{u}, \vec{v}$ and $\vec{w}$ be three vectors in 3-D space and c be a scalar. Then:

- $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$
- $\vec{v}\times\vec{u} = -\vec{u}\times\vec{v}$
- If θ is the angle between $\vec{u}$ and $\vec{v}$, $(0 \le \theta \le \pi)$ then $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$
- The non-zero vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u}\times\vec{v} = 0$
- The length of $\vec{u}\times\vec{v}$ equals the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$
- $(c\vec{u})\times\vec{v} = c(\vec{u}\times\vec{v}) = \vec{u}\times(c\vec{v})$
- $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v}+\vec{u}\times\vec{w}$
- $(\vec{v}+\vec{w})\times\vec{u} = \vec{v}\times\vec{u}+\vec{w}\times\vec{u}$
- $\vec{u}\cdot(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\cdot\vec{w}$
- $\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{u}\cdot\vec{w})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w}$ [You do not need to memorize this identity.]
- $|\vec{u}\cdot(\vec{v}\times\vec{w})|$ is the volume of the parallelepiped determined by vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$

Definition: A non-zero vector $\vec{u} = < a, b, c >$ is called a **direction vector** of a line L if $\vec{u}$ is parallel to L. Numbers a, b and c are called **direction numbers** of L.

Equations of a Line: Let L be a line passing through (x_0, y_0, z_0) with direction vector $\vec{u} = < a, b, c >$. The following are different equations of L.

- $\vec{r} = \vec{r_0} + t\vec{u}$, where $\vec{r_0} = < x_0, y_0, z_0 >$, and t is a real number. (vector equation of L)
- $x = x_0 + ta,\ y = y_0 + tb,\ z = z_0 + tc,$ where t is a real number. (parametric equations of L)
- $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$. If $a = 0$ this should be written as $x = x_0$ and $\frac{y-y_0}{b} = \frac{z-z_0}{c}$ (symmetric equations of L)

Note: To find an equation of a line through two points first you need to find its direction vector by subtracting these two points and then use one of the above formulas.

Let $\vec{a} = < a_1, a_2, a_3 >$ and $\vec{b} = < b_1, b_2, b_3 >$ be two vectors. The line segment from (a_1, a_2, a_3) to (b_1, b_2, b_3) is given by $\vec{r} = (1 - t)\vec{a} + t\vec{b}$ where $0 \le t \le 1$.

Note: To check if two lines are parallel find their direction vectors and check if they are scalar multiples of one another.

Definition: Two lines are called **skew lines** if they are not parallel and they do not intersect.

Definition: A vector $\vec{n}$ is called a **normal vector** of a plane if it is orthogonal to the plane, i.e. $\vec{n}$ is orthogonal to all vectors in that plane.

Equations of a Plane: Let $A(x_0, y_0, z_0)$ be a point on a plane P and $\vec{n} = < a, b, c >$ be a normal vector of P. Two equations of P are as follows:

- $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}$ where $\vec{r_0} = < x_0, y_0, z_0 >$ (vector equation of P)
- $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ (scalar equation of P)

The second equation can be rewritten as $ax + by + cz + d = 0$ where $d = - ax_0 - by_0 - cz_0$. This equation is called a **linear equation** of P.

Note: To check if two planes are parallel check if their normal vectors are scalar multiples of one another.

Note: To find an equation of a plane passing through three given points A, B and C, find two vectors inside this plane by connecting two of these points (e.g. $\vec{AB}$ and $\vec{AC}$). Then find the cross product of these vectors to find a normal vector to this plane. Then use the above formula.

Note: The distance D from a point (x, y, z) to the plane $ax + by + cz + d = 0$ is calculated by:

$$D = \frac{|ax+by+cz+d|}{\sqrt{a^2+b^2+c^2}}$$

Chapter 13

Definition: A **vector function** is a function whose domain is a subset of real numbers and whose range is a set of vectors.

Definition: If $\vec{r}(t) = < f(t),\ g(t),\ h(t) >$ is a vector function, then

$$\lim_{t\to a} \vec{r}(t) = < \lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t) > .$$

Definition: A vector function $\vec{r}$ is called **continuous** at a if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$.

Definition: If $\vec{r}(t) = < f(t),\ g(t),\ h(t) >$ is a continuous vector function on an interval I, then the set of all points (x, y, z) in the space such that $x = f(t)$, $y = g(t)$ and $z = h(t)$ where $t \in I$ is called a **space curve**. These equations are called parametric equations of this space curve.

Note: To identify a space curve it is best to find a relation between x, y and z. For example to identify the curve $< \sin t,\ \cos t,\ \cos t >$ notice that the curve is on the cylinder $x^2 + y^2 = 1$ and on the plane $y = z$.

Definition: Derivative and integral of a vector function $\vec{r}(t) = < f(t),\ g(t),\ h(t) >$ is defined similar to its limits by looking at the components:

$$\vec{r}\,'(t) = < f'(t),\ g'(t),\ h'(t) >$$

$$\int_a^b \vec{r}(t)\, dt = < \int_a^b f(t)\, dt,\ \int_a^b g(t)\, dt,\ \int_a^b h(t)\, dt >$$

Definition: The **tangent line** to the space curve $\vec{r}(t)$ at a point P is a line through P and parallel to the tangent vector $\vec{r}\,'(t)$.

Differentiation Rules: Assume $\vec{u}$ and $\vec{v}$ are two vector functions, c is a scalar and f is a real-valued function. Then

- $\frac{d}{dt}(\vec{u}(t) \pm \vec{v}(t)) = \vec{u}\,'(t) \pm \vec{v}\,'(t)$

- $\frac{d}{dt}(c\vec{u}(t)) = c\vec{u}'(t)$
- $\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$
- $\frac{d}{dt}(\vec{u}(t) \cdot \vec{v}(t)) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
- $\frac{d}{dt}(\vec{u}(t) \times \vec{v}(t)) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$
- $\frac{d}{dt}(\vec{u}(f(t))) = f'(t)\vec{u}'(f(t))$

For a space curve given by $\vec{r}(t)$ where $a \le t \le b$, its length is evaluated by $\int_a^b |\vec{r}'(t)|\, dt$. The **arc length function** s is given by $s(t) = \int_a^t |\vec{r}'(u)|\, du$.

To **parametrize a curve with respect to arc length** evaluate s from the above formula. Then find t in terms of s by solving the equation for t and then use that to evaluate $\vec{r}$ as a function of s.

Definition: If the position vector of a moving particle at time t is given by $\vec{r}(t)$ its **velocity** is given by $\vec{v}(t) = \vec{r}'(t)$. Its **speed** is the scalar $|\vec{v}(t)|$. Its **acceleration** is given by $\vec{a}(t) = \vec{r}''(t) = \vec{v}'(t)$.

Chapter 14

Definition: A **function** f **of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number $f(x, y)$. The **graph** of f is the set of all (x, y, z) in the 3-D space when $z = f(x, y)$.

Definition: The **level curves** of a function f of two variables are the curves with equations $f(x, y) = c$, where c is a constant.

Note: Functions of 3 or more variables are defined in the same manner. We can define **level surfaces** of a 3 variable function is a surface given by $f(x, y, z) = c$, where c is a constant.

Definition: Let f be a function of two variables. We say $\lim_{(x, y) \to (a, b)} f(x, y) = L$ if $f(x, y)$ can be made arbitrarily close to L (i.e. $|f(x, y) - L|$ can be made arbitrarily small) when (x, y) is made sufficiently close to (a, b), but not equal to (a, b) (i.e. $(x - a)^2 + (y - b)^2$ is made sufficiently small but not zero).

Note: Assume

- $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a curve C_1 and
- $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a curve C_2 and
- $L_1 \neq L_2$

Then $\lim_{(x, y)\to(a, b)} f(x, y)$ does not exists.

Definition: A function $f(x, y)$ is called **continuous** at (a, b) if $\lim_{(x, y)\to(a, b)} f(x, y) = f(a, b)$.

Note: Limits and continuity are defined in the similar manner for functions of 3 or more variables.

Definition: A function f is called a **polynomial of two variables** if $f(x, y)$ is the sum of terms of the form $cx^n y^m$, where c is a constant and n and m are non-negative integers. A **rational function of 2 variables** is the ratio of two polynomials of 2 variables. Similarly you can define polynomials and rational function of 3 or more variables.

Note: All polynomials and rational functions of 2 or more variable are continuous on their domains.

Note: To evaluate $\lim_{(x, y)\to(a, b)} f(x, y)$ try the following:

1. Use the direct substitution property (i. e. number plugging) if the function is continuous.
2. If $(a, b) \neq (0, 0)$, take $u = x - a$ and $v = y - b$ and write $f(x, y)$ as a function $g(u, v)$ of u and v. Then we can write the limit as $\lim_{(x, y)\to(a, b)} f(x, y) = \lim_{(u, v)\to(0, 0)} g(u, v)$. So we need to evaluate limits when approaching $(0, 0)$.
3. When approaching $(0, 0)$, try evaluating the limit along different lines through the origin. i.e. approach the origin along $y = mx$ and $x = 0$. If you get different limits then the limit does not exists and we are done. If all limits are the same try different curves like $y = x^2$, $x = y^2$, etc. If two of the limits are different then the limit does not exist. If all limits are the same then suspect the limit exists and try to prove it using the next steps.
4. Write x and y in polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. As $(x, y) \to (0, 0)$, we know $r \to 0$. Simplify this expression and try evaluating this limit using methods for evaluating limits of functions of one variable r, e. g. the Squeeze Theorem. Make sure to notice θ changes and could be any angle. If you want to plug in $r = 0$ and the expression involves θ, you need to first use the Squeeze Theorem to get functions depending only on r.

Definition: **Partial derivative of $f(x, y)$ with respect to x at (a, b)** is given by

$$f_x(a, b) = g'(a) \text{ where } g(x) = f(x, b)$$

or

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

Similarly you can define the partial derivative of f with respect to y at (a, b) denoted by $f_y(a, b)$.

For a function f of two variables x and y, the partial derivative functions f_x and f_y are defined by $f_x(x, y) = \lim\limits_{h\to 0} \frac{f(x+h, y)-f(x, y)}{h}$ and $f_x(x, y) = \lim\limits_{h\to 0} \frac{f(x, y+h)-f(x, y)}{h}$. Alternative notations for partial derivatives are f_x, $D_x f$, $D_1 f$ and $\frac{\partial f}{\partial x}$.

Note: To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Similar to the above definitions we can define partial derivatives for functions of 3 or more variables. For instance derivative of a function of 3 variables with respect to x is defined as

$$f_x(x, y, z) = \lim_{h\to 0} \frac{f(x+h, y, z)-f(x, y, z)}{h}$$

Definition: For a function of 2 variables f, the functions f_x and f_y are functions of 2 variables. Partial derivatives of f_x and f_y are called **second partial derivatives** of f. We use the following notations:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = f_{11} = (f_x)_x$$
$$\frac{\partial^2 f}{\partial y^2} = f_{yy} = f_{22} = (f_y)_y$$
$$\frac{\partial^2 f}{\partial x\,\partial y} = f_{yx} = f_{21} = (f_y)_x$$
$$\frac{\partial^2 f}{\partial y\,\partial x} = f_{xy} = f_{12} = (f_x)_y$$

Note that for evaluating f_{xy} (or $\frac{\partial^2 f}{\partial y \partial x}$) we first differentiate f with respect to x and then we differentiate f_x with respect to y.

Clairaut's Theorem: Suppose $f(x, y)$ is defined on a disk D that contains (a, b). If the functions f_{xy} and f_{yx} are continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

Tangent Plane: Suppose f has continuous partial derivatives. Assume $P(x_0, y_0, z_0)$ is a point on the surface $z = f(x, y)$. An equation of the tangent plane to the surface $z = f(x, y)$ at P is $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

Definition: **Linear Approximation** or **Tangent Plane Approximation** of a 2 variable function f at (a, b) is the approximation given by $f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$.

Definition: If $z = f(x, y)$, then f is **differentiable** at (a, b) if $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$ can

be expressed in the form:
$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$ where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Theorem: If f_x and f_y exist and are continuous near (a, b), then f is differentiable at (a, b).

The Chain Rule:

1. Assume $f(x, y)$ is differentiable and $x = g(t)$ and $y = h(t)$ are differentiable functions of t. Then f is a differentiable function of t and $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.
2. Assume $f(x, y)$ is differentiable and $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then f is a differentiable function of s and t and $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$ and $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.
3. Suppose f is a differentiable function of $x_1, x_2, \cdots, x_n$ and each x_i is a differentiable function of $t_1, t_2, \cdots, t_m$. Then f is a differentiable function of $t_1, t_2, \cdots, t_m$ and $\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}$.

Implicit Differentiation: If y as a function of x is given by $F(x, y) = 0$ then $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

Definition: The **directional derivative** of a function $f(x, y)$ at (x_0, y_0) in the direction of a unit vector $\vec{u} = < a, b >$ is defined by $D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$.

Note: If a non-zero vector $\vec{u}$ is not a unit vector, to find the directional derivative of $f(x, y)$ at (x_0, y_0) in the direction of $\vec{u}$, you need to evaluate $D_{\vec{v}} f(x_0, y_0)$ where $\vec{v} = \vec{u}/|\vec{u}|$.

Definition: The **gradient** of a function $f(x, y)$ is defined as $\nabla f(x, y) = < f_x(x, y), f_y(x, y) >$.

Theorem: If $f(x, y)$ is differentiable then it has a directional derivative in any direction and for a unit vector $\vec{u}$, we have $D_{\vec{u}} f(x, y) = < f_x(x, y), f_y(x, y) > \cdot \vec{u} = \nabla f(x, y) \cdot \vec{u}$.

Similarly one can define directional derivatives for functions of 3 or more variables as follows:

Definition: For a unit vector $\vec{u}$ in 3-D space and a function f of 3 variables x, y and z we define the directional derivative of f at (x_0, y_0, z_0) in the direction $\vec{u}$ by ,

$$D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h}$$

where $\vec{a} = < x_0, y_0, z_0 >$.

Theorem: Assume (x, y) is a point on the plane. Assume $f(x, y)$ is a differentiable function. The maximum value of $D_{\vec{u}} f(x, y)$ is $|\nabla f(x, y)|$ and occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(x, y)$. The same holds for functions of 3 variables.

Equation of tangent planes to level surfaces: Let F be a 3 variable function. Assume c is a constant and $P(x_0, y_0, z_0)$ is a point on the level surface $F(x, y, z) = c$. Then $\nabla F(x_0, y_0, z_0)$ is a normal line to the tangent plane to this level surface at P. An equation of this tangent plane is given by $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$.

Equation of normal line to level surface $F(x, y, z) = c$ is given by:

$$\frac{x-x_0}{F_x(x_0, y_0, z_0)} = \frac{y-y_0}{F_y(x_0, y_0, z_0)} = \frac{z-z_0}{F_z(x_0, y_0, z_0)}.$$

Definition: A function f of two variables has a **local minimum** at (a, b) if $f(a, b) \le f(x, y)$ for any (x, y) near (a, b). The number $f(a, b)$ is called a **local minimum value**. Similarly f has a **local maximum** at (a, b) if $f(a, b) \ge f(x, y)$ for any (x, y) near (a, b). The number $f(a, b)$ is called a **local maximum value**. Similar to functions of one variable if these inequalities hold for *all* values of (x, y) we say f has an **absolute minimum** or **absolute maximum** at (a, b).

Definition: A point (a, b) is called a **critical point** of f, if $f_x(a, b) = f_y(a, b) = 0$ or if one of these partial derivatives does not exist.

Theorem: If f has a local maximum or minimum at (a, b), then f has a critical point at (a, b).

Second Derivatives Test: Assume the second partial derivatives of f are continuous on a disk centered at (a, b). Suppose (a, b) is a critical point of f and let

$$D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

1. If $D > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b).
2. If $D > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b).
3. If $D < 0$, then $f(a, b)$ is not a local maximum or a local minimum. In this case we say f has a **saddle point** at (a, b).

Note: If $D = 0$, the test is inconclusive, i.e. f could have a local minimum, a local maximum or a saddle point at (a, b).

Note: To remember the formula of D you can write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}\, f_{yy} - (f_{xy})^2$$

Definition: A subset E of plane or space is called **bounded** if there is a circle or a sphere that contains all points of $E.$ The subset E is called **closed** if it contains its boundary.

Extreme Value Theorem: Let E be a closed and bounded subset of the plane or 3-D space. If f is a 2 variable or 3 variable function continuous on $E,$ then there are points $x, y \in E$ such that f has an absolute maximum at x and an absolute minimum at y on $E.$

Note: Let E be a closed and bounded subset of the plane. Assume f is a continuous function on $E.$ To find the absolute maximum and minimum values of f on $E,$

1. Find the values of f at the critical points of f in $E.$
2. Find the extreme values of f on the boundary of $E.$
3. The largest value of the values from the above 2 steps is the absolute maximum value of f on E and the smallest value of the values of the above 2 steps is the absolute minimum value of f on $E.$

Method of Lagrange Multipliers: To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$:

1. Find all values of x, y and z and λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = k.$
2. Evaluate f at all points (x, y, z) that result from the first step. The largest of these values is the maximum value of f and the smallest value is the minimum value of $f.$

Note: The Method of Lagrange Multipliers can be used only when the extreme values exist and $\nabla g \neq \vec{0}$ on the surface $g(x, y, z) = k.$

Note: If there are 2 constraints $g(x, y, z) = k$ and $h(x, y, z) = c,$ then we need to solve the following equation in the first step: $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z).$ The second step remains the same. In this case we need to make sure the extreme values exist and $\nabla g(x, y, z)$ and $\nabla h(x, y, z)$ are not zero and are not parallel.

Chapter 15

Definition: Let $R = I \times J$ be a rectangle in the xy – plane. Let n and m be two positive integers. Divide the interval I into n sub-intervals of equal length with endpoints $x_0, x_1, \cdots, x_n$ and divide J into m sub-intervals of equal length whose endpoints are $y_0, y_1, \cdots, y_m.$ These points divide R into mn rectangles. Pick a sample point (x_{ij}^*, y_{ij}^*) inside the rectangle number $(i, j).$ Let ΔA be the area of each of these sub-rectangles. The double integral of f over R is defined to be

$$\int\int_R f(x, y)\, dA = \lim_{m,\, n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

Definition: The sum $\sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*)\Delta A$ is called a **double Riemann Sum.**

Note: If $f(x, y) \geq 0$ then the volume of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is given by the double integral $\int\int_R f(x, y)\, dA.$

Midpoint Rule: $\int\int_R f(x, y)\, dA \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(\overline{x_i}, \overline{y_j})\Delta A$ where $\overline{x_i}$ and $\overline{y_j}$ are the midpoints of $i-$th and $j-$th sub-intervals of I and J, respectively.

Definition: Let D be any bounded plane region. Assume R is a rectangle containing $D.$ For any function f over D define a new function $F(x, y)$ over R to be the same as $f(x, y)$ when $(x, y) \in D$ and zero when $(x, y) \notin D.$ The double integral of f over D is defined to be the double integral of F over R: $\int\int_D f(x, y)\, dA = \int\int_R F(x, y)\, dA.$

Definition: For any region $R,$ the **average value** of f over D is defined as $f_{ave} = \frac{1}{A(D)} \int\int_D f(x, y)\, dA,$ where $A(D)$ is the area of $D.$

Properties of Double Integrals: For a constant c and two functions f and g we have:

- $\int\int_D f(x, y) + g(x, y)\, dA = \int\int_D f(x, y)\, dA + \int\int_D g(x, y)\, dA$
- $\int\int_D cf(x, y)\, dA = c\int\int_D f(x, y)\, dA$
- $\int\int_D f(x, y)\, dA \geq \int\int_D g(x, y)\, dA$ if $f(x, y) \geq g(x, y)$ for any $(x, y) \in D$
- $\int\int_D f(x, y)\, dA = \int\int_{D_1} f(x, y)\, dA + \int\int_{D_2} f(x, y)\, dA,$ where D is the union of D_1 and D_2 and they do not overlap except possibly at their boundaries.
- $\int\int_D 1\, dA = A(D),$ the area of region $D.$

Definition: Starting with an integrable function of two variables $f(x, y)$ on a rectangle $R = [a, b] \times [c, d],$ we can integrate f with respect to y to get a function of $x,$ i. e. define $A(x) =$

$\int_c^d f(x, y)\, dy$. Since this, itself, is a function of x, we can integrate that with respect to x to get a constant number, i. e. $\int_a^b A(x)\, dx = \int_a^b [\int_c^d f(x, y)\, dy]\, dx$. This is called an **iterated integral**.

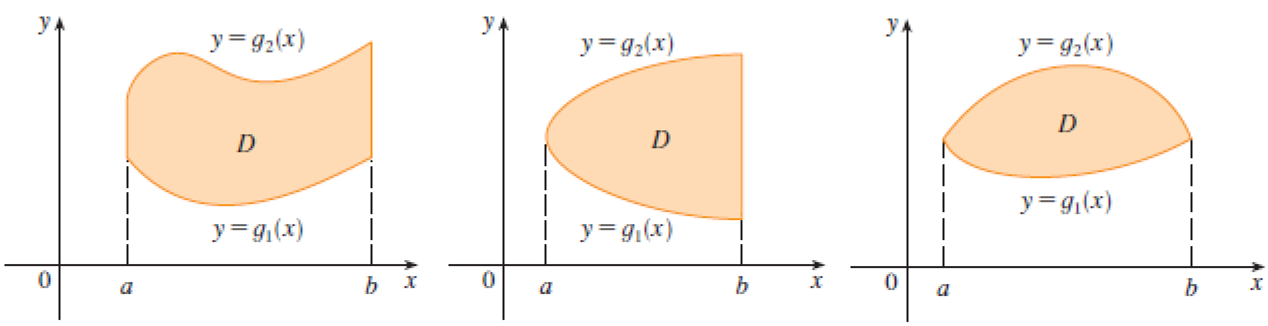
Fubini's Theorem: Assume a, b, c and d are constants. If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dxdy.$$

Note: For two continuous functions f and g we have,

$$\iint_R f(x)g(y)\, dA = \int_a^b f(x)\, dx \times \int_c^d g(y)\, dy.$$

Definition: A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, i. e. there are two constants a and b and two continuous functions $g_1(x)$ and $g_2(x)$ such that $D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$.



Some type I regions

Definition: A region of **type II** is a region on the plane that can be expressed as

$$D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

for two constants c and d and two functions $h_1(y)$ and $h_2(y)$.

Note: The double integral over a type I region D as above is evaluated by,

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

The double integral over a type II region D as above is evaluated by,

$$\int\int_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

For a type I region, to find a and b, find the maximum and minimum possible values of x – coordinates of points inside D. Fixing a number x between a and b, find the maximum and minimum values of y, i.e. look at the vertical line through $(x, 0)$ and find functions $g_1(x)$ and $g_2(x)$.

Similarly for type II regions to find c and d, find maximum and minimum values of y – coordinates of points in D and then look at horizontal lines.

Change to Polar Coordinates in Double Integrals: Let f be a continuous function on a polar rectangle R given by $0 \le a \le r \le b$ and $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$. Then

$$\int\int_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta,\ r\sin\theta)\, r\, dr\, d\theta.$$

Note: One common error when switching from rectangular coordinates (x, y) to polar coordinates (r, θ) is forgetting the additional factor of r on the right hand side of the above equality.

Note: If f is continuous on a polar region $D = \{(r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$, then

$$\int\int_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\ r\sin\theta)\, r\, dr\, d\theta.$$

Note: To find α and β for a region D, find maximum and minimum possible values of θ for all points in D. Make sure all values between α and β are possible values of θ. Then fix θ and look at the ray through the origin corresponding to θ. Find the maximum and minimum values of r for such θ. These would give us the functions $h_2(\theta)$ and $h_1(\theta)$. Make sure any value between $h_1(\theta)$ and $h_2(\theta)$ is an r value for some point in D.

Chapter 16

Definition: Let D be subset of the xy – plane (or the xyz – space). A 2-dimensional (or 3-dimensional) **vector field** F is a function that assigns a vector $F(x, y)$ (or $F(x, y, z)$) to any (x, y) (or (x, y, z)) in D.

Note: For any scalar function $f(x, y, z)$ the gradient ∇f is a vector field.

Definition: A vector field F is called **conservative** if it is the gradient of a scalar function, i. e. $F = \nabla f$ for some function f. This function f is called a **potential function** for F.

Definition: A plane or space curve C given by $\vec{r}(t)$ over an interval I is called smooth if $\vec{r}'(t)$ exists for all $t \in I$ and $\vec{r}'(t) \neq \vec{0}$.

Assume C is a smooth curve given by $\vec{r}(t) = < x(t), y(t) >, \; t \in [a, b]$. Divide the interval $[a, b]$ into sub-intervals of equal length and in each sub-interval pick a smaple real number t_i^* and set $x_i^* = x(t_i^*)$ and $y_i^* = y(t_i^*)$. Let $\Delta s_1, \Delta s_2, \cdots, \Delta s_n$ be the lengths of corresponding sub-arcs.

Definition: If f is defined on a smooth curve C given by $\vec{r}(t) = < x(t), y(t) >$, then the line integral of f along C with respect to arc length, x or y are defined as follows:

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i \text{ (Line integral with respect to arc length.)}$$

$$\int_C f(x, y)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i \text{ (Line integral with respect to } x\text{.)}$$

$$\int_C f(x, y)\, dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i \text{ (Line integral with respect to } y\text{.)}$$

Definition: A curve C is called **piecewise smooth**, if it is a finite union of smooth curves. If $C = C_1 \cup C_2 \cup \cdots \cup C_n$ where C_i's are smooth curves, then the integral of a function f over C is defined to be the sum of integrals of f over C_i's.

Evaluating Line Integrals:

- $\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$
- $\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$
- $\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$

Definition: Line integrals in space are defined similarly. Assume the space curve C is given by $\vec{r}(t),\; a \le t \le b$ and $f(x, y, z)$ is a function defined on C, the integral of f along C is evaluated by:

- $\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\, |\vec{r}'(t)|\, dt$ (Line integral with respect to arc length.)

- $\int_C f(x, y, z)\, dx = \int_a^b f(x(t), y(t), z(t))\, x'(t)\, dt$ (Line integral with respect to x.)
- $\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\, y'(t)\, dt$ (Line integral with respect to y.)

Line Integrals of Vector Fields: Let F be a continuous vector field defined on a smooth curve C given by a vector function $\vec{r}(t),\ a \le t \le b$. Then the line integral of F along C is defined as,

$$\int_C F \cdot d\vec{r} = \int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_C F \cdot \vec{T}\, ds, \text{ where } \vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|.$$

Note: When line integrals with respect to x, y or z occur together we may write them as one integral as below: $\int_C P\,dx + Q\,dy + R\,dz$ is the same as $\int_C P\,dx + \int_C Q\,dy + \int_C R\,dz.$

The Fundamental Theorem of Line Integrals: Let C be a smooth curve given by $\vec{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient is continuous on C. Then $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$

Definition: Let F be a vector field with domain D. The line integral $\int_C F \cdot d\vec{r}$ is said to be **independent of path** if $\int_{C_1} F \cdot d\vec{r} = \int_{C_2} F \cdot d\vec{r}$ for any two path C_1 and C_2 in D that have the same initial and terminal points.

Definition: A curve C is called **closed** if its terminal point is the same as its initial point.

Theorem: $\int_C F \cdot d\vec{r}$ is independent of path in D if and only if $\int_C F \cdot d\vec{r} = 0$ for any closed path C in D.

Definition: A plane set D is said to be **open** if for every point P in D, there is a disk around P that lies entirely in D. It is said to be **connected** if any two points inside D can be joined by a path inside D.

Theorem: Suppose F is a vector field that is continuous on an open and connected plane set D. If $\int_C F \cdot d\vec{r}$ is independent of path in D, then F is a conservative vector field on D.

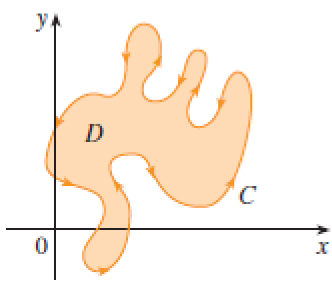
Theorem: If $F(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a conservative vector field and P and Q have continuous first-order partial derivatives on a domain D. Then throughout D we have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Definition: A curve is called **simple** if it does not intersect itself anywhere between its endpoints.

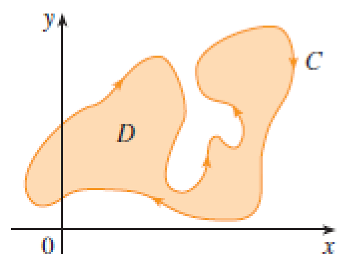
Definition: A plane region D is called **simply-connected** if it is connected and each simple closed curve in D encloses only points of D.

Theorem: Let $F = < P(x, y), Q(x, y) >$ be a vector field on an open, simply-connected plane region D. Suppose P and Q have continuous first-order partial derivatives and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D, then F is conservative.

Definition: **Positive orientation** of a simple closed curve C is a single counterclockwise traversal of C.



(a) Positive orientation



(b) Negative orientation

Green's Theorem: Let C be a positively oriented piecewise-smooth simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region containing D, then:

$$\int_C P\,dx + Q\,dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Sometimes we use the notation $\oint_C P\,dx + Q\,dy$ to indicate C is positively oriented. We may also use $\int_{\partial D} P\,dx + Q\,dy$ to indicate a positive orientation of the boundary of D is used.

Note: When using the Green's Theorem make sure all conditions are satisfied. A very common error is to use the Green's Theorem when the vector field is not defined or is not continuous inside D.

Definition: Let D be a plane region (possibly with holes). An orientation of the boundary ∂D is a

positive orientation if the region is on the left when this boundary is traversed.

Note: Green's Theorem is true for more general regions and a positive orientation discussed above. In general Green's Theorem can be stated as,

$$\int_{\partial D} P\,dx + Q\,dy = \int\int_{D} \left(\tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y}\right) dA.$$

Definition: For a 3-D vector field $F = \langle P, Q, R \rangle$ we define **curl of** F, denoted by $curl\,F$, to be the following vector field.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$= \operatorname{curl} \mathbf{F}$$

Theorem: If f is a 3 variable function whose second order partial derivatives are continuous, then $curl(\nabla f) = \vec{0}$.

Theorem: If F is a vector field defined on the whole 3-D space whose components have continuous partial derivatives and $curl\,F = \vec{0}$, then F is conservative.

Note: To find a potential function for a conservative vector field $F = \langle P, Q, R \rangle$,

1. Write $P = f_x$, $Q = f_y$, $R = f_z$
2. Regard y and z as constants and integrate P with respect to x. You will then find $f = \int P\,dx + g(y, z)$. Notice that since we regard y and z as constants we need to add a function of y and z (i.e. $g(y, z)$) instead of a constant.
3. Differentiate the above identity with respect to y to get f_y. Use that to solve $f_y = Q$ by simplifying both sides first and then integrating with respect to y (considering z as a constant). Doing that you will be able to evaluate $g(y, z)$ up to a function $h(z)$ of only z. Now plug that back into the equation for f to evaluate f.
4. Use the expression that you get for f and the last identity $f_z = R$ to evaluate a potential function f by integrating both sides with respect to z.

Definition: For a vector field $F = \langle P, Q, R \rangle$ define $div\,F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

Using the “del” notation we have: $div\ F = \nabla \cdot F$.

Theorem: If F is a 3-D vector field whose components have continuous second-order partial derivatives, then $div\ (curl\ F) = 0$.

Definition: The set of all points (x, y, z) such that $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$ where u, v are in a plane set D form a surface S called a **parametric surface** and these equations are called **parametric equations** of S.

Note: Lets consider the surface S obtained by rotating the curve $y = f(x) \geq 0$ for $a \leq x \leq b$ about the $x-$axis. Parametric equations of this surface of revolution is given by:

$$x = x,\ y = f(x)\cos\theta,\ z = f(x)\sin\theta.$$

Definition: A parametric surface S given by the position vector $\vec{r}(u, v)$ is called **smooth** if $\vec{r_u} \times \vec{r_v} \neq \vec{0}$ for any u and v, where $\vec{r_u}$ is the vector whose components are partial derivatives of components of $\vec{r}$ with respect to u and similarly for $\vec{r_v}$.

Definition: The tangent plane to S is a plane that contains the tangent vectors $\vec{r_u}$ and $\vec{r_v}$.

Note: A normal vector to the tangent plane is given by $\vec{r_u} \times \vec{r_v}$.

Definition: Let S be a smooth parametric surface given by $\vec{r}(u, v)$ where $(u, v) \in D$. Assume S is covered just once as (u, v) ranges throughout the parameter domain D. Then the **surface area** of S is evaluated by

$$A(S) = \int\int_D |\vec{r_u} \times \vec{r_v}|\ dA.$$

Surface area of the graph of a function: Let S be the surface $z = f(x, y)$, where $(x, y) \in D$. The area of S is evaluated by $A(S) = \int\int_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}\ dA$.

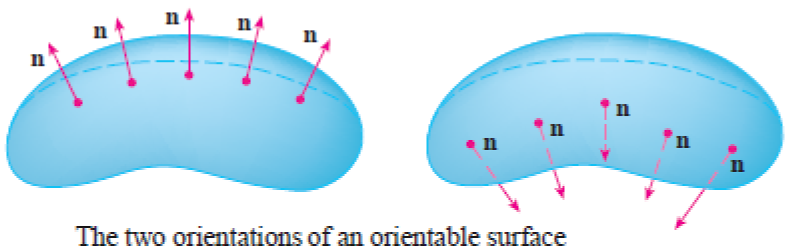
Surface Integrals: Let S be a smooth surface given by $\vec{r}(u, v)$ where $(u, v) \in D$ and assume S is covered only once as (u, v) ranges through D. Then

$$\int\int_S f(x, y, z)\ dS = \int\int_D f(\vec{r}(u, v))\ |\vec{r_u} \times \vec{r_v}|\ dA.$$

If S is the surface given by $z = g(x, y)$, then

$$\int\int_S f(x, y, z)\ dS = \int\int_D f(x, y, g(x, y))\sqrt{1 + (\tfrac{\partial z}{\partial x})^2 + (\tfrac{\partial z}{\partial y})^2}\ dA.$$

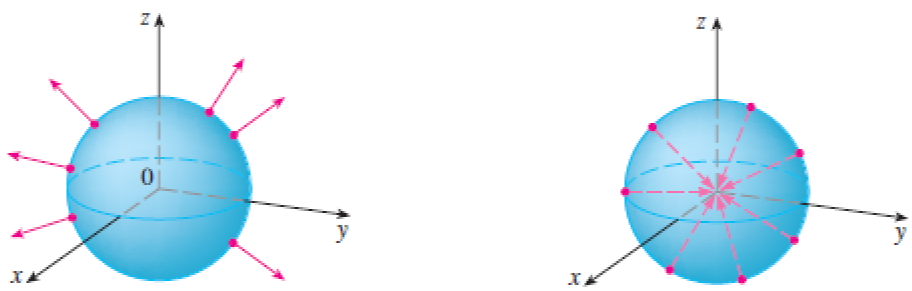
Definition: If it is possible to choose a unit normal vector $\vec{n}(x, y, z)$ at every point (x, y, z) on a surface S so that $\vec{n}$ varies continuously over S, then S is called an **oriented surface** and the given choice of $\vec{n}$ provides S with an **orientation**. For any oriented surface there are two possible orientations.



The two orientations of an orientable surface

For a smooth parametric surface S given by $\vec{r}(u, v)$, an orientation of S is given by $\vec{n} = (\vec{r}_u \times \vec{r}_v)/|\vec{r}_u \times \vec{r}_v|$. We can take $-\vec{n}$ to get another orientation of the same surface. If S is the graph of $g(x, y)$, then $\vec{n} = (-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k})/\sqrt{1 + (\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2}$ gives an orientation of S. Since in this orientation the $\vec{k}$ – component is positive this orientation is called the **upward orientation**.

Definition: A surface S is called a **closed surface** if it is the boundary of a solid E. A **positive orientation** of a closed surface S is the one that normal vectors point outward. Inward orientation is called a negative orientation.



Positive orientation

Negative orientation

Definition: Let F be a continuous vector field defined on an oriented surface S with unit normal vector $\vec{n}$, then the **surface integral of F over S** is

$$\int\int_S F \cdot dS = \int\int_S F \cdot \vec{n}\, dS.$$

This integral is also called the **flux** of F across S.

If the parametric surface S is given by $\vec{r}(u, v)$ where $(u, v) \in D$ then

$$\int\int_S F \cdot dS = \int\int_D F \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

If S is given by $z = g(x, y)$, where $(x, y) \in D$ and $F = \langle P, Q, R \rangle$, then

$$\iint_S F \cdot dS = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA.$$

Definition: Let S be an oriented surface with an orientation $\vec{n}$ and let C be the boundary of S. An orientation of C is called a **positive orientation** if when you walk in the positive direction around C with your head pointing in the direction of $\vec{n}$, then the surface S will always be on your left.

Note: Notice that an orientation for the boundary C is positive only *relative* to an orientation of S. if you change the orientation of S from $\vec{n}$ to $-\vec{n}$, then you need to change the orientation of C to get a positive orientation.

Stoke's Theorem: Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let F be a vector field whose components have continuous partial derivatives on an open region in 3-D space that contains S. Then

$$\int_C F \cdot d\vec{r} = \iint_S curl\, F \cdot dS$$

Note: Given a surface S with an orientation, its boundary along with a positive orientation relative to the orientation of S is usually denoted by ∂S.

Sections 15.6-15.8

Surface Areas of Graphs: The area of a surface given by $z = f(x, y)$, $(x, y) \in D$ is evaluated by

$$A(S) = \iint_D \sqrt{1 + (f_x(x, y))^2 + (f_y(x, y))^2}\, dA$$

Fubini's Theorem for Triple Integrals: Let a, b, c, d, e and f be constants. If g is a continuous function on the rectangular box $B = [a, b] \times [c, d] \times [e, f]$, then

$$\iiint_B g(x, y, z)\, dV = \int_a^b \int_c^d \int_e^f g(x, y, z)\, dz\, dy\, dx.$$

Note: In the above integration you can change the order of integration, but if you do make sure to change the limits of integration, too.

Definition: A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$$

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \le x \le u_2(y, z)\}$$

A solid region E is of **type 3** if it is of the form

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \le y \le u_2(x, z)\}.$$

Evaluating Triple Integrals for Regions of Type 1: Let E be a solid of type 1 given above, then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\right] dA$$

The inside integral is a single integral that may be evaluated using methods of integration. The outside double integral may be evaluated using methods of evaluating double integrals such as polar coordinates or Fubini's Theorem. If D is given by

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}.$$

Then

$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx.$$

Similarly we can evaluate triple integrals over regions of type 2 and 3.

Note: The volume of a solid E is the triple integral of 1 over E: $\iiint_E 1\, dV = V(E)$.

If the density of a solid E at (x, y, z) is given by $\rho(x, y, z)$, then its mass is evaluated by

$$m(E) = \iiint_E \rho(x, y, z)\, dV$$

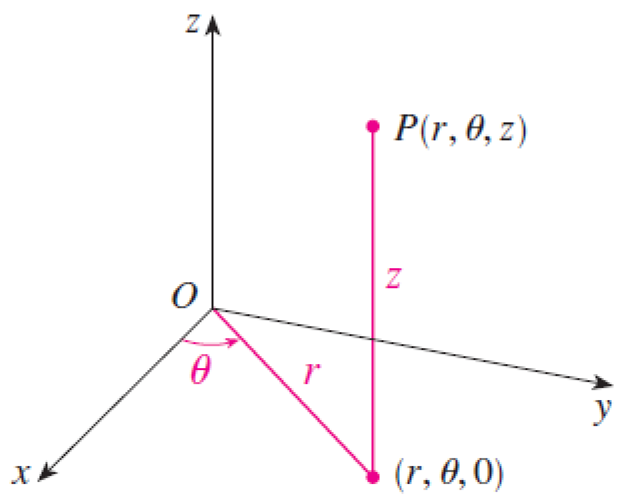
If $(\bar{x}, \bar{y}, \bar{z})$ is **the center of mass** or the **centroid** of this solid then

$$\bar{x} = \frac{1}{m(E)} \iiint_E x\rho(x, y, z)\, dV$$

$$\bar{y} = \frac{1}{m(E)} \iiint_E y\rho(x, y, z)\, dV$$

$$\bar{z} = \frac{1}{m(E)} \iiint_E z\rho(x, y, z)\, dV$$

Definition: Any point P in three-dimensional space can be represented by an ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy – plane and is the directed distance from the xy – plane to P. The ordered triple $r, \theta,$ and z are called the **cylindrical coordinates** of P.



Cylindrical coordinates and Cartesian coordinates are related by the following equations:

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z,$$

$$r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x},\ z = z.$$

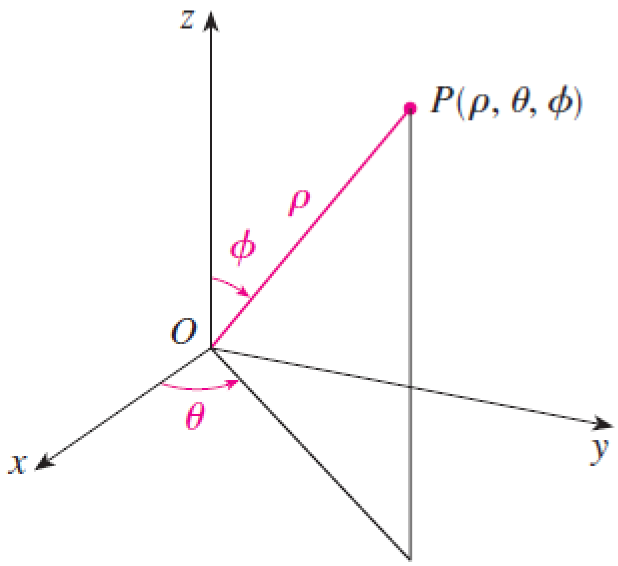
Evaluating Triple Integrals with Cylindrical Coordinates: Let E be a type 1 region given by $E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$. Assume that D is a polar regions given by $D = \{(r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le\ h_2(\theta)\}$. Then

$$\iiint_E f(x, y, z)\, dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos x,\ r\sin x)}^{u_2(r\cos x,\ r\sin x)} f u_1(r\cos x,\ r\sin x,\ z)\, r\, dz\, dr\, d\theta.$$

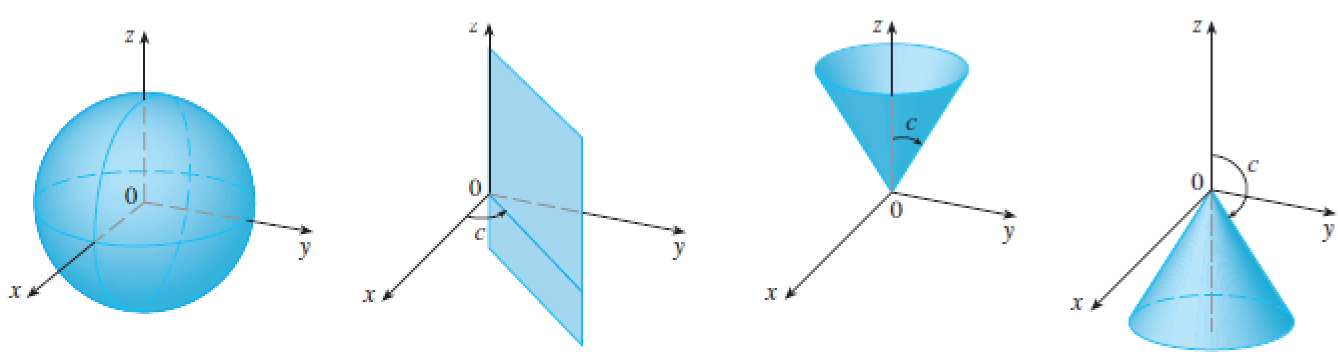
This formula is called the **formula for triple integration in cylindrical coordinates**.

Note: The above formula is used mostly when the projection of the solid on one of the coordinate planes is a polar region (such as a circle, a half-circle, a washer, a sector of a circle, etc.).

Definition: The **spherical coordinates** (ρ, θ, φ) of a point P in space are shown in the following picture, where ρ is the distance from the origin O to P, θ is the same angle as in cylindrical coordinates, and φ is the angle between the positive z – axis and the line segment OP. Note that $\rho \geq 0$, $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi$.



For a constant c, graphs of $\rho = c$, $\theta = c$ and $\varphi = c$ are spheres, half-planes and cones:



$\rho = c$, a sphere

$\theta = c$, a half-plane

$0 < c < \pi/2$

$\pi/2 < c < \pi$

$\phi = c$, a half-cone

Spherical coordinates and Cartesian coordinates are related by the following formulas:

$$x = \rho\ \sin\varphi\ \cos\theta,\ y = \rho\ \sin\varphi\ \sin\theta,\ z = \rho\ \cos\varphi,\ \rho^2 = x^2 + y^2 + z^2.$$

Definition: A solid E in the space is called a **spherical wedge** if there are constants $a,\ b,\ c,\ d$, α and β such that:

$$E = \{(\rho,\ \theta,\ \varphi)\ |\ a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \varphi \le d\}.$$

Evaluating Triple Integrals with Spherical Coordinates:

- If E is the spherical wedge given above, then
$$\iiint_E f(x,\ y,\ z)\ dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)\ \rho^2\sin\varphi\ d\rho\ d\theta\ d\varphi.$$

  Note: A common error is forgetting the factor $\rho^2\sin\varphi$ when switching from Cartesian coordinates to spherical coordinates. Also make sure the order of integration coresponds to the limits of integrals.

- If E is a more general region given by
$$E = \{(\rho,\theta,\ \varphi)\ |\ a \le \varphi \le b,\ \alpha \le \theta \le \beta,\ g_1(\theta,\ \varphi) \le \rho \le g_2(\theta,\ \varphi)\}.$$
  Then
$$\iiint_E f(x,\ y,\ z)\ dV = \int_a^b \int_\alpha^\beta \int_{g_1(\theta,\ \varphi)}^{g_2(\theta,\ \varphi)} f(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)\ \rho^2\sin\varphi\ d\rho\ d\theta\ d\varphi.$$

- If E is a more general region given by
$$E = \{(\rho,\theta,\ \varphi)\ |\ a \le \varphi \le b,\ g_1(\varphi) \le \theta \le g_2(\varphi),\ h_1(\theta,\ \varphi) \le \rho \le h_2(\theta,\ \varphi)\}.$$
  Then
$$\iiint_E f(x,\ y,\ z)\ dV = \int_a^b \int_{g_1(\varphi)}^{g_2(\varphi)} \int_{h_1(\theta,\ \varphi)}^{h_2(\theta,\ \varphi)} f(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)\ \rho^2\sin\varphi\ d\rho\ d\theta\ d\varphi.$$

Note: We generally use spherical coordinates to evaluate triple integrals when the solid is similar to a cone, a sphere or the solid is formed by intersecting spheres, cones and planes.

Note: To find the limits of integration, $a,\ b,\ g_1(\varphi),\ g_2(\varphi),\ \ h_1(\theta,\ \varphi)$ and $h_2(\theta,\ \varphi)$ follow these steps:

1. Find the maximum and minimum possible values for φ. These values are a and b above. Make sure all of the values between a and b are possible values for φ.
2. Consider a fixed angle φ -that gives us a cone- and find all possible values of θ. The maximum and minimum values would give us the functions $g_1(\varphi)$ and $g_2(\varphi)$. Make sure

all values between $g_1(\varphi)$ and $g_2(\varphi)$ are possible values of θ. These functions $g_1(\varphi)$ and $g_2(\varphi)$ may or may not depend on φ, but they should not depend on ρ.

3. Considering a fixed value for θ and a fixed value for φ -which determines a half-line through the origin- find maximum and minimum values of ρ. These would determine $h_1(\theta, \varphi)$ and $h_2(\theta, \varphi)$. Make sure all values between $h_1(\theta, \varphi)$ and $h_2(\theta, \varphi)$ are possible values of ρ.

Note: You can use a different order of integration, but to do that you need to follow the above 3 steps with a different order for coordinates, ρ, θ and φ.

Section 16.9

The Divergence Theorem: Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let F be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\int\int_S F \cdot dS = \int\int\int_E div\, F\; dV.$$

Note: When using the Divergence Theorem make sure all conditions are satisfied. A very common error is to use the Divergence Theorem when the vector field is not defined or is not continuous inside E.

Methods of Evaluating Surface Integrals of Vector Fields: To evaluate $\int\int_S F \cdot dS$,

1. If S is a closed surface consider using the Divergence Theorem. This is particularly useful when the vector field F is complicated but its divergence has a simpler formula.
2. If S is not closed but the divergence of F is simple consider closing the surface with a simple surface S_1. Then use the Divergence Theorem to evaluate the integral over $S \cup S_1$ and subtract $\int\int_{S_1} F \cdot dS$ from this integral.
3. If S the graph of a function with upward orientation, consider using the formula
$$\int\int_S F \cdot dS = \int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA.$$
If the orientation is downward multiply the above quantity with a negative sign.
4. If F is the curl of a vector field, consider using the Stoke's Theorem.

Methods of Evaluating Line Lntegrals of Vector Fields: To evaluate a line integral $\int_C F \cdot d\vec{r}$,

1. If you can parametrize C consider using the formula $\int_C F \cdot d\vec{r} = \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t)\, dt.$
2. If C is a closed plane curve consider using the Green's Theorem.
3. If C in a plane curve but not closed, consider closing it with a simple curve (or a line segment) C_1 and use the Green's Theorem to evaluate the line integral over $C \cup C_1$ and C_1 and take the difference of these two numbers.
4. If C is a closed space curve consider using the Stoke's Theorem.
5. If C in a space curve but not closed, consider closing it with a simple curve (or a line segment) C_1 and use the Stoke's Theorem to evaluate the line integral over $C \cup C_1$ and C_1 and take the difference of these two numbers.
6. If F is conservative, i.e. $F = \nabla f$, then consider using the Fundamental Theorem of Line Integrals.