Math 246 Notes

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This note may contain some typos. Feel free to message me if you see any typos.

Contents

1 Week 1

Differential Equation: An equation involving derivatives of one or more functions.

Example 1.1. (a) $\left(\frac{dx}{dt}\right)^2 + x\sin t = \cos x$.

(b) $\frac{\partial y}{\partial t}\frac{\partial y}{\partial s} + y\frac{\partial z}{\partial t} = \sin(st)$.

(c) $y'' + ty' + y = \cos t$.

Ordinary Differential Equation (ODE): A differential equation that involves no partial derivatives. Those that involve partial derivatives are called **Partial Differential Equations** or PDE's.

Order of a differential equation is the largest derivative that appears in the differential equation.

Example 1.2. Determine if each equation in the previous example is an ODE or a PDE. Determine their orders.

Linear Differential Equation: A differential equation that can be written in a way that each of its terms either does not contain any unknown variables or is the product of some derivative of only one of the unknown variables and a function of the independent variables.

Example 1.3. Determine which one of the equations in Example 1.1 are linear and why.

Example 1.4. Write down the general form of an n-th order linear ODE with one unknown variable y that is a function of one independent variable t.

1.1 First-Order Equations

Recall that first-order equations are the ones that involve only the first derivatives of unknown functions. We focus on solving first-order ODE's of the form $\frac{dy}{dt} = f(t, y)$. Depending on the type of function $f(t, y)$ we will employ different strategies.

1.1.1 Explicit First-Order Equations

Explicit first-order ODE's are the ones of the form $\frac{dy}{dt} = f(t)$.

Example 1.5. Solve $\frac{dy}{dt} = \sin t$.

Example 1.6. Solve $\frac{dy}{dt} = \frac{1}{t^2 - t}$, and $y(2) = 1$.

Differential equations such as the ones in Example 1.6 that involve given "initial values" are called **Initial Value Problems** or **IVP**'s.

Example 1.7. Solve the IVP

$$\frac{dy}{dt} = e^{t^2}, y(0) = 1.$$

The **interval of definition** of a solution to an IVP is the largest interval over which the solution is defined.

Example 1.8. Find the interval of definition of the solution to the IVP

$$\frac{dy}{dt} = \frac{t+1}{\sqrt[3]{t^2 - 1}}, y(0) = 10.$$

Example 1.9. Suppose y is the solution to the IVP given by $\frac{dy}{dt} = \sqrt[5]{t^3 - 4t}$, $y(1) = -1$. Determine where $y(t)$ is increasing and where it is decreasing.

1.1.2 First-Order Linear Equations

The general form of a first-order linear equation is $p(t)\frac{dy}{dt} + q(t)y = f(t)$, where $p(t)$ is not the zero function. The functions $p(t)$ and $q(t)$ are called **coefficients**, and $f(t)$ is called the **forcing** function. If the forcing is zero we say the equation is **homogeneous**, otherwise we say it is **non-homogeneous**.

Dividing both sides by the leading coefficient we can re-write any first-order linear equation of the following form, called the **normal form**:

$$\frac{dy}{dt} + q(t)y = f(t)$$

Example 1.10. Solve the equation $t\frac{dy}{dt} + y = 1$.

Example 1.11. Solve the IVP $y' + y = t$, $y(0) = 1$.

To solve the equations $y' + p(t)y = f(t)$, we need to write the left hand side as a derivative of a single function. To do that we multiply both sides by a factor called an **integrating factor**. The equation then would look like $\mu y' + \mu p(t)y = \mu f(t)$. In order for the left hand side to follow the form of the product rule we need to have $\mu' = \mu p(t)$. This can be achieved by taking $\mu = e^{\int p(t)dt}$. Remember that we only need *one* integrating factor!

Example 1.12. Solve $(t^2 + 1)y' + ty = 0$.

Theorem 1.1. *Suppose functions $p(t)$ and $f(t)$ are continuous over the interval (a, b). Let t_0 be a point in (a, b) and y_0 be a real number. Then, the IVP*

$$\frac{dy}{dt} + p(t)y = f(t), y(t_0) = y_0$$

has a unique solution defined over (a, b).

Example 1.13. Find the interval of definition of the solution to the IVP $(1 - t)y' + e^t y = \sin t, y(0) = 1$.

Example 1.14. Suppose y_1 and y_2 are solution of a first-order linear homogeneous equation. Show that for every constant c, the function $y_1 + cy_2$ is also a solution of the same equation.

For more examples, check the textbook.

1.2 Summary

- The solution to the IVP $\frac{dy}{dt} = f(t), y(t_0) = y_0$ is given by $y(t) = y_0 + \int_{t_0}^{t} f(x)\, dx$.
- To find the interval of definition of the solution to the IVP $\frac{dy}{dt} = f(t), y(t_0) = y_0$ we find the largest open interval containing t_0 where $f(t)$ is defined and continuous.
- To solve a first-order linear equation:
  - Write down the equation in the normal form $y' + p(t)y = f(t)$.
  - Find an integrating factor $\mu = e^{\int p(t)}$.
  - The equation can then be written as $\frac{d}{dt}(\mu y) = \mu f(t)$.
  - Solve the equation by integrating both sides.

- If there is an initial condition use $\mu y = C + \int_{t_0}^{t} \mu f(s) ds$, and find C by substituting $t = t_0$ and $y = y_0$.

- To find the interval of definition of a first-order linear IVP:
  - Write the equation in the normal form $y' + p(t)y = f(t)$.
  - Find the largest open interval containing the initial value t_0 for which $p(t)$ and $f(t)$ are defined and continuous.

2 Week 2

2.1 Separable Equations

A first-order equation is called **separable** if it can be written in the form

$$\frac{dy}{dt} = f(t)g(y).$$

The name "separable" refers to the fact that we can separate the variables and write the differential equation in the form

$$\frac{dy}{g(y)} = f(t)\ dt.$$

The solution can then be obtained by simply integrating both sides.

Example 2.1. Solve the equation $\frac{dy}{dt} = y^2 - 1$.

Equations of the form $\frac{dy}{dt} = g(y)$ are called **autonomous** since the derivative of y is "self-governed" by the function y and is independent of t. Notice that because we divided both sides by $y^2 - 1$, there are some solutions that are missing. In fact $y = 1$ and $y = -1$ are two constant functions that satisfy the above equation. This brings us to the following definition.

Given a separable equation $\frac{dy}{dt} = f(t)g(y)$, any constant y_0 that satisfies $g(y_0) = 0$ gives a constant solution $y = y_0$ to this differential equation. This solution is called a **stationary solution**. Other names for stationary solutions are **fixed points** or **equilibrium points**.

Example 2.2. Find all solutions of the equation including its stationary solutions.

$$\frac{dy}{dt} = \frac{3t + ty}{y + t^2 y}.$$

Example 2.3. Given a constant y_0, solve the initial value problem and find its interval of definition:

$$\frac{dy}{dt} = y^2,\ y(0) = y_0.$$

Example 2.4. Solve the initial value problem:

$$\frac{dy}{dt} = 3y^{2/3},\ y(0) = 0.$$

Surprisingly these are many solution of this IVP that we can not obtain using the methods discussed before!

Theorem 2.1 (Existence-Uniqueness Theorem for Separable Equations)**.** *Suppose $f(t), g(y)$ are continuous functions over (t_L, t_R), and (y_L, y_R), respectively, and let g be differentiable at its zeros in (y_L, y_R). Then, for every t_0 in (t_L, t_R) and every y_0 in (y_L, y_R), there is a unique solution to the IVP*

$$\frac{dy}{dt} = f(t)g(y), \quad y(t_0) = y_0$$

for as long as t stays in the interval (t_L, t_R) and y stays in the interval (y_L, y_R).

Example 2.5. For what values of y_0 can we guarantee that the following IVP has a unique solution near $(0, y_0)$?

$$\frac{dy}{dt} = t(y-1)^{1/3}, \quad y(0) = y_0.$$

2.2 General Theory for $\frac{dy}{dt} = f(t,y)$

We are interested in initial value problems of the form

$$\frac{dy}{dt} = f(t,y), \quad y(t_0) = y_0$$

that have a unique solution.

First, recall that partial derivative of a multivariable function is its derivative when all other variables are considered constant. Partial derivative of a function f with respect to a variable x is denoted by f_x or $\partial_x f$ or $\frac{\partial f}{\partial x}$.

Example 2.6. Evaluate f_x, f_y, and f_z, where $f(x,y,z) = x + x \ln y + z^2 x$.

We say a point (t,y) is an interior point of a region S in the ty-plane if there is a circle around (t,y) that is fully contained in S.

Theorem 2.2 (Existence-Uniqueness Theorem for First-Order IVP's)**.** *Suppose $f(t,y)$ is a function defined over a region S in the ty-plane such that*

- *f is continuous over S, and*
- *f_y exists and is continuous over S.*

Then, for any initial time t_0 and initial y_0 for which (t_0, y_0) is in the interior of S, the initial value problem

$$\frac{dy}{dt} = f(t,y), \; y(t_0) = y_0$$

has a unique solution for as long at (t,y) remains in S.

Example 2.7. Determine for which values of t_0 and y_0 the following initial value problem is guaranteed to have a unique solution:

$$\frac{dy}{dt} = \frac{ln|y|}{1+t^2-y^2}, \quad y(t_0) = y_0.$$

2.3 Phase-Line Portraits

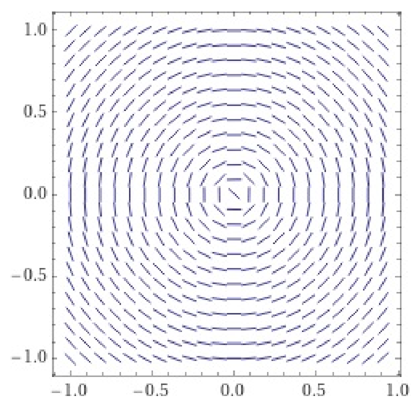
Example 2.8. Draw a phase line portrait for the equation $\frac{dy}{dt} = 4y^2 - y^4$.

A stationary solution $y = y_0$ is called **stable** if all solutions near y_0 move towards y_0. It is called **unstable** if all solution near y_0 move away from y_0. It is said to be **semistable** if some solutions near y_0 move towards it while others move away from y_0.

Example 2.9. Consider the differential equation $\frac{dy}{dt} = \frac{y - y^3}{(y+2)^2}$.

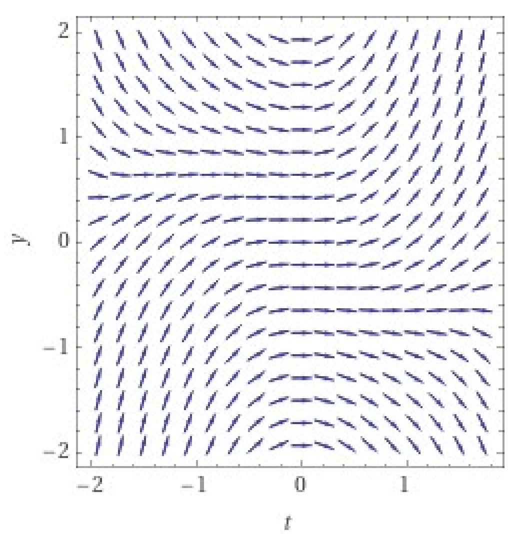
1. Sketch the phase-line portrait of this equation.
2. Classify all of the stationary solutions.
3. Evaluate $\lim_{t\to\infty} y(t)$, and $\lim_{t\to-\infty} y(t)$, where y is the solution satisfying $y(0) = 5$.

Example 2.10. Consider the differential equation $y\frac{dy}{dt} + t = 0$. Draw the direction field of this equation.



Example 2.11. Draw the direction field of the equation

$$\frac{dy}{dt} = sin^2 t + ty.$$



Existence-Uniqueness Theorems tell us that solution curves do not intersect.

To sketch solutions that are explicit we use the command **fplot** in MATLAB.

To produce counter plots $H(X, Y) = c$ in MATLAB we use **meshgrid** or **contour**.

We produce direction fields using **quiver** or **meshgrid**.

Check the online textbook for more examples and MATLAB examples.

2.4 Summary

- To solve a separable equation of the form $\frac{dy}{dt} = f(t)g(y)$:
  - Find all stationary solutions by solving $g(y) = 0$.
  - For non-stationary solutions: separate the variables and rewrite the equation as $\frac{dy}{g(y)} = f(t)dt$. Then integrate both sides.
- For a separable IVP to have a unique solution we need $f(t)$ and $g(y)$ to be continuous, and $g(y)$ to be differentiable at all of its zeros.
- To check if the initial value problem $\frac{dy}{dt} = f(t, y),\ y(t_0) = y_0$ has a unique solution we need to make sure:
  - $f(t, y)$ is continuous.
  - f_y exists and is continuous.
  - (t_0, y_0) is an interior point.
- To draw the phase-line portrait of an autonomous equation $\frac{dy}{dt} = g(y)$:
  - Find all stationary solutions by solving $g(y) = 0$.
  - Find all points of discontinuity of $g(y)$ and the points where $g(y)$ is undefined.
  - Plot these points on a number line.
  - Determine the sign of $g(y)$ in each interval. Those values determine if y is increasing or decreasing. Indicate that using arrows.
- A stationary solution $y = y_0$ is called **stable** if all solutions near y_0 move towards y_0. It is called **unstable** if all solution near y_0 move away from y_0. It is said to be **semistable** if some solutions near y_0 move towards it while others move away from y_0.

3 Week 3

3.1 Applications

The general strategy for many of these application problems is that

$$\text{Rate} = \text{Rate In - Rate Out.}$$

3.1.1 Tank Problems

Example 3.1. A tank contains 200 liters of salt solution at concentration of 7 g/L. A solution at concentration of 0.5 g/L flows into the tank at the rate of 3 L/min. At the same time the well-mixed solution flows out of the tank at the rate of 3 L/min. How long will it be before the solution is at concentration 1 g/L?

Example 3.2. Use the same settings as in the previous problem except assume the rate of flow out of the tank is 2 L/min. Write down a differential equation that governs the amount of salt in the tank.

Example 3.3. A circular cylindrical tank with an open top has a circular base of radius 1 m, and a height of 2 m. The water pours into the tank at the rate of 7 L/min and drains at the rate of $5\sqrt{h}$ L/min, where h is the height of water. Will the tank overflow?

3.1.2 Population Dynamics

Populations are often modeled by differential equations. One model is the following

$$\frac{dp}{dt} = R(p)p - h(t).$$

$R(p)$ is the growth rate and $h(t)$ is due to predators.

Example 3.4. In the absence of predators the population of mosquitoes increases at a rate proportional to its population and doubles every three weeks. There are 250,000 mosquitoes initially when a flock of birds arrive that eats 80,000 mosquitoes per week. How many mosquitoes remain after two weeks?

3.1.3 Motions

An object falling in air will be governed by the equation

$$\frac{dv}{dt} = g - kv^2,$$

where k is a constant depending on the density of air and the shape of the object.

Example 3.5. A skydiver with mass 60 kg jumps from an airplane. Assuming $k = 0.002$ 1/m, find the terminal velocity of the skydiver.

3.1.4 Loans

If $B(t)$ indicates the loan balance after t years, r is the annual interest rate, and P is the annual payment, then the equation governing the loan balance is given by

$$\frac{dB}{dt} = rB - P.$$

Example 3.6. A car buyer has a \$4,000 down payment and can afford a constant rate of \$ 250 per month. 5-year fixed rate loans at an interest rate of 5% per year compounded continuously are available. What is the most expensive car that the buyer can afford? (Assume the payment is done continuously.)

3.2 Numerical Methods

Most differential equations cannot be analytically solved. So, we don't have any choice but to approximate solutions. In this section we will discuss appropximating solutions to first order IVP's given below:

$$\begin{cases} \dfrac{dy}{dt} = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

The goal is to find an approximate value for $y(t)$, where t is a point near t_0.

3.2.1 Explicit Euler's Method

In this method we will use the fact that the slope at any point is given by $f(t, y)$.

To approximate $y(t)$, we divide the interval $[t_0, t]$ into N subintervals of equal width $h = \dfrac{t - t_0}{N}$. We evaluate approximate values for $y(t_n)$ where $t_n = t_0 + nh$ recursively by $y_{n+1} = y_n + f(t_n, y_n)h$. Then, y_N is an approximation for $y(t)$.

Example 3.7. Estimate $y(0.2)$ and $y(-0.1)$, where $\dfrac{dy}{dt} = t^2 + y^2$ and $y(0) = 1$. Use step size $h = 0.1$.

Another way of looking at the solutions of the differential equation $\dfrac{dy}{dt} = f(t, y)$ is

$$y(t + h) - y(t) = \int_t^{t+h} y'(s)\ ds = \int_t^{t+h} f(s, y(s))\ ds.$$

We could use methods that we learned in single variable calculus to approximate this integral. Each method would give us a new formula.

Using the left endpoint method we obtain $\displaystyle\int_t^{t+h} f(s, y(s))\ ds \approx f(t, y(t))h$, which yields $y(t + h) \approx y(t) + hf(t, y(t))$ which is the same formula as in the Euler's Explicit method.

Using the midpoint method, the trapezoidal method, and the Simpson's method for approximating this integral we obtain different estimates. The formulas for all of these estimates can be found below:

Trapezoidal Method:

$$f_n = f(t_n, y_n), \quad y_{n+1} = y_n + \frac{h}{2}(f_n + f(t_{n+1}, y_n + hf_n))$$

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Midpoint Method:

$$\begin{aligned} f_n &= f(t_n, y_n), & y_{n+\frac{1}{2}} &= y_n + \tfrac{h}{2} f_n \\ f_{n+\frac{1}{2}} &= f(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}), & y_{n+1} &= y_n + h f_{n+\frac{1}{2}} \end{aligned}$$

Runge-Kutta Method: If we use the Simpson's method to estimate the integral $\int_t^{t+h} f(s, y)\, ds$ we obtain a much more complicated formula, with far better results. This method is called the Runge-Kutta method.

Example 3.8. Estimate $y(0.1)$ with 1 step in Midpoint method for the solution to the IVP:

$$\frac{dy}{dt} = t^2 + y^2, y(0) = 1.$$

3.3 Error Analysis

A function $f(h)$ is said to be of order h^2 (written as $O(h^2)$) if the function is roughly a constant multiple of h^2 for small values of h. This means the function goes to zero as fast as h^2 when h approaches zero. The errors in all of the methods above are listed below:

Method	Global Error
Explicit Euler	$O(h)$
Midpoint	$O(h^2)$
Trapezoidal	$O(h^2)$
Runge-Kutta	$O(h^4)$

Example 3.9. If we want to make sure that the error in estimating the value of the solution to a first order equation is divided by 16, what changes do we need to make if we are using each of the following methods?

(a) Euler's method.

(b) Trapezoidal method.

(c) Runge-Kutta method.

3.4 Summary

- The main formula for solving application problems is: Rate=Rate In - Rate Out.
- Make sure you know the formulas for Explicit Euler's, Midpoint and Trapezoidal methods.
- Error for Euler's method is $O(h)$, for midpoint and trapezoidal methods the errors are $O(h^2)$, and for the Runge-Kutta method the error is $O(h^4)$.

4 Week 4

4.1 Exact Equations

We will continue focusing on solving equations of the form $\frac{dy}{dt} = f(t, y)$. Assume the solution is implicitly given by $H(t, y) = c$. By the chain rule we get $H_t + H_y \frac{dy}{dt} = 0$. This is often written as

$$H_t dt + H_y dy = 0. \ (*)$$

If we can write a first order equation in the form $(*)$ then we can solve it. Note that $\frac{\partial H_t}{dy} = \frac{\partial H_y}{\partial t}$.

An equation $M(t, y)dt + N(t, y)dy = 0$ is called **exact** if $M_y = N_t$. To solve such an equation we will find H for which $H_t = M$ and $H_y = N$.

Example 4.1. Solve the equation $\frac{dy}{dx} = -\frac{xy^2 + y + e^x}{x^2 y + x}$.

Example 4.2. Solve the initial value problem

$$\frac{dy}{dt} + \frac{e^t y + 2t}{2y + e^t} = 0, \quad y(0) = 0.$$

4.2 Integrating Factors

Sometimes equations that are not exact can be turned into exact equations by multiplying by a factor called an **integrating factor**.

Example 4.3. Solve the equation $(2e^x + y^3)dx + 3y^2 dy = 0$.

Example 4.4. Solve the equation $2ty + (2t^2 - e^y)\frac{dy}{dt} = 0$

Example 4.5. Solve the IVP $(4xy + 3y^3)dx + (x^2 + 3xy^2)dy = 0$.

4.3 Higher-Order Linear Equations

An n-th order linear differential equation in normal form is an equation of the form:

$$y^{(n)} + a_n(t)y^{(n-1)} + \cdots + a_2(t)y' + a_1(t)y = f(t).$$

$f(t)$ is called the forcing and $a_i(t)$'s are called coefficients.

We would also like to explore initial value problems of the form:

$$\begin{cases} y^{(n)} + a_n(t)y^{(n-1)} + \cdots + a_2(t)y' + a_1(t)y = f(t). \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \\ \vdots \\ y^{(n-1)}(t_0) = y_{n-1} \end{cases}$$

Theorem 4.1 (Existence-Uniqueness Theorem for Linear Equations.)**.** *Assume all coefficients and the forcing of a linear equation in normal form are continuous over an interval* (a, b)*. Then, for every initial time* t_0 *in* (a, b) *and real numbers* $y_0, y_1, \ldots, y_{n-1}$*, the initial value problem*

$$\begin{cases} y^{(n)} + a_n(t)y^{(n-1)} + \cdots + a_2(t)y' + a_1(t)y = f(t). \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \\ \vdots \\ y^{(n-1)}(t_0) = y_{n-1} \end{cases}$$

has a unique solution that is defined over (a, b)*.*

Example 4.6. Find the interval of definition of the solution to the initial value problem:

$$(t^2 - 1)y'' - \frac{1}{t^2 - 4}y' + y = t^2 - 3, \ y(0) = y'(0) = 1.5.$$

Example 4.7. Show that $\sin(t^3)$ is not a solution to any linear differential equation of the form $y''' + a(t)y'' + b(t)y' + c(t)y = 0$, where all coefficients are continuous over $(-1, 1)$.

4.4 Linear Differential Operators

Recall that linear equations are those of the form:

$$\frac{d^n y}{dt^n} + a_n(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_2(t)\frac{dy}{dt} + a_1(t)y = f(t) \qquad (*)$$

To simplify this, we denote $\frac{d}{dt}$ by D, so instead of $\dfrac{d^k y}{dt^k}$ we write $D^k[y]$. The equation $(*)$ is then written as $L[y] = f(t)$, where $L = D^n + a_n(t)D^{n-1} + \cdots + a_2(t)D + a_1(t)$.

Example 4.8. Let y_1 and y_2 be two solutions to the linear homogeneous equation $y'' + t^2 y' + \sin(t^3)y = 0$. Show that $y = c_1 y_1 + c_2 y_2$ is also a solution to this differential equation, where c_1, c_2 are constants.

In general for constants $c_1, \ldots, c_n$ and functions $y_1, \ldots, y_n$ we have

$$L[c_1 y_1 + \cdots + c_n y_n] = c_1 L[y_1] + \cdots + c_n L[y_n].$$

Consequently if $y_1, \ldots, y_n$ are solutions to $L[y] = 0$ the linear combination $c_1 y_1 + \cdots + c_n y_n$ is also a solution to $L[y] = 0$.

Example 4.9. Given that $\sin t$ and $\cos t$ are two colutions of the equation $y'' - y = 0$, find a solution to the IVP

$$y'' - y = 0, \ y(0) = 1, \ \text{and } y'(0) = 2.$$

In general if $y_1, \ldots, y_n$ are solutions to $L[y] = 0$, to solve the IVP

$$L[y] = 0, \ y(t_0) = y_0, y'(t_0) = y_1, \ldots, y^{(n-1)}(t_0) = y_{n-1}$$

we set $y = c_1 y_1 + \cdots + c_n y_n$, and find the constants $c_1, \ldots, c_n$ using the initial conditions.

So, we would like to see when the system given below has a solution for constants $c_1, \ldots, c_n$:

$$\begin{cases} c_1 y_1(t_0) + \cdots + c_n y_n(t_0) = y_0 \\ c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) = y_1 \\ \vdots \\ c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) = y_{n-1} \end{cases}$$

Theorem 4.2. *Given constants a_{ij}, and b_j, consider the following system:*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

This system has a unique solution if and only if the matrix of the coefficients given below has non-zero determinant.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Theorem 4.3. *Given constants a_{ij}, consider the following system:*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0 \end{cases}$$

This system has a non-zero solution if and only if the matrix of the coefficients given below has zero determinant.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Determinant of a 2×2 matrix is given by $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$.

4.5 Summary

- An equation $Mdt + Ndy = 0$ is exact if $M_y = N_t$.
- To solve an exact equation $Mdt + Ndy = 0$ we will find H for which $H_t = M$, and $H_y = N$. The solutions then are given by $H(t, y) = c$.

- To solve equations using the integrating factor method:
  - First check if the equation is exact.
  - If it is not, multiply both sides by μ and set up the equation $(\mu M)_y = (\mu N)_t$.
  - Find an appropriate μ. Test if $\mu_y = 0$ would yield a function of t for μ, or if $\mu_t = 0$ would yield a function of y for μ.
  - Multiply both sides of the equation by μ, and solve the resulting equation using the method for exact equations.
- To find the interval of definition of a linear IVP:
  - Write the equation in the normal form.
  - Find the largest open interval containing the initial value t_0 for which all coefficients and the forcing are defined and continuous.

5 Week 5

5.1 Wronskian

Suppose $Y_1, \ldots, Y_n$ are solutions to an order n homogeneous linear differential equation $L[y] = 0$. In order for the system

$$\begin{cases} c_1 Y_1(t_0) + \cdots + c_n Y_n(t_0) = y_0 \\ c_1 Y_1'(t_0) + \cdots + c_n Y_n'(t_0) = y_1 \\ \vdots \\ c_1 Y_1^{(n-1)}(t_0) + \cdots + c_n Y_n^{(n-1)}(t_0) = y_{n-1} \end{cases}$$

to have a unique solution for each choice of $y_0, \ldots, y_{n-1}$ we need

$$\det \begin{pmatrix} Y_1(t_0) & \cdots & Y_n(t_0) \\ Y_1'(t_0) & \cdots & Y_n'(t_0) \\ & \vdots & \\ Y_1^{(n-1)}(t_0) & \cdots & Y_n^{(n-1)}(t_0) \end{pmatrix} \neq 0$$

The above determinant is called the **Wronskian** of $Y_1, \ldots, Y_n$ at t_0. In general the Wronskian is defined and denoted as follows:

$$W[Y_1, \ldots, Y_n](t) = \det \begin{pmatrix} Y_1(t) & \cdots & Y_n(t) \\ Y_1'(t) & \cdots & Y_n'(t) \\ & \vdots & \\ Y_1^{(n-1)}(t) & \cdots & Y_n^{(n-1)}(t) \end{pmatrix}.$$

It turns out that the Wronskian of n solutions of a linear homogeneous differential equation is either always zero or never zero.

In fact the Wronskian satisfies the differential equation $\frac{dW}{dt} + a_n(t)W = 0$. This is called **Abel's Theorem**.

Example 5.1. Suppose the Wronskian of three solutions of the equation $y''' - ty'' + y = 0$ satisfies $W(0) = 1$. Find $W(t)$.

Definition 5.1. Solutions $Y_1, \ldots, Y_n$ are said to form a **fundamental set of solutions** if their Wronskian is nonzero.

When $Y_1, \ldots, Y_n$ form a fundamental set of solutions, the **general solution** is given by $y = c_1 Y_1 + \cdots + c_n Y_n$.

Example 5.2. Consider the equation $y'' + 4y = 0$. Find a general solution to this equation. Hint: $\sin(2t)$ and $\cos(2t)$ are two solutions of this equation.

5.2 Natural Fundamental Set of Solutions

Everything that we have done so far has been based on the assumption that a fundamental set of solutions exists, but does it? The Existence-Uniqueness Theorem guarantees that given any set of proper initial values a unique solution exists. Let $N_0(t), N_1(t), \ldots, N_{n-1}(t)$ be solutions to $L[y] = 0$ given the initial values given below:

$$\begin{array}{llll} N_0(t_0) = 1, & N_0'(t_0) = 0, & \cdots & N_0^{(n-1)}(t_0) = 0 \\ N_1(t_0) = 0, & N_1'(t_0) = 1, & \cdots & N_1^{(n-1)}(t_0) = 0 \\ & & \vdots & \\ N_{n-1}(t_0) = 0, & N_{n-1}'(t_0) = 0, & \cdots & N_{n-1}^{(n-1)}(t_0) = 1 \end{array}$$

The Wronskian of these solutions is

$$\det \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = 1$$

This set is called the **Natural Fundamental Set of Solutions at** t_0.

To find a natural fundamental set of solutions we solve the initial value problem

$$L[y] = 0, \; y(t_0) = y_0, y'(t_0) = y_1, \ldots, y^{(n-1)}(t_0) = y_{n-1}.$$

The coefficients of $y_0, y_1, \ldots, y_{n-1}$ would be the elements of the natural fundamental set of solutions.

Example 5.3. Given that e^t and e^{-2t} are solutions to the equation $y'' + y' - 2y = 0$, find the natural fundamental set of solutions of this equation at $t = 0$.

5.3 Sample Examples

Example 5.4. Write the solution to the following IVP as an integral: $\dfrac{dy}{dt} = e^{t^2}, \ \ y(1) = -1.$

Example 5.5. In the absence of predators the population of mosquitoes in a certain area would increase at a rate proportional to its current population such that it would triple every five weeks. There are 85,000 mosquitoes in the area when a flock of birds arrives that eats 25,000 mosquitoes per week. Write down an initial-value problem that governs $M(t)$, the population of mosquitoes in the area after the flock of birds arrives. Do not solve!

Example 5.6. Draw a complete Phase-Line portrait for the differential equation $\dfrac{dy}{dt} = \dfrac{y(y+3)}{(y-1)^2}$. Identify and classify the stationary points. Find $\lim\limits_{t\to-\infty} y(t)$ if we know $y(0) = 2$.

Example 5.7. Find the interval of definition of the solution to the following IVP:

$$(t-1)y''' - \frac{t+2}{t^2-3t}y'' - \sin(t)y = \cos^{-1}(t/5), \quad y(2) = 4.$$

Example 5.8. Solve the differential equation: $(x^2 + 3xy^2)\frac{dy}{dx} = -(4xy + 3y^3)$.

5.4 Summary

- Wronskian is given by $W[Y_1, \dots, Y_n](t) = \det \begin{pmatrix} Y_1(t) & \cdots & Y_n(t) \\ Y_1'(t) & \cdots & Y_n'(t) \\ & \vdots & \\ Y_1^{(n-1)}(t) & \cdots & Y_n^{(n-1)}(t) \end{pmatrix}$.
- If the Wronskian is nonzero then the general solution is given by $y = c_1Y_1 + \cdots + c_nY_n$.
- Abel's Theorem states that Wronskian satisfies $W' + a_n(t)W = 0$.
- To find the natural fundamnetal set of solution at t_0, we solve the IVP $L[y] = 0, \ y(t_0) = y_0, y'(t_0) = y_1, \dots, y^{(n-1)}(t_0) = y_{n-1}$. The coefficients of $y_0, y_1, \dots, y_{n-1}$ would be the elements of the natural fundamental set of solutions.

6 Week 6

6.1 Linear Independence

Functions $Y_1, \dots, Y_n$ are said to be **linearly independent** if whenever $c_1Y_1 + \cdots + c_nY_n = 0$ for some constants $c_1, \dots, c_n$ we have $c_1 = \cdots = c_n = 0$. Another way of thinking about linearly independent functions is that none of them can be written as a linear combination of the others. Functions that are not linearly independent are said to be **linearly dependent**.

Example 6.1. Functions $1, t, t^2$ are linearly independent. However $\sin^2 t, \cos^2 t$ and $\cos(2t)$ are linearly dependent.

Theorem 6.1. *Suppose $Y_1, \ldots, Y_n$ are solutions to a linear homogeneous differential equation with continuous coefficients. Then $Y_1, \ldots, Y_n$ form a fundamental set of solutions precisely when they are linearly independent.*

6.2 Homogeneous Linear Equations with Constant Coefficients

In this section we will be solving linear equations of the form $L[y] = 0$, where $L = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$, and a_i's are all constant.

Example 6.2. Solve the equation $y'' - 5y' + 6y = 0$.

The polynomial $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ is called the **characteristic polynomial** corresponding to L. Since $D[e^{zt}] = ze^{zt}$ we conclude that $L[e^{zt}] = p(z)e^{zt}$. This means $L[e^{zt}] = 0$ if $p(z) = 0$. This is the main idea of the above method.

Example 6.3. Find a general solution for $y'' + 4y' + 4y = 0$.

Repeatedly differentiating $L[e^{zt}] = p(z)e^{zt}$ with respect to z we obtain the following

$$\begin{aligned}
&L[e^{zt}] = p(z)e^{zt} \\
&L[te^{zt}] = p'(z)e^{zt} + p(z)te^{zt} \\
&L[t^2e^{zt}] = p''(z)e^{zt} + 2p'(z)te^{zt} + p(z)t^2e^{zt} \\
&L[t^3e^{zt}] = p'''(z)e^{zt} + 3p''(z)te^{zt} + 3p'(z)t^2e^{zt} + p(z)t^3p(z) \\
&\vdots
\end{aligned} \quad .$$

The coefficients above are those in the Pascal's triangle. These identities are called **Key Identities**.

When z is a double root of p this yields $L[te^{zt}] = 0$, and when z has multiplicity 3 we obtain $L[t^2e^{zt}] = 0$. Using these facts we conclude that when a root z has multiplicity, in order to get more solutions we need to multiply e^{zt} by powers of t.

Example 6.4. Find a general solution to $(D + 1)^3 D^2 (D - 1)y = 0$.

When the characteristic polynomial has a non-real root $a + bi$, we will need to substitute $z = a + bi$ into the Key Identity $L[e^{zt}] = p(z)e^{zt}$. To find solutions we need the following identity about complex numbers:

$$e^{a+ib} = e^a e^{ib} = e^a(\cos b + i \sin b) = e^a \cos b + ie^a \sin b.$$

This gives us $L[e^{(a+bi)t}] = 0$ or $L[e^{at}\cos(bt) + ie^{at}\sin(bt)] = 0$. Looking at the real and imaginary parts of this equality we obtain

$$L[e^{at}\cos(bt)] = L[e^{at}\sin(bt)] = 0.$$

Example 6.5. Solve: $(D + 1)^2 D(D^2 + 2D + 2)y = 0$.

When the complex roots have multiplicity we multiply the solutions by powers of t to get new solutions.

Example 6.6. Find a general solution for $y^{(4)} + 2y'' + y = 0$.

6.3 Non-homogeneous Linear Equations

Recall that a non-homogeneous linear equation is an equation of the form $L[y] = f(t)$, where $f(t)$ is a non-zero forcing and L is a linear differential operator.

Note that because of linearity if $L[y_1] = 0$ and $L[y_2] = f(t)$, then $L[y_1 + y_2] = L[y_1] + L[y_2] = 0 + f(t) = f(t)$. So our strategy in solving a non-homogeneous equation is to (1) find the general solution Y_H to $L[y] = 0$, (2) find a particular (i.e. some) solution Y_p for $L[y] = f(t)$, and (3) Setting $y = Y_H + Y_P$ we obtain the general solution to the non-homogeneous equation $L[y] = f(t)$.

Example 6.7. We know $y = 5t$ is a solution to $y'' + 2y = 10t$. Find the general solution to this equation.

Example 6.8. Given that e^t satisfies $y'' + 2y' + 2y = 5e^t$, solve the initial value problem

$$y'' + 2y' + 2y = 5e^t,\ y(0) = 1, y'(0) = -1.$$

6.4 Finding Y_P for Linear Equations with Constant Coefficients

Example 6.9. Find a particular solution to the equation $y'' + 2y = e^{5t}$.

The idea is to write down the Key Identities, substitute appropriate z values and use them to obtain the forcing. Recall the Key Identities are the ones listed below:

$$\begin{aligned}
&L[e^{zt}] = p(z)e^{zt} \\
&L[te^{zt}] = p'(z)e^{zt} + p(z)te^{zt} \\
&L[t^2e^{zt}] = p''(z)e^{zt} + 2p'(z)te^{zt} + p(z)t^2e^{zt} \\
&L[t^3e^{zt}] = p'''(z)e^{zt} + 3p''(z)te^{zt} + 3p'(z)t^2e^{zt} + p(z)t^3p(z) \\
&\vdots
\end{aligned}.$$

Example 6.10. Solve: $y'' - 6y' + 9y = 4e^{3t}$.

Example 6.11. Find a particular solution for $y'' + 2y' + 10y = \cos(2t)$.

In general this method works well if the forcing has the following form:

$$f(t) = (\text{polynomial}) \cdot e^{\mu t}\cos(\nu t) + (\text{polynomial}) \cdot e^{\mu t}\sin(\nu t).$$

In which case we write down the Key Identities and substitute $z = \mu + i\nu$. Then we take appropriate linear combinations to obtain the forcing.

Example 6.12. Find a particular solution for $y'' + 2y' + 10y = 4te^{2t}$.

Example 6.13. Find a particular solution for $y'' + y = \sin t + t$.

Example 6.14. Solve the initial value problem $y'' + 4y = t\cos(2t),\ y(0) = y'(0) = 1$.

6.5 Summary

- To solve a homogeneous linear differential equation with constant coefficients:
  - Write down the characteristic polynomial $p(z)$ of the linear operator.
  - Find all roots of $p(z)$.
  - For every real root r with multiplicity m include $e^{rt}, \ldots, t^{m-1}e^{rt}$ in a fundamental set of solutions.
  - For a non-real root $a + bi$ include $e^{at}\cos(bt)$ and $e^{at}\sin(bt)$ in the fundamental set of solutions.
  - If a non-real root has multiplicity similar to above multiply $e^{at}\cos(bt)$ and $e^{at}\sin(bt)$ by powers of t to get the appropriate number of solutions.
  - Steps above create a fundamental set of solutions. Take a linear combination to get the general solution.
- To solve a non-homogeneous equation $L[y] = f(t)$:
  - Solve $L[y] = 0$ to find Y_H.
  - Find a particular solution Y_P for $L[y] = f(t)$.
  - $y = Y_H + Y_P$ is the general solution to $L[y] = f(t)$.
- One strategy that we use to find a particular solution is to write the Key Identities and substitute appropriate z values in order to obtain the given forcing.

7 Week 7

7.1 Undetermined Coefficients

The method of undetermined coefficients is used to find particular solutions to linear equations with constant coefficients. The main idea is to guess the form of a solution Y_P, substitute into the equation and find the unknown coefficients.

Example 7.1. Find a particular solution for the equation $y'' - 6y' + 9y = 4e^{3t}$.

Note that as before, we are dealing with forcings of the form

$$f(t) = (\text{polynomial}) \cdot e^{\mu t}\cos(\nu t) + (\text{polynomial}) \cdot e^{\mu t}\sin(\nu t).$$

The form of a particular solution depends on two things: the forcing, and the characteristic polynomial. To find the form of a particular solution (1) Find the degree of the polynomial in the forcing, and (2) Find the smallest m for which $p^{(m)}(\mu + i\nu) \neq 0$. A particular solution is given by

$$Y_P = (A_0 t^{m+d} + \cdots + A_d t^m)e^{\mu t}\cos(\nu t) + (B_0 t^{m+d} + \cdots + B_d t^m)e^{\mu t}\sin(\nu t).$$

Example 7.2. Find a particular solution using the method of undetermined coefficients: $y'' + 2y' + 10y = \cos(2t)$.

Example 7.3. Using the method of undetermined coefficients find a particular solution for $y'' + 2y' + 10y = 5e^{-t}\cos(3t)$.

Example 7.4. Using the method of undetermined coefficients find a particular solution for $y^{(4)} + 25y'' = 2t + t\cos(2t)$.

7.2 Green Functions

We notice that all of the previous techniques require the forcing to have a specific form. The method of Green functions gives us a way to find a particular solution regardless of whether or not the forcing is of the form given above.

A particular solution to $L[y] = f(t)$ is given by $Y_p(t) = \displaystyle\int_{t_0}^{t} g(t-s)f(s)\ ds$, where $g(t)$ is the solution to the initial value problem

$$L[g] = 0,\ \ g(0) = g'(0) = \cdots = g^{(n-2)}(0) = 0,\ \ g^{(n-1)}(0) = 1.$$

(Recall that L has constant coefficients.)

g is called the Green function associated with L.

Example 7.5. Find a particular solution for the equation $y'' - y = \dfrac{2e^t}{e^{2t}+1}$.

7.3 Applications: Mechanical Viberations

In this section we will model motion of a spring that is hanging from a support with a mass attached to the bottom. We will consider several forces that are exerted on this system. These forces are:

- Gravitational force F_{grav} that depends on the mass hanging from the spring.
- Damp force F_{damp} that depends on the medium that the spring is placed in.
- Spring force F_{spr} that depends on the spring, and
- External force F_{ext} that is any other force that is exerted on the system.

We are only going to model the vertical position $y(t)$ of the mass on the y-axis. Suppose y_0 and y_r are the positions of the bottom of spring without and with a mass, respectively. Newton's third law of motion gives

$$my'' = F_{grav} + F_{spr} + F_{damp} + F_{ext} \quad (*)$$

Gravitational force is given by $F_{grav} = -mg$.

By Hooke's Law we know $F_{spr} = k(y_0 - y)$, where k is the spring constant (or coefficient).

Damp force is often modeled by $F_{damp} = -cy'$, where $c \geq 0$ is called the damping coefficient.

Since y_r is the rest position, we must have $mg = k(y_0 - y_r)$ and thus $F_{grav} = k(y_r - y_0)$.

Substituting these into $(*)$ we obtain the following:

$$my'' = k(y_r - y_0) + k(y_0 - y) - cy' + F_{ext} \Rightarrow my'' + cy' + k(y - y_r) = F_{ext}$$

We now let $h(t) = y(t) - y_r$ be the displacement from the rest position. This simplifies our equation into the following:

$$\boxed{mh'' + ch' + kh = F_{ext}}$$

This equation is a second order linear equation with constant coefficients that can be solved using the methods that we learned. We will take a few cases to explore this further.

7.3.1 Unforced, Undamped Motion

In this case we assume there is no external force and we ignore the damping force, i.e. the resistance of the medium is assumed to be zero.

The equation for displacement then becomes, $mh'' + kh = 0$. The characteristic equation is $mz^2 + k = 0$, and its roots are $\pm\sqrt{\frac{-k}{m}} = \pm i\sqrt{\frac{k}{m}}$, which are not real. For simplicity let $\omega_0 = \sqrt{\frac{k}{m}}$. The general solution is $h(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$.

Example 7.6. A mass of 10 grams stretches a spring 5 cm when at rest. At time $t = 0$ the mass is set in motion from its rest with downward velocity of 35 cm/sec. Assuming no external forces exist and neglecting the damping:

(a) What is the displacement of mass at time t?

(b) What is the amplitude, phase, frequency, and period of the motion?

(c) At what time does the mass first return to its rest position?

To find the amplitude, phase, and period write down the displacement in the form $h(t) = A\cos(\omega t - \delta)$ with $0 \leq \delta < 2\pi$ and $A > 0$. Amplitude is A, phase is δ radians, period is $2\pi/\omega$ seconds, and frequency is ω radians/seconds.

7.3.2 Unforced, Damped Motion

In this case $F_{ext} = 0$ but $c \neq 0$. This gives us a homogeneous linear equation with constant coefficients $mh'' + ch' + kh = 0$. Its characteristic polynomial is $mz^2 + cz + k$ and its roots can be obtained using the quadratic formula. We will illustrate that with some examples.

Example 7.7. Assume a 10 lb weight stretches a spring 6 inches. This system is placed in a medium, and there are no external forces. At time $t = 0$ the mass is at rest position when it is being given an initial upward velocity of 1 ft/sec. In each of the following cases find a formula for the displacement of the mass at time t. (Recall: $g = 32\ ft/sec^2$.)

(a) When the mass is moving 2 ft/sec the medium exterts a force at 26 lbs.

(b) When the mass is moving 2 ft/sec the medium exterts a force at 6 lbs.

(c) When the mass is moving 2 ft/sec the medium exterts a force at 10 lbs.

When both roots are real we say the system is overdamped. When the roots are equal we say the system is critically damped and when the roots are non-real we say the system is underdamped.

Example 7.8. Assume a 2 kg mass streches a spring 10 cm. The system is placed in a medium, and there is no external force. What should the damping coefficient be in order to make sure the system is critically damped? (Recall: $g = 9.8\ \ m/sec^2$.)

7.3.3 Forced, Undamped Motion

In this case $c = 0$ but $F_{ext} \neq 0$. This gives a second order non-homogeneous linear equation: $mh'' + kh = F_{ext}$. As usual, we solve this by finding a general solution to the homogeneous equation $mh'' + kh = 0$ and a particular solution to the non-homogeneous equation. Many of the problems that we deal with will have forcing of the form $F_{ext} = F\cos(\omega t)$.

7.3.4 Forced, Damped Motion

In this case $c, F_{ext} \neq 0$. We solve the second order non-homogeneous equation $mh'' + ch' + kh = F_{ext}$ as usual.

7.4 Summary

- To find a particular solution using the method of undetermined coefficients follow the steps below:
  - Break up the forcing into functions of the form $f(t) = (\text{polynomial}) \cdot e^{\mu t}\cos(\nu t) + (\text{polynomial}) \cdot e^{\mu t}\sin(\nu t)$. If that is not possible, then this method does not work.
  - Find the degree d of the polynomial in the forcing.
  - Let m be the smallest integer for which $p^{(m)}(\mu + i\nu) \neq 0$.
  - Use $Y_P = (A_0 t^{m+d} + \cdots + A_d t^m)e^{\mu t}\cos(\nu t) + (B_0 t^{m+d} + \cdots + B_d t^m)e^{\mu t}\sin(\nu t)$.
  - Substitute into the equation and find the constants A_j, B_j.
- If the forcing is $f(t) = f_1(t) + \cdots + f_k(t)$, then we find a particular solution for each $L[y] = f_1(t), \ldots, L[y] = f_k(t)$. Then we add all of these solutions to obtain a particular solution for $L[y] = f(t)$.

- To find a particular solution for $L[y] = f(t)$ using Green functions:
  - Find a general solution for $L[y] = 0$.
  - Solve the initial value problem $L[g] = 0$, $g(0) = g'(0) = \cdots = g^{(n-2)}(0) = 0$, $g^{(n-1)}(0) = 1$.
  - $Y_P(t) = \int_{t_0}^{t} g(t-s)f(s)\ ds$.
- Spring motions are described by the equation $mh'' + ch' + kh = F_{ext}$, where m is the mass, c is the damping coefficient, k is the spring coefficient, and F_{ext} is the external force.
  - $c \geq 0$, and can be found using $F_{drag} = -ch'$.
  - $k \geq 0$, and can be found using Hooke's Law: $mg = k(y_0 - y_r)$.
  - F_{ext} and m are typically given.
- When both roots are real we say the system is overdamped.
- When the roots are equal we say the system is critically damped.
- When the roots are non-real we say the system is underdamped.

8 Week 8

8.1 Variable Coefficients

We will now discuss another method for finding particular solutions to second order equations. The coefficients may be non-constants.

In this method we assume we know a fundamental set of solutions for the homogeneous equation. We will use that to find a particular solution for the non-homogeneous equation.

Suppose Y_1, Y_2 form a FSoS to $y'' + p(t)y' + q(t)y = 0$. We know the general solution to this equation is $c_1Y_1 + c_2Y_2$. We will allow the constants c_1, c_2 to change to find a particular solution to

$$y'' + p(t)y' + q(t)y = f(t).$$

Suppose $y = u_1Y_1 + u_2Y_2$. We have

$$y' = u_1Y_1' + u_1'Y_1 + u_2Y_2' + u_2'Y_2.$$

Letting $u_1'Y_1 + u_2'Y_2 = 0$ we obtain

$$y' = u_1Y_1' + u_2Y_2'.$$

Differentiating again we get

$$y'' = u_1Y_1'' + u_1'Y_1' + u_2Y_2'' + u_2'Y_2'.$$

If we substitute into the original equation we get:

$$\begin{aligned} y'' + p(t)y' + q(t)y &= u_1Y_1'' + u_1'Y_1' + u_2Y_2'' + u_2'Y_2' + p(t)(u_1Y_1' + u_2Y_2') + q(t)(u_1Y_1 + u_2Y_2) \\ &= u_1(Y_1'' + p(t)Y_1' + q(t)Y_1) + u_2(Y_2'' + p(t)Y_2' + q(t)Y_2) + u_1'Y_1' + u_2'Y_2' \\ &= 0 + 0 + u_1'Y_1' + u_2'Y_2'. \end{aligned}$$

We need this to be equal to $f(t)$. Therefore, to find a particular solution we need to solve the following system:

$$\begin{cases} u_1'Y_1 + u_2'Y_2 = 0 \\ u_1'Y_1' + u_2'Y_2' = f(t) \end{cases}$$

Example 8.1. Given that $t, t^2 - 1$ are solutions to the homogeneous equation associated to

$$(t^2 + 1)y'' - 2ty' + 2y = (t^2 + 1)^2 e^t,$$

solve the above non-homogeneous equation.

8.2 Laplace Transforms

Laplace transform, typically denoted by $\mathcal{L}$, assigns to any function f a new function defined below:

$$F(s) = \int_0^\infty e^{-st} f(t)\ dt.$$

This function is often denoted by $\mathcal{L}[f(t)](s)$.

Example 8.2. For a constant c, evaluate $\mathcal{L}[e^{ct}]$.

Properties of Laplace Transforms: For every two functions $f(t), g(t)$ whose Laplace are $F(s)$ and $G(s)$, and every constant c we have the following:

- $\mathcal{L}[f(t) + g(t)] = F(s) + G(s)$.
- $\mathcal{L}[cf(t)] = cF(s)$.
- $\mathcal{L}[e^{ct} f(t)](s) = F(s - c)$.
- $\mathcal{L}[f'(t)] = sF(s) - f(0)$.

Example 8.3. Evaluate $\mathcal{L}[e^{(a+ib)t}]$. Use that to find $\mathcal{L}[e^{at}\cos(bt)$ and $\mathcal{L}[e^{at}\sin(bt)]$.

We can define Laplace transforms for some functions that are not continuous. One example of such functions are those involving the so called **unit step function** $u(t)$ defined below:

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Example 8.4. Show that $\mathcal{L}[u(t - c)f(t - c)] = e^{-cs}\mathcal{L}[f(t)](s)$, if $c > 0$.

Example 8.5. Using the method of Laplace transforms, solve the initial value problem

$$y' - 2y = 5e^t,\ y(0) = 3.$$

It turns out we can extract the function $f(t)$ given its Laplace $F(s)$. This is written as $\mathcal{L}^{-1}[F(s)] = f(t)$.

8.3 Summary

- To find a particular solution to a second order linear equation:
  - We need a FSoS Y_1, Y_2 for the homogeneous equation.
  - Make sure the equation is in normal form. Let $f(t)$ be the forcing.
  - Solve the system:
$$\begin{cases} u_1'Y_1 + u_2'Y_2 = 0 \\ u_1'Y_1' + u_2'Y_2' = f(t) \end{cases}$$
  - This gives $Y_P = u_1Y_1 + u_2Y_2$. Remember that you only need *one* u_1, u_2 and not all of them.
- Laplace transform of $f(t)$ is given by $F(s) = \int_0^\infty e^{-st} f(t)\, dt$.
- Laplace transform $\mathcal{L}$ is linear.

9 Week 9

The following Table of Laplace Transforms can be used during exams and quizzes.

$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
$e^{at}t^n$, a is real and n is a non-negative integer	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}\sin(bt)$, a, b are real	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$, a, b are real	$\frac{s-a}{(s-a)^2+b^2}$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
$u(t-c)f(t-c), c > 0$	$e^{-cs}F(s)$
$e^{at}f(t)$	$F(s-a), s > a$

Example 9.1. Find the inverse Laplace of $\frac{2}{s^2-1}$.

Example 9.2. Find the Laplace of $f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ t+1 & \text{if } 1 \le t \end{cases}$

Example 9.3. Solve the initial value problem using the method of Laplace transforms: $y'' + 4y = f(t)$, where

$$f(t) = \begin{cases} t^2 & 0 \le t < 2 \\ 2t & 2 \le t < 4 \\ 4 & 4 \le t \end{cases}$$

Example 9.4. Find $y(t)$, where $Y(s) = \frac{5}{s^4 + 13s^2 + 36}$.

9.1 Evaluating Green Functions

Recall that Green Function associated to a linear differential operator satisfies the following:

$$L[g] = 0, \;\; g(0) = g'(0) = \cdots = g^{(n-2)}(0) = 0, \;\; g^{(n-1)}(0) = 1.$$

The method of Laplace transforms allows us to find Green Functions. Suppose

$$L = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n.$$

Let $G(s)$ be the Laplace of $g(t)$. Thus,

$$\begin{gathered}
\mathcal{L}[g] = G(s) \\
\mathcal{L}[g'] = sG(s) - g(0) = sG(s), \\
\mathcal{L}[g''] = s^2 G(s) - sg(0) - g'(0) = s^2 G(s), \\
\vdots \\
\mathcal{L}[g^{(n)}] = s^n G(s) - s^{n-1} g(0) - \cdots - sg^{(n-2)}(0) - g^{(n-1)}(0) = s^n G(s) - 1.
\end{gathered}$$

Substituting this into $L[g] = 0$ we obtain the following:

$$(s^n + a_1 s^{n-1} + \cdots + a_n)G(s) - 1 = 0 \Rightarrow G(s) = \frac{1}{p(s)}, \text{ where } p(s) \text{ is the characteristic polynomial of } L.$$

Therefore, to find the Green function associated to L we need to find $\mathcal{L}^{-1}(p(s))$.

Example 9.5. Find the Green function for the linear operator $L = D^2 + 2D - 3$.

9.2 First-Order Linear Systems

A first order system is one of the form

$$\begin{cases}
\dfrac{dx_1}{dt} = f_1(t, x_1, \ldots, x_n) \\
\vdots \\
\dfrac{dx_n}{dt} = f_n(t, x_1, \ldots, x_n)
\end{cases}$$

We typically write the above system in a more compact form: $\dfrac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$, where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{f} = (f_1, \ldots, f_n)$.

We say a point $(t_0, \mathbf{x}_0)$ is an interior point of a region S in $\mathbb{R} \times \mathbb{R}^n$ if there is a box $(a, b) \times (a_1, b_1) \times \cdots \times (a_n, b_n)$ around $(t_0, \mathbf{x}_0)$ that is contained in S.
Similar to differential equations, we need existence-uniqueness Theorems that guarantee solutions exist. One such theorem is the following:

Theorem 9.1. *Consider the initial value problem*

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}), \;\; \mathbf{x}(t_0) = \mathbf{x}_0.$$

Suppose S is a region consisting of all points $(t, \mathbf{x})$ for which the following conditions are both satisfied:

- *All components of* $\mathbf{f}(t, \mathbf{x})$ *are continuous, and*
- *All components of all partial derivatives* $\dfrac{\partial \mathbf{f}}{\partial x_j}$ *are continuous.*

Suppose the initial value $(t_0, \mathbf{x}_0)$ *is an interior point of* S*. Then, the above initial value problem has a unique solution.*

Example 9.6. Consider the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{x-y} \\ \dfrac{dy}{dt} = \sqrt{t^2-1} \end{cases} \qquad x(t_0) = x_0,\ y(t_0) = y_0.$$

Determine all values of t_0, x_0, y_0 for which the existence-uniqueness theorem guarantees a unique solution exists.

Recasting a higher-order system as a first-order system is to write a first-order system that is equivalent to the given system.

Example 9.7. Recast the following IVP as a first-order initial value problem:

$$y'' + yy' + ty = \cos t,\ y(0) = 1, y'(0) = -1.$$

Example 9.8. Recast the system as a first-order system:

$$y_1'' + f(y_1, y_2) = 0,\ y_2'' + g(y_1, y_2) = 0.$$

Example 9.9. Write down a linear equation as a first-order system:

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = f(t).$$

9.3 Summary

- Laplace transform has an inverse: $\mathcal{L}^{-1}[F(s)] = f(t)$.
- You should be able to use the table of Laplace transforms to evaluate Laplace or Laplace inverse of functions.
- $G(s) = \dfrac{1}{p(s)}$, where p is the characteristic polynomial, and $g(t)$ is the Green function.
- To check a first-order initial value problem has a unique solution find the region S for which:
  - $\mathbf{f}$ is continuous.
  - All partials of $\mathbf{f}$ are continuous.
  - The initial value is in the interior of S.
- Every system can be written as a first-order system. To do that for any instance of $y^{(n)}$ introduce new variables: $x_1 = y, x_2 = y', \ldots, x_n = y^{(n-1)}$.

10 Week 10

Example 10.1. Find the general solution of the equation $y''' + 2y'' + y' = e^t$.

Example 10.2. Give an example of a linear equation with constant coefficients whose general solution is $y = c_1e^t + c_2e^{2t} + t$.

Example 10.3. Assume a 4 kg mass stretches a spring 9.8 cm, and that when the mass is moving with a velocity of 5 cm/s the medium in which the mass is moving exerts a force of 3 N. Is this system overdamped, critically damped, or underdamped? Note: $g = 9.8\ m/sec^2$

Example 10.4. Find the Laplace of the solution to the initial value problem

$$y'' - 2y' + y = e^t, y(0) = 1, y'(0) = -1.$$

Example 10.5. Write down the form of one solution to the equation

$$y'' + 2y' + y = t^2e^{-t}.$$

Do not solve!

10.1 Explicit Euler's Method

Suppose we want to estimate $\mathbf{x}(t)$, the value of the solution $\mathbf{x}$ to the following initial value problem at t:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}),\ \mathbf{x}(t_0) = \mathbf{x}_0.$$

We employ a method similar to before: Divide $[t_0, t]$ into N subintervals of equal width:

$$t_0 < t_1 < \cdots < t_N = t,\ h = \frac{t - t_0}{N},\ t_n = t_0 + nh.$$

Then, we estimate $\mathbf{x}(t_n)$ by a recursion similar to what we had before:

$$\mathbf{x}(t_{n+1}) = \mathbf{x}(t_n) + h\mathbf{f}_n,\ \mathbf{f}_n = \mathbf{f}(t_n, \mathbf{x}_n)$$

Example 10.6. Consider the second order initial value problem $y'' = e^y + 1, y(0) = 1, y'(0) = 0$. Using Euler's Explicit method with 1 step approximate $y(1)$.

10.2 Tank Problems

Example 10.7. Consider two interconnected tanks filled with salt water. The first tank contains 42 liters and the second 25 liters. Salt water with a concentration of 9 g/L flows into the first tank at 5 L/h. Well-stirred mixture flows from the first tank into the second tank at a rate of 7 L/h, from the second tank into the first at a rate of 3 L/h, from the first tank into a drain at a rate of 1 L/h, and from the second tank into a drain at a rate of 4 L/h. At time $t = 0$, there are 76 grams of salt in the first tank and 23 grams of salt in the second tank. Give an IVP that governs the amount of salt in each tank. Do not solve!

10.3 Matrices

You need to be familiar with matrix operations, and determinant of a matrix. You also need to be able to find inverse of a 2×2 matrix.

Example 10.8. Given

$$A = \begin{pmatrix} t+1 & 1 \\ 0 & -t \end{pmatrix}, B = \begin{pmatrix} t & t+1 \\ t & 0 \end{pmatrix}.$$

(a) Find AB, BA, and $A + B$.

(b) Find $\det A$.

(c) For what values of t is t invertible?

(d) Find the inverse of A.

10.4 Summary

11 Week 11

11.1 Linear Systems

A first-order n-dimensional linear system is a system of the following form:

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}(t)x_1(t) + \cdots + a_{1n}(t)x_n(t) + f_1(t) \\ \\ \dfrac{dx_2}{dt} = a_{21}(t)x_1(t) + \cdots + a_{2n}(t)x_n(t) + f_2(t) \\ \qquad\qquad\vdots \\ \dfrac{dx_n}{dt} = a_{n1}(t)x_1(t) + \cdots + a_{nn}(t)x_n(t) + f_n(t) \end{cases}$$

This system can be written in matrix form as follows:

$$\underbrace{\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}'} = \underbrace{\begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}}_{\text{coefficient matrix } A(t)} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} + \underbrace{\begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}}_{\text{forcing } \mathbf{f}(t)}.$$

This is often written in a more compact form $\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{f}(t)$. The sqaure matrix $A(t)$ is called the **coefficient matrix** and $\mathbf{f}(t)$ is called the **forcing**. We say this system is **homogeneous** if the forcing is zero. Otherwise, we say the system is **non-homogeneous**.

Example 11.1. Find the coefficient matrix and forcing of the system:

$$\begin{cases} \dfrac{dx}{dt} = t^2 x - (\cos t)y + 1 \\ \dfrac{dy}{dt} = ty - (\sin t)x + t \end{cases}$$

Existence and Uniqueness Theorem: An initial value problem $\dfrac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{f}(t),\ \mathbf{x}(t_0) = \mathbf{x}_0$ has a unique solutions as long as:

- All entries of $A(t)$ and $\mathbf{f}(t)$ are continuous, and
- the initial value t_0 is in the interval.

Example 11.2. Find the interval of definition of the solution to the system

$$\begin{cases} \dfrac{dx}{dt} = (\tan t)x + \dfrac{1}{\sqrt{t^2 - 3}} y \\ \dfrac{dy}{dt} = \dfrac{t-1}{t-3} x + y \\ x(2) = 1,\ y(2) = -1 \end{cases}$$

Similar to what we did before, to find the general solution to the non-homogeneous system $\dfrac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{f}(t)$:

- We will find the general solution $X_H(t)$ of the corresponding homogeneous system.
- We will find a particular solution $X_P(t)$ to the non-homogeneous system.
- Then we add them up to get the general solution to the non-homogeneous system.

11.2 Linear Homogeneous Systems

We will now focus on those systems with $\mathbf{f}(t) = \mathbf{0}$. Much of the theory is similar to what we did for linear differential equations.

Suppose $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$ are solutions to a homogeneous system. Then, every linear combination of them $c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$ is also a solution. This linear combination can be written as

$$c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) = (\mathbf{x}_1(t) \cdots \mathbf{x}_n(t)) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \Psi(t)\mathbf{c}.$$

For this to produce all solutions we need each initial value $\Psi(t_0)\mathbf{c} = \mathbf{x}_0$ to have a solution. Thus, the determinant of $\Psi(t_0)$ must be non-zero. Similar to linear differential equations, this determinant is called **Wronskian**.

$$W[\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)] = \det(\mathbf{x}_1(t) \cdots \mathbf{x}_n(t)).$$

When the Wronskian is non-zero we say $\mathbf{x}_1, \dots, \mathbf{x}_n$ form a **Fundamental Set of Solutions**. The matrix $\Psi(t) = (\mathbf{x}_1(t) \cdots \mathbf{x}_n(t))$ is said to be the **Fundamental Matrix**.

Example 11.3. Suppose $\begin{pmatrix} 1+t^2 \\ t \end{pmatrix}, \begin{pmatrix} t \\ 1 \end{pmatrix}$ are solutions to a first-order 2-dimensional linear system.

(a) Find a fundamental matrix for this equation.

(b) Find the general solution.

(c) Find the coefficient matrix.

The **Natural Fundamental Matrix** at initial time t_0 is a fundamental matrix $\Phi(t)$ for which $\Phi(t_0) = I$, the identity matrix.

Start with a fundamental matrix $\Psi(t)$. We know $\Phi'(t) = A(t)\Phi(t)$, and thus $\Phi(t) = \Psi(t)B$ for some matrix B. Substituting $t = t_0$ we obtain $\Phi(t_0) = \Psi(t_0)B$, which means $I = \Psi(t_0)B$, and thus $B = [\Psi(t_0)]^{-1}$. This gives us the following equality:

$$\Phi(t) = \Psi(t)[\Psi(t_0)]^{-1}.$$

The solution to the IVP $\mathbf{x}' = A(t)\mathbf{x}, \mathbf{x}(t_0) = x_0$ is given by $\Phi(t)\mathbf{x}_0$, since $\Phi(t_0)\mathbf{x}_0 = I\mathbf{x}_0 = \mathbf{x}_0$.

Example 11.4. Suppose we know $\begin{pmatrix} e^{5t} \\ e^{5t} \end{pmatrix}, \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$ is a FSoS for

$$\mathbf{x}' = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \mathbf{x}.$$

Find a Natural Fundamental Matrix at initial time $t_0 = 0$. Use that to solve the IVP

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t),\ \mathbf{x}(0) = \begin{pmatrix} 6 \\ -2 \end{pmatrix}.$$

11.3 Non-homogeneous Systems

As discussed, to solve $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$ we need the general solution $X_H(t)$ to the homogeneous system and one particular solution $X_P(t)$ to the non-homogeneous system. We already studied $X_H(t)$. In order to find $X_P(t)$ we will use a method similar to the method of Variation of Parameters. We let $X_P(t) = \Psi(t)\mathbf{u}(t)$ for a vector $\mathbf{u}(t)$. We then substitute into the system:

$$X_P'(t) = \Psi'(t)\mathbf{u}(t) + \Psi(t)\mathbf{u}'(t) = A(t)\Psi(t)\mathbf{u}(t) + \Psi(t)\mathbf{u}'(t).$$

On the other hand

$$A(t)X_P(t) + \mathbf{f}(t) = A(t)\Psi(t)\mathbf{u}(t) + \mathbf{f}(t).$$

In order for $X_P(t)$ to be a solution we need the following:

$$\Psi(t)\mathbf{u}'(t) = \mathbf{f}(t).$$

Multiplying by the inverse of $\Psi(t)$ from the left we obtain:

$$\mathbf{u}'(t) = [\Psi(t)]^{-1}\mathbf{f}(t).$$

Example 11.5. Suppose we know $\begin{pmatrix} e^{5t} \\ e^{5t} \end{pmatrix}, \begin{pmatrix} e^{t} \\ -e^{t} \end{pmatrix}$ is a FSoS for the first-order 2-dimensional linear homogeneous system

$$\mathbf{x}' = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \mathbf{x}.$$

Find the general solution to the system

$$\mathbf{x}' = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \mathbf{x} + \frac{e^{2t}}{1+e^{2t}} \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

11.4 Homogeneous Linear Systems with Constant Coefficients

In this section we will focus on equations of the form $\mathbf{x}' = A\mathbf{x}$, where A is a constant matrix. Recall that one solution for the differential equation $y' = ay$ is $y = e^{at}$. Note that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

We will see that if we let

$$e^{tA} = I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots.$$

The derivative of this matrix is

$$\frac{d}{dt}(e^{tA}) = \frac{A}{1!} + \frac{2tA^2}{2!} + \frac{3t^2A^3}{3!} + \cdots = A\left(I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \cdots\right) = Ae^{tA}.$$

To summarize we showed $\frac{d}{dt}(e^{tA}) = Ae^{tA}$, which means each column of e^{tA} is a solution to $\mathbf{x}' = A\mathbf{x}$. Since $e^{0A} = I$, the matrix e^{tA} is the Natural Fundamental matrix at initial time $t_0 = 0$.

We will now focus on finding the matrix exponential e^{tA}.

As usual, we will use $D = \frac{d}{dt}$. We note that $D(e^{tA}) = Ae^{tA}$. Similar to what we did before, if $p(z)$ is a polynomial, then

$$p(D)e^{tA} = p(A)e^{tA}.$$

Note that if $p(A) = 0$, then $p(D)e^{tA} = 0$. Thus, all entries of e^{tA} satisfy the differential equation $p(D)y = 0$. We will now find these entries by finding initial conditions for the matrix e^{tA}. We will have the following:

$$e^{tA}|_{t=0} = I,\ \frac{d}{dt}(e^{tA})|_{t=0} = A,\ \frac{d^2}{dt^2}(e^{tA})|_{t=0} = A^2, \ldots.$$

Thus, in order to find e^{tA} we need to do the following:

- Find a polynomial $p(z)$ for which $p(A) = 0$.
- Find a NFSoS $N_0(t), N_1(t), \ldots, N_m(t)$ for $p(D)y = 0$.
- $e^{tA} = N_0(t)I + N_1(t)A + \cdots + N_m(t)A^m$.

Such a polynomial $p(z)$ can be found using the Cayley-Hamilton Theorem stated below:

Cayley-Hamilton Theorem: If $p(z) = \det(A - zI)$, then $p(A) = 0$.

Example 11.6. Find a polynomial $p(z)$ for which $P(A) = 0$, where $A = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$.

Example 11.7. Compute e^{tA}, where $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

Note that e^{tA} is a natural fundamental matrix at initial time $t_0 = 0$, since $e^{0A} = I$. This allows us to solve systems with constant coefficients.

Example 11.8. Solve the initial value problem:

$$\begin{cases} x' = 2x + y \\ y' = x + 2y \end{cases}, x(0) = 1,\ y(0) = -1.$$

11.5 Summary

- To find the interval of definition of a linear initial value problem:
  - Find all points of discontinuity of all coefficients and the forcing.
  - Place all of these on a number line.
  - Determine the largest interval that contains the initial value t_0.
- $W[\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)] = \det\left(\mathbf{x}_1(t) \cdots \mathbf{x}_n(t)\right)$.
- When Wronskian is not zero, we say we have a FSoS.
- Placing a FSoS into columns of a matrix we get a Fundamental Matrix $\Psi(t)$.
- The general solution is given by $\Psi(t)\mathbf{c}$.
- To find the coefficient matrix $A(t)$ of a homogeneous equation, use $\Psi'(t) = A(t)\Psi(t)$. Make sure you are multiplying both sides by the inverse of $\Psi(t)$ from the right: $A(t) = \Psi'(t)[\Psi(t)]^{-1}$.
- Given a fundamental matrix $\Psi(t)$, the natural fundamental matrix at time t_0 is given by $\Psi(t)[\Psi(t_0)]^{-1}$.
- The IVP $\mathbf{x}' = A(t)\mathbf{x}, \mathbf{x}(t_0) = \mathbf{x}_0$ can be solved by the formula $\Phi(t)\mathbf{x}_0$, where $\Phi(t)$ is the natural fundamental matrix at time t_0.
- When the forcing is $\mathbf{f}(t)$, a paricular solution can be found by solving $\mathbf{u}'(t) = [\Psi(t)]^{-1}\mathbf{f}(t)$ and setting $\mathbf{X}_P = \Psi(t)\mathbf{u}(t)$.
- To evaluate e^{tA}:
  - Evaluate the polynomial $p(z) = \det(A - zI)$.

- Find all roots of $p(z) = 0$ and use that to find Y_H for $p(D)y = 0$.
- Solve the IVP $p(D)y = 0, y(0) = y_0, y'(0) = y_1, \ldots$.
- Coefficients of $y_0, y_1, \ldots$ are elements of a NFS.
- $e^{tA} = N_0 I + N_1 A + \cdots$.

12 Week 12

12.1 Eigen Methods

First, we need a bit of linear algebra.

Let A be a square matrix, and $\mathbf{v}$ be a non-zero vector, and λ be a scalar. If $A\mathbf{v} = \lambda\mathbf{v}$, then we say λ is an **eigenvalue**, $\mathbf{v}$ is an **eigenvector**, and $(\lambda, \mathbf{v})$ is an **eigenpair** for the matrix A.

$A\mathbf{v} = \lambda\mathbf{v}$ can be written as $A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$ or $(A - \lambda I)\mathbf{v} = \mathbf{0}$. If the matrix $A - \lambda I$ were to have an inverse, we would have $(A - \lambda I)^{-1}(A - \lambda I)\mathbf{v} = \mathbf{0}$, which means $\mathbf{v} = \mathbf{0}$, which contradicts our assumption that $\mathbf{v}$ is non-zero. Therefore, $A - \lambda I$ does not have an inverse, which implies $\det(A - \lambda I) = 0$.

Example 12.1. Find two eigenpairs, one for each eigenvalue of $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$.

Example 12.2. Find two eigenpairs, one for each eigenvalue of $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Note: If $(\lambda, \mathbf{v})$ is an eigenpair for a real matrix A, then so is $(\overline{\lambda}, \overline{\mathbf{v}})$.

Recall that our main objective was to solve $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$. Looking back at differential eqautions of the form $y' = ay$ we know a solution is of the form e^{at}. We guess that a solution to $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ might be of the form $\mathbf{x} = e^{\lambda t}\mathbf{v}$. Substituting this into the system we obtain

$$\lambda e^{\lambda t}\mathbf{v} = Ae^{\lambda t}\mathbf{v} \Rightarrow e^{\lambda t}\lambda\mathbf{v} = e^{\lambda t}A\mathbf{v} \Rightarrow \lambda\mathbf{v} = A\mathbf{v}.$$

This precisely means that $(\lambda, \mathbf{v})$ is an eigenpair for A. So, in order to find solutions for the system we will need to find eigenpairs.

Example 12.3. Using the eigen method, find the general solution of the system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, where $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$. Use that to find e^{tA}.

Example 12.4. Using the eigen method, find the general solution of the system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, where $A = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$.

Example 12.5. Using eigen-method find the general solution of the system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, where $A = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$.

In the above example we are unable to find enough eigen vectors to produce the general solution to $\mathbf{x}' = A\mathbf{x}$.

Assume the polynomial $p(z) = \det(A - zI)$ is of the form $(z - \lambda)^2$, but there are not two non-parallel eigenvectors. Assume $(\lambda, \mathbf{v})$ is an eigenpair for A. We know $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}$ is one solution. The second solution can be found as follows:

- Find a non-zero vector $\mathbf{w}$ that is not parallel to $\mathbf{v}$.
- $\mathbf{x}_2(t) = e^{\lambda t}\mathbf{w} + te^{\lambda t}(A - \lambda I)\mathbf{w}$.

Finish solving the previous example.

12.2 Method of Laplace Transforms

This method will be used to find particular solutions to non-homogeneous equations or to solve initial value problems.
Laplace is defined component-wise. In other words, $\mathcal{L}\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] = \begin{pmatrix} \mathcal{L}[x_1] \\ \mathcal{L}[x_2] \end{pmatrix}$.
Suppose $\mathbf{x}$ is a solution to $\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{f}$. Using properties of Laplace transform we obtain $\mathcal{L}[\frac{d\mathbf{x}}{dt}] = A\mathcal{L}[\mathbf{x}] + \mathcal{L}[\mathbf{f}]$. Similar to before we obtain $sX(s) - \mathbf{x}(0) = AX(s) + F(s)$, where $X(s), F(s)$ are the Laplace transforms of $\mathbf{x}(t)$ and $\mathbf{f}(t)$, respectively. This yields $(sI - A)X(s) = F(s) + \mathbf{x}(0)$. Multiplying by the inverse we obtain

$$X(s) = (sI - A)^{-1}(F(s) + \mathbf{x}(0)).$$

Taking the Laplace inverse we can find the solution to the initial value problem $\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{f}, \mathbf{x}(0) = \mathbf{x}_0$, and thus a particular solution.

Example 12.6. Solve the initial value problem using Laplace method:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^t \\ 0 \end{pmatrix}, \ \mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

12.3 Summary

- Eigenvalues of a matrix A are found by solving $\det(A - \lambda I) = 0$.
- After finding an eigenvalue λ we will find an eigenvector by solving $(A - \lambda I)\mathbf{v} = \mathbf{0}$.
- If $(\lambda, \mathbf{v})$ is a real eigenpair for A, then $e^{\lambda t}\mathbf{v}$ is a solution to $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$.
- If λ is not real, then we will find two solutions by looking at real and imaginary part of $e^{\lambda t}\mathbf{v}$.
- If $\det(A - zI) = (z - \lambda)^2$, then:
  - Set $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}$.
  - Find a non-zero vector $\mathbf{w}$ that is not parallel to $\mathbf{v}$.
  - $\mathbf{x}_2(t) = e^{\lambda t}\mathbf{w} + te^{\lambda t}(A - \lambda I)\mathbf{w}$.

  - The general solution is $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$.

- To solve the initial value problem $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t), \mathbf{x}(0) = \mathbf{x}_0$ using the method of Laplace transforms:

  - Find $F(s)$ the Laplace transform of $\mathbf{f}(t)$.
  - Set $X(s)$ to be the Laplace transform of $\mathbf{x}$.
  - By taking Laplace of both sides we obtain $sX(s) - \mathbf{x}_0 = AX(s) + F(s)$.
  - Re-arranging we will obtain $X(s) = (sI - A)^{-1}(F(s) + \mathbf{x}_0)$.
  - Taking the inverse Laplace we can find $\mathbf{x}$.

13 Week 13

13.1 Phase-Plane Portraits

In this section we will focus on graphical understanding of solutions to 2-dimensional linear systems with constant coefficients:

$$\begin{cases} x' = a_{11}x + a_{12}y \\ y' = a_{21}x + a_{22}y \end{cases}$$

This can also be written as:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

For every solution $(x(t), y(t))$, the curve $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ in the xy-plane is called an **orbit**.

For each real eigenpair $(\lambda, \mathbf{v})$ the solution $\mathbf{x}(t) = ce^{\lambda t}\mathbf{v}$ with c as a constant, is called an **eigensolution** of this system. Each eigensolution is represented by a line through the origin and is called an **eigensolution orbit**.

If λ is positive, the value of $\mathbf{x}(t) = ce^{\lambda t}\mathbf{v}$ would diverge to infinity as $t \to \infty$. This is indicated on the eigensolution by an arrow. In other words, the arrow indicates how the eigensolution is traversed as t increases.

If λ is negative, the value of $ce^{\lambda t}\mathbf{v}$ approaches zero as t tends to infinity. This is also indicated by an arrow towards the origin.

When $\lambda = 0$. then we indicate that by placing circles on the eigensolution.

Example 13.1. Sketch the eigensolution orbits of the system

$$\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}\mathbf{x}.$$

After drawing the eigensolution orbits, if any, we will draw sample orbits in each region. Remember that because of the uniqueness of solutions to initial value problems the orbits do not intersects.

Example 13.2. Draw a phase-plane portrait for each of the system $\mathbf{x}' = A\mathbf{x}$, where A is each of the following:

(a) $A = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}$.

(b) $A = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}$.

(c) $A = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$.

When eigenvalues are non-real, there are no eigensolutions.

Example 13.3. Draw a phase-plane portrait for each of the following systems:

(a) $\begin{cases} x' = x + y \\ y' = -x + y \end{cases}$

(b) $\begin{cases} x' = y \\ y' = -4x \end{cases}$

A system is said to be **stable** if solutions that start near the origin, stay near the origin. We say the origin is **attracting** if every solution that starts near the origin approaches the origin.

Example 13.4. Sketch the phase-plane portrait of the system $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix} \mathbf{x}$. Determine the stability of this system. Is the origin attracting?

Check Part III, 7.2 pages 10-15.

13.2 Summary

- To draw a phase-plane portrait:
  - Find eigenpairs of A.
  - For real eigenpairs draw the eigensolution orbits $e^{\lambda t}\mathbf{v}$.
  - For each eigensolution orbit indicate with arrows the behavior of the solution as t increases.
  - Sketch sample orbits and indicate with arrows the behavior of the solution as t increases.
  - When we have non-real eigenvalues $a \pm bi$, the orbits would be either ellipses (when $a = 0$) or spirals (when $a \neq 0$).

  - Indicate the direction of spirals and their orientations. If $a < 0$ then the arrows should be inwards, towards the origin, and if $a > 0$ the arrows should point outwards. To determine the orientation of spiral pick a sample point (say $(1, 0)$ or $(0, 1)$) and see if x or y increases or decreases. Put these two pieces of information together to determine if the spiral is clockwise or counter-clockwise, and then specify how the solution behaves with arrows.

- The origin is stable if eigenvalues have non-positive real parts, otherwise it is unstable.
- The origin is attracting if eigenvalues have negative real parts.

14 Week 14

14.1 Non-linear Systems

14.1.1 Stationary and Semi-Stationary Solutions

A solution is called **stationary** if it is independent of time t, i.e. it is constant. Stationary solutions are sometimes called **equilibrium solutions**, or **critical points**.

Example 14.1. Find all stationary solutions of the system

$$\begin{cases} x' = y^2 - 1 \\ y' = x^2 y - x \end{cases}$$

To find stationary solutions of the system

$$x' = f(x, y), \ y' = g(x, y)$$

we solve the system

$$\begin{cases} f(x, y) = 0 \\ g(x, y) = 0 \end{cases}$$

Semi-stationary solutions are those that one of x or y is constant.

Example 14.2. Find semi-stationary solutions of the system

$$x' = x^2 - xy + x - y, y' = y(x^2 - 2x + 3).$$

14.2 Phase-Plane Portrait for Non-linear Systems

To be able to draw phase-plane portraits for non-linear systems we first find all stationary solutions. Then, we will approximate the system with a linear system near each stationary solution, and then we draw the phase-plane portrait near each of the stationary solutions.

Linearization of functions $f(x, y)$ and $g(x, y)$ are given below. These are also sometimes called tangent plane approximations:

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0),$$
$$g(x,y) \approx g(x_0,y_0) + g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0).$$

The system can now be written as follows:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{pmatrix} + \begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}.$$

Assuming (x_0,y_0) is a stationary point, we have $f(x_0,y_0) = g(x_0,y_0) = 0$, and thus we obtain the following:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}.$$

Setting $\tilde{x} = x - x_0, \tilde{y} = y - y_0$ we obtain the following linear system:

$$\frac{d}{dt}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}.$$

Example 14.3. Consider the system

$$\frac{dx}{dt} = y, \ \frac{dy}{dt} = 4x - x^3.$$

(a) Find all stationary solutions of this system.

(b) Write down the linearization of this system near each stationary solution.

(c) Sketch the phase-plane portrait of this system near each stationary solution.

(d) Classify each stationary solution as stable or unstable, and also as attracting or not attracting.

14.3 Orbit Equations

Given a system

$$\frac{dx}{dt} = f(x,y), \ \frac{dy}{dt} = g(x,y)$$

we can eliminate t by combining the two equations as $f(x,y)\frac{dy}{dt} = g(x,y)\frac{dx}{dt}$, which can be written as

$$f(x,y)dy - g(x,y)dx = 0.$$

This equation is called the **orbit equation**. Sometimes this equation can be solved and the solution gives us some orbits.

Example 14.4. Sketch the global phase-plane portrait for to the system

$$\frac{dx}{dt} = ye^{1+x^2+y^2}, \ \frac{dy}{dt} = -xe^{1+x^2+y^2}.$$

A system whose orbit equation is exact is called **Hamiltonian**.

Example 14.5. Show the following system is Hamiltonian:

$$\frac{dx}{dt} = 2xy, \ \frac{dy}{dt} = 3 - y^2.$$

Find the equation of all orbits of this system as curves in the xy-plane.

Example 14.6. Draw the global phase-plane portrait of the following system:

$$\frac{dx}{dt} = y, \ \frac{dy}{dt} = x.$$

14.4 Summary

- Consider the system $x' = f(x, y), y' = g(x, y)$:
  - To find stationary solutions we set $f(x, y) = g(x, y) = 0$ and solve.
  - To find semi-stationary solutions we will find those fixed values of x for which $f(x, y) = 0$ for all y, and those fixed values of y for which $g(x, y) = 0$ for all x. We use this information and solve the differential equation for the other variable.
- The linearization of the system $x' = f(x, y), y' = g(x, y)$ at a stationary point (x_0, y_0) is given by

$$\frac{d}{dt}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}.$$

- To sketch the phase-plane portrait of a non-linear system near a stationary solution find the linearization of the system and draw the phase-plane of the linearization.
- The orbit equation associated to the system $x' = f(x, y), y' = g(x, y)$ is given by

$$f(x, y)dy - g(x, y)dx = 0.$$

- The orbit equation sometimes can be solved and that gives us equations of orbits in the xy-plane.
- Drawing the curves obtained from solving the orbit equation in a plane gives us the phase-plane portrait.

A Appendix: Complex Numbers

Any number of the form $a + ib$, where a, b are real is called a **complex number**. The real numbers a and b are called the **real part** and **imaginary part** of the given complex number. Note that imaginary part of a real number is *real*! The **conjugate** of a complex number $a + ib$ is $a - ib$.

Addition, subtraction and multiplication are done by combining like terms, distributive law and use of the fact that $i^2 = -1$. To divide by a complex number we multiply both numerator and denominator by the conjugate of the denominator.

Example A.1. Find $z + w, z - w, zw$ and z/w, where $z = 1 + 2i$ and $w = 2 - i$.

Example A.2. Suppose a, b are real numbers for which $a + b + ai + 2bi = 4 + 3i$. Find a and b.

Solution. Comparing the real parts of both sides we obtain the following system:

$$\begin{cases} a + b = 4 \\ a + 2b = 3 \end{cases}$$

Subtracting the two equations we obtain $b = -1$. Substituting yields $a = 5$. $\square$