Math 410 Summary and Examples

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Contents

Preface

How to use these notes

Here is how you would want to use these notes:

- In your first reading you may skip the Preliminaries Chapter. You should come back to it if you are unclear on properties of real numbers or methods of proof.
- Each chapter has four types of problems:
  - **Warm-Ups**: Basic problems that should be easy for a student who is well-prepared for this class.
  - **Examples**: The most common types of problems that appear in this class.
  - **Practice Problems**: Problems without solutions.
  - **Challenge Problems**: Problems for students who want to get more out of this class.

Chapter 0

Preliminaries

If you see any errors or typos feel free to message me. No typo is too small to report!

These notes will regularly be updated.

0.1 Basic Properties of Real Numbers

We will start our journey with assuming some basic properties of real numbers without proof. These are called "Axioms" that we assume for real numbers: Field Axioms, The Positivity Axiom, and The Completeness Axiom. We will list these axioms below. The construction of real numbers is not dealt with in this class.

Definition 0.1. There is a set $\mathbb{R}$, called **the set of real numbers**, along with two binary operations called **multiplication** and **addition** that assign to every two $a, b \in \mathbb{R}$, two real numbers ab (or $a \cdot b$) and $a + b$ for which they satisfy the following:

Field Axioms: Let $a, b, c \in \mathbb{R}$.

- $(a + b) + c = a + (b + c)$, and $(ab)c = a(bc)$. [Multiplication and addition are associative.]
- $ab = ba$, and $a + b = b + a$. [Multiplication and addition are commutative.]
- There are two distinct elements $0, 1 \in \mathbb{R}$ (called zero and one, respectively) for which $a + 0 = 1a = a$. [Additive and multiplicative identity.]
- There is $d \in \mathbb{R}$ for which $a + d = 0$. [Additive inverse.]
- If $a \neq 0$, then there is $d \in \mathbb{R}$ for which $ad = 1$. [Multiplicative inverse.]
- $a(b + c) = (ab) + (ac)$. [Distributive property.]

Positivity Axiom: There is a subset $\mathcal{P}$ of real numbers called the set of positive numbers that satisfies the following properties:

(a) $\mathcal{P}$ is closed under addition and multiplication, i.e. if $a, b \in \mathcal{P}$, then so are ab and $a + b$; and

(b) For every $a \in \mathbb{R}$ precisely one of the following holds:

- $a = 0$
- $a \in \mathcal{P}$
- $-a \in \mathcal{P}$

Completeness Axiom: This will be discussed in the next chapter.

Definition 0.2. Every element of $\mathbb{R}$ is called a **real number** or simply a **number**.

Theorem 0.1. *Let a be a real numbers.*

(a) 0 and 1 are unique. In other words, the additive identity and multiplicative identity are unique in $\mathbb{R}$.

(b) The additive inverse of a is unique.

(c) If $a \neq 0$, then the multiplicative inverse of a is unique.

Proof. (a) Suppose 0 and $0'$ are both additive identities. Since $0'$ is an additive identity $0 + 0' = 0$. Since 0 is an additive identity and addition is commutative, we have $0 + 0' = 0'$, and thus $0 = 0'$, which implies the additive identity is unique. The proof of uniqueness of the multiplicative identity is similar.

(b) Suppose b and c are both additive inverses of a, we have

$$\begin{aligned} b &= b + 0 && \text{0 is the additive identity} \\ &= b + (a + c) && c \text{ is an additive inverse of } a \\ &= (b + a) + c && \text{associativity of addition} \\ &= (a + b) + c && \text{addition is commutative} \\ &= 0 + c && b \text{ is an additive inverse of } a \\ &= c + 0 && \text{addition is commutative} \\ &= c && \text{0 is the additive identity,} \end{aligned}$$

and thus the additive inverse is unique.

(c) Similar to above.

□

Definition 0.3. Every element of $\mathcal{P}$ is called a **positive** number and every real number that is either zero or positive is called a **nonnegative** number. Similarly, every number whose additive inverse is positive is called **negative** and every number that is either negative or zero is called **nonpositive**.

Notation: The additive inverse of a real number a is denoted by $-a$. The multiplicative inverse of a nonzero real number b is denoted by $\frac{1}{b}$. The product of a and $\frac{1}{b}$ is denoted by $\frac{a}{b}$ and is called the **quotient of** a **by** b or **the division of** a **by** b. Similarly a **minus** b or **subtraction of** b **from** a is defined as $a + (-b)$ and is denoted by $a - b$.

Theorem 0.2. *For every three real numbers a, b, c,*

(a) $0a = 0$

(b) $-(-a) = a$, *and if* $a \neq 0$, *then* $\frac{1}{(\frac{1}{a})} = a$.

(c) $(-1)a = -a$

(d) $(-a)b = a(-b) = -(ab)$

(e) $(-a)(-b) = ab$

(f) *If* $b \neq 0$, *then* $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$, *and* $\frac{-a}{-b} = \frac{a}{b}$.

(g) $a(b - c) = ab - ac$.

Proof. (a) $0a = (0 + 0)a = 0a + 0a$. Adding the additive inverse of $0a$ to both sides we get $0a = 0$.

(b) Since $a + (-a) = 0$, the number $-a$ is the additive inverse of a, and thus, by uniqueness $-(-a) = a$. Similar for the multiplicative inverse.

(c) $a + (-1)a = a(1 + (-1)) = a0 = 0$. Thus, by uniqueness of additive inverse $(-1)a = -a$.

(d) $(-a)b + ab = ((-a) + a)b = 0b = 0$. Thus, by uniqueness of additive inverse $(-a)b = -(ab)$. Similar for the other one.

(e) By the previous parts $(-a)(-b) = -(a(-b)) = -(-(ab)) = ab$.

(f) By definition of quotient and the previous parts $\frac{-a}{b} = (-a)\frac{1}{b} = -(a\frac{1}{b}) = -\frac{a}{b}$. Note that $(-b)\frac{a}{-b} = a$, which implies $-b(\frac{a}{-b}) = a$ or $b\frac{a}{-b} = -a$. This means $\frac{a}{-b} = \frac{-a}{b}$. For the last part we will use the previous parts twice: $\frac{-a}{-b} = -\frac{a}{-b} = -(-\frac{a}{b}) = \frac{a}{b}$.

(g) $a(b - c) = a(b + (-c)) = ab + a(-c) = ab + (-ac) = ab - ac$. Here we used the definition of subtraction, distributive law, part (d) and the definition of subtraction, in order.

$\square$

Definition 0.4. The set of **natural numbers**, denoted by $\mathbb{N}$, is defined as

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

Similarly, we define the set of **integers** by

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}.$$

Every element of $\mathbb{Z}$ is called an **integer**.

Definition 0.5. A real number is called **rational** or a **fraction** if it is of the form $\dfrac{m}{n}$, where m, n are integers and $n \neq 0$. Otherwise, it is called **irrational**. The set of all rational numbers is denoted by $\mathbb{Q}$.

Definition 0.6. For every nonzero real number a, we define a^n for integers n recursively by:

- $a^0 = 1$; and

- $a^n = a^{n-1} \cdot a$, for every natural number n; and
- $a^{-n} = \dfrac{1}{a^n}$, for every natural number n.

Theorem 0.3 (Zero Product Property)**.** *If $ab = 0$ for two real numbers a and b, then $a = 0$ or $b = 0$. If $a \neq 0$ and $b \neq 0$, then $ab \neq 0$.*

Proof. Suppose $ab = 0$. If $a = 0$, we are done. So, assume $a \neq 0$. Multiplying both sides by the multiplicative inverse of a, we obtain $1b = 0$, or $b = 0$, as desired. The second statement is the contrapositive of the first. □

Theorem 0.4 (Addition, Multiplication and Representation of Fractions)**.** *Let m, n, k, ℓ be real numbers with $k, \ell \neq 0$. Then*

(a) $\dfrac{m}{k} + \dfrac{n}{\ell} = \dfrac{m\ell + nk}{k\ell}$,

(b) $\dfrac{m}{k}\dfrac{n}{\ell} = \dfrac{mn}{k\ell}$, *and*

(c) $\dfrac{m}{k} = \dfrac{m\ell}{k\ell}$.

Proof. (a) By distributive property we obtain $k\ell(\frac{m}{k} + \frac{n}{\ell}) = k\ell\frac{m}{k} + k\ell\frac{n}{\ell}$. By definition of $\frac{m}{k}$ and $\frac{n}{\ell}$, we have $k\frac{m}{k} = m$ and $\ell\frac{n}{\ell} = n$. Therefore, $k\ell(\frac{m}{k} + \frac{n}{\ell}) = m\ell + nk$. Thus, by definition of quotient $\frac{m}{k} + \frac{n}{\ell} = \frac{m\ell+nk}{k\ell}$.

(b) $k\ell\frac{m}{k}\frac{n}{\ell} = k\frac{m}{k}\ell\frac{n}{\ell} = mn$, thus, by definition of quotient $\frac{m}{k}\frac{n}{\ell} = \frac{mn}{k\ell}$.

(c) By definition of quotient $k\ell\frac{m\ell}{k\ell} = m\ell$. Using properties of real numbers we obtain $(k\frac{m\ell}{k\ell} - m)\ell = 0$. The Zero Product Property implies $k\frac{m\ell}{k\ell} - m = 0$. This implies $k\frac{m\ell}{k\ell} = m$, or $\frac{m\ell}{k\ell} = \frac{m}{k}$.

□

0.2 Basics of Inequalities

Inequalities are extensively used in Real Analysis. It is very important to be comfortable with basic properties of inequalities. The main properties of inequalities are listed below. This is long and you may want to skip this in a first reading. If you do so, you should come back to it as needed.

Definition 0.7. For two real numbers a and b we write $a > b$ or $b < a$, if $a - b \in \mathcal{P}$. We write "$a \geq b$" or "$b \leq a$" instead of "$a > b$ or $a = b$".

Definition 0.8. Given a real number x, we define the **absolute value of** x, denoted by $|x|$, as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Theorem 0.5 (The Triangle Inequality)**.** *For every two real numbers a, b we have $|a + b| \leq |a| + |b|$.*

Some particular identities come in handy.

Theorem 0.6 (Difference and sum of n-th powers)**.** *For every two real numbers a, b and every positive integer n we have:*

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$

If n is odd, then

$$a^n - b^n = (a+b)(a^{n-1} - a^{n-2}b + \cdots - ab^{n-2} + b^{n-1}).$$

Theorem 0.7 (Geometric series sum)**.** *Let a, r be real numbers and n be a positive integer. If $r \neq 1$, then*

$$a + ar + \cdots + ar^{n-1} = \frac{a - ar^n}{1-r}.$$

Theorem 0.8 (The Binomial Theorem)**.** *For every two real numbers a, b and every positive integer n we have:*

$$(a+b)^n = a^n + \binom{n}{1} b^{n-1}b + \cdots + \binom{n}{n-1} ab^{n-1} + b^n.$$

Theorem 0.9 (Properties of inequalities)**.** *Let a, b, c, d be real numbers. Then, [Translation in words is included in brackets!]*

(1) If $a > b$ (resp. $a \geq b$) and $c > 0$, then $ac > bc$ (resp. $ac \geq bc$). [Multiplying both sides of an inequality by a positive number does not change the direction.]

(2) If $a > b$ (resp. $a \geq b$) and $c < 0$, then $ac < bc$ (resp. $ac \leq bc$). [Multiplying both sides of an inequality by a negative number flips the direction.]

(3) If $a > b$ (resp. $a \geq b$) and $b > c$ (resp. $b \geq c$), then $a > c$ (resp. $a \geq c$). ["$>$" is a transitive relation.]

(4) If $a > b$ (resp. $a \geq b$) and $c > d$ (resp. $c \geq d$), then $a + c > b + d$ (resp. $a + c \geq b + d$). [One may add inequalities.]

(5) If $a > b > 0$ (resp. $a \geq b > 0$) and $c > d > 0$ (resp. $c \geq d > 0$), then $ac > bd$ (resp. $ac \geq bd$). [One may multiply inequalities involving positive numbers.]

(6) $1 > 0$. [Duh!]

(7) If $a > 0$ (resp. $a < 0$), then $\frac{1}{a} > 0$ (resp. $\frac{1}{a} < 0$). [Multiplicative inverse of a number has the same sign.]

(8) If $a > b$ (resp. $a \geq b$) and $ab > 0$, then $\frac{1}{a} < \frac{1}{b}$ (resp. $\frac{1}{a} \leq \frac{1}{b}$). [Taking the multiplicative inverse of both sides of an inequality flips the direction, as long as both sides have the same sign.]

(9) $a^2 \geq 0$. [The Trivial Inequality.] Furthermore if n is an even natural number, then $a^n \geq 0$ and if n is odd, then either both a and a^n are nonpositive or both are nonnegative.

(10) If $a < 0$ and $b < 0$, then $ab > 0$ and if $a < 0$ and $b > 0$, then $ab < 0$. [Back-to-Elementary-School Inequality!]

(11) *If $a > b \geq 0$ (resp. $a \geq b \geq 0$) and n is a natural number, then $a^n > b^n$ (resp. $a^n \geq b^n$). [x^n is strictly increasing over $[0, \infty)$.]*

(12) *If $0 \geq a > b$ (resp. $0 \geq a \geq b$) and n is an even natural number, then $a^n < b^n$ (resp. $a^n \leq b^n$). [x^n is strictly decreasing over $(-\infty, 0]$, when n is even.]*

(13) *If $a < b$ (resp. $a \leq b$) and n is an odd natural number, then $a^n < b^n$ (resp. $a^n \leq b^n$). [x^n is strictly increasing when n is odd.]*

(14) *Let n be an even positive integer. Then $a^n < b^n$ (resp. $a^n \leq b^n$) iff $|a| < |b|$ (resp. $|a| \leq |b|$). [n-the root function is strictly increasing over $[0, \infty)$.]*

(15) *Let n be a positive odd integer. Then $a^n < b^n$ (resp. $a^n \leq b^n$) iff $a < b$ (resp. $a \leq b$). [n-th root function is strictly increasing when n is odd.]*

(16) *Assume $b \geq 0$. Then $|a - c| < b$ (resp. $|a - c| \leq b$) iff $-b + c < a < c + b$ (resp. $-b + c \leq a \leq c + b$). [Opening absolute value inequalities.]*

(17) *Assume $a < b < 0$ (resp. $a \leq b < 0$), then $|a| > |b|$ (resp. $|a| \geq |b|$). [Absolute value is strictly decreasing over $(-\infty, 0)$.]*

(18) *Precisely one of these three statements is true: "$a > b$", "$a < b$", "$a = b$". ["$>$" is a total order.]*

Proof. Let $\mathcal{P}$ be the set of all positive real numbers.

(1) By definition, $a - b, c \in \mathcal{P}$. By properties of $\mathcal{P}$, $(a - b)c \in \mathcal{P}$. Thus, $ac - bc \in \mathcal{P}$, which by definition we obtain $ac > bc$.

(2) By definition $a - b, -c \in \mathcal{P}$. By properties of $\mathcal{P}$, $(a - b)(-c) \in \mathcal{P}$, which implies $bc - ac \in \mathcal{P}$. Therefore $bc > ac$.

(3) By definition $a - b, b - c \in \mathcal{P}$. Since $\mathcal{P}$ is closed under addition, $(a - b) + (b - c) \in \mathcal{P}$. Thus $a - c \in \mathcal{P}$, which implies $a > c$.

(4) By definition $a - b \in \mathcal{P}$ and $c - d \in \mathcal{P}$. Since $\mathcal{P}$ is closed under addition, $(a - b) + (c - d) \in \mathcal{P}$. This implies $a + c - (b + d) \in \mathcal{P}$. Therefore $a + c > b + d$.

(5) Since $a > b$ and $c > 0$, by part (1), $ac > bc$. Similarly applying part (1) to $b > 0$ and $c > d$, we obtain $bc > bd$. Thus by transitivity of $>$ we conclude $ac > bd$.

(6) Since $1 \neq 0$, by properties of $\mathcal{P}$, $1 \in \mathcal{P}$ or $-1 \in \mathcal{P}$, but not both. If $-1 \in \mathcal{P}$, then $(-1)(-1) \in \mathcal{P}$ which implies both 1 and -1 are in $\mathcal{P}$, which is a contradiction. Therefore $1 \in \mathcal{P}$, which means $1 > 0$.

(7) Since $\frac{1}{a} \neq 0$, by properties of $\mathcal{P}$, either $\frac{1}{a} \in \mathcal{P}$ or $-\frac{1}{a} \in \mathcal{P}$, but not both. Assume $-\frac{1}{a} \in \mathcal{P}$. Since $a \in \mathcal{P}$ and $\mathcal{P}$ is closed under multiplication, $-1 = a(-\frac{1}{a}) \in \mathcal{P}$ which contradicts (6).

(8) By definition $a - b \in \mathcal{P}$ and $ab \in \mathcal{P}$. Thus by (7), $\frac{1}{ab} \in \mathcal{P}$. Therefore $(a-b)(\frac{1}{ab}) \in \mathcal{P}$, which implies $\frac{1}{b} - \frac{1}{a} \in \mathcal{P}$. Hence $\frac{1}{b} > \frac{1}{a}$.

(9) If $a = 0$, then $a^2 = 0$ which implies $a^2 \geq 0$.
If $a > 0$, then $a^2 > 0$ by properties of $\mathcal{P}$.
If $a < 0$, then multiplying by -1 we get $-a > 0$. Thus $(-a)^2 > 0$. Therefore $a^2 > 0$.
If $n = 2k$ for some natural number k, then $a^n = (a^k)^2 \geq 0$.

(10) Assume $a < 0$ and $b < 0$. Use part (2) to obtain $ab > 0 \cdot b = 0$. The second one is similar.

(11) $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$. The second parenthesis is in $\mathcal{P}$ by properties of $\mathcal{P}$. Therefore $a^n > b^n$.

(12) We notice that $-b > -a > 0$ by (2). Therefore by (11), we obtain $(-b)^n > (-a)^n$. Since n is even $(-a)^n = a^n$ and $(-b)^n = b^n$. Thus $b^n > a^n$.

(13) If $a, b > 0$, then this follows from (11). If $a < 0 < b$, then clearly $a^n < 0 < b^n$. If $a < b < 0$, the proof is similar to (12).

(14) Notice that since n is an even integer, $a^n = (-a)^n = |a|^n$ and $b^n = (-b)^n = |b|^n$. If $|a| < |b|$, then $|a|^n < |b|^n$ by (11). Which implies $a^n < b^n$. Similarly if $|a| \geq |b|$, then $|a|^n \geq |b|^n$, by (11). Which implies $|a| < |b|$ iff $a^n < b^n$.

(15) If $a < b$, then by (13), $a^n < b^n$. If $a \geq b$, then by (13) we obtain $a^n \geq b^n$. Therefore $a < b$ iff $a^n < b^n$.

(16) By (14), $|a - c| < b$ iff $(a-c)^2 < b^2$. This is equivalent to $(a - c - b)(a - c + b) < 0$. Which is true iff exactly one of $a - c - b$ and $a - c + b$ is positive and the other is negative. Since $a - c - b \leq a - c + b$, this is equivalent to $a - c - b < 0$ and $a - c + b > 0$ which is equivalent to $-b + c < a < c + b$.

(17) By (12), $a^2 > b^2$. By (14), we obtain $|a| > |b|$.

(18) Apply the Positivity Axiom to $a - b$. We conclude precisely one of these is true: $a - b \in \mathcal{P}$ or $a - b = 0$ or $b - a \in \mathcal{P}$. Therefore precisely one of these is true: $a > b$ or $a = b$ or $a < b$.

□

Since the properties of inequalities are more straight forward when numbers are nonnegative, **often times it is convenient to use absolute values when dealing with inequalities**.

Definition 0.9. Given two real numbers $a < b$, the intervals (a, b), $(a, b]$, $[a, b)$, $[a, b]$, (a, ∞), $[a, \infty)$, $(-\infty, a)$, $(-\infty, a]$, and $(-\infty, \infty)$ are defined as below:

- $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$
- $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
- $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$

- $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
- $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$
- $[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$
- $(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}$
- $(-\infty, a] = \{x \in \mathbb{R} \mid x \leq a\}$
- $(-\infty, \infty) = \mathbb{R}$.

0.3 Methods of Proof

0.3.1 Mathematical Induction

Let $P(n)$ be a statement that depends on a natural number n. To prove $P(n)$ we prove $P(1)$ (the basis step) and then we prove that if $P(n)$ for some natural number n, then $P(n+1)$ (the inductive step).

Definition 0.10. A nonempty subset S of $\mathbb{R}$ is said to be **closed under addition** if $a + b \in S$ for every $a, b \in S$. It is called **closed under multiplication** if $ab \in S$ for every $a, b \in S$. We say S is **closed under subtraction** is $a - b \in S$ for all $a, b \in S$.

Theorem 0.10. *$\mathbb{N}$ and $\mathbb{Z}$ are closed under addition, and multiplication.*

Proof. Suppose $m, n \in \mathbb{N}$. We will prove by induction on n that $m + n \in \mathbb{N}$.
Basis step: Since $\mathbb{N}$ is inductive, $m + 1 \in \mathbb{N}$.
Inductive step: Suppose $m + n \in \mathbb{N}$. Since $\mathbb{N}$ is inductive, $(m + n) + 1 \in \mathbb{N}$. Thus, $m + (n + 1) \in \mathbb{N}$.

Therefore, $m + n \in \mathbb{N}$, for all $m, n \in \mathbb{N}$.

Now, we will prove by induction on n that $mn \in \mathbb{N}$.
Basis step: For $n = 1$, we have $mn = m$ which is in $\mathbb{N}$.
Inductive step: Suppose $mn \in \mathbb{N}$. We have $m(n + 1) = mn + m$. By inductive hypothesis $mn \in \mathbb{N}$. By assumption $m \in \mathbb{N}$. By the previous part $mn + m \in \mathbb{N}$. This completes the proof of the claim that $\mathbb{N}$ is closed under multiplication.
The proof that $\mathbb{Z}$ is closed under addition and multiplication is left as an exercise. $\square$

Theorem 0.11. *$\mathbb{Q}$ is closed under addition, subtraction and multiplication. Furthermore, if a, b are rational and $b \neq 0$, then $\dfrac{a}{b}$ is also rational.*

Proof. Suppose $a, b \in \mathbb{Q}$. By definition $a = \dfrac{m}{n}$ and $b = \dfrac{r}{s}$, for some $m, n, r, s \in \mathbb{Z}$, with $n, s \neq 0$. $a + b = \dfrac{m}{n} + \dfrac{r}{s} = \dfrac{sm + rn}{sn}$. Since $\mathbb{Z}$ is closed under multiplication and addition, $sm + rn$ and sn are both integers. By Zero Product Property $sn \neq 0$. Therefore, $a + b \in \mathbb{Q}$. Similarly $a - b, ab \in \mathbb{Q}$ and if $b \neq 0$, then $\dfrac{a}{b} \in \mathbb{Q}$. $\square$

0.3.2 Proof by Contradiction

Often times to prove a number is irrational we use the method of proof by contradiction. The reason is that, being irrational is the same as "not being rational" and showing a negative is often done by contradiction.

Example 0.1. *Let x be an irrational number and a, b, c, d be rational numbers for which c and d are not both zero.*

(i) Prove that $cx + d \neq 0$.

(ii) Prove that $\dfrac{ax+b}{cx+d}$ is rational if and only if $ad - bc = 0$.

Solution. (i) We prove $cx + d \neq 0$ by contradition. Suppose $cx + d = 0$. If $c = 0$, then $d = 0$, which is a contradiction. Otherwise, $x = -\dfrac{d}{c}$, which is rational. This contradiction shows that $cx + d \neq 0$.

(ii) Let $\dfrac{ax+b}{cx+d} = r$. Suppose $r \in \mathbb{Q}$. Multiplying both sides by $cx + d$ we obtain $ax + b = rcx + rd$, thus $x(a - rc) = rd - b$ (*). If $a \neq rc$, then $x = \dfrac{rd-b}{a-rc}$. Therefore, x is rational, which is a contradiction.
If $a = rc$, then $rd = b$ by (*). Multiplying the equalities we obtain $rad = rbc$, therefore $r(ad - bc) = 0$, which implies $r = 0$ or $ad - bc = 0$. The first implies $ax + b = 0$, which implies $a = b = 0$ by a proof similar to (i). Therefore in both cases $ad - bc = 0$.
Now suppose $ad - bc = 0$. By assumption either c or d is nonzero. If $c \neq 0$, then

$$r = \frac{acx+bc}{c(cx+d)} = \frac{acx+ad}{c(cx+d)} = \frac{a(cx+d)}{c(cx+d)} = \frac{a}{c} \in \mathbb{Q}.$$

If $d \neq 0$, then,

$$r = \frac{adx+bd}{d(cx+d)} = \frac{bcx+bd}{d(cx+d)} = \frac{b(cx+d)}{d(cx+d)} = \frac{b}{d} \in \mathbb{Q}.$$

□

Now, you might ask how I knew I had to multiply the numerator and denominator by c or d? Good question! In fact, that is not what I did first! The first and perhaps the most natural way is to solve $ad = bc$ for one of the variables, say a, and substitute into $r = \dfrac{ax+b}{cx+d}$ and show the outcome is rational. But when you do that, you end up multiplying the numerator and denominator by c or d in order to simplify the fraction. But then, why not do this multiplication from the beginning? That makes our solution neater!

0.4 Warm-ups

Example 0.2. *Prove that for every real number x, we have $x < x + 1$.*

Solution. By properties of inequality we know $0 < 1$. Adding x to both sides we obtain $x < x + 1$. □

Example 0.3. *Prove that if x^2 is irrational, then x is also irrational.*

Solution. On the contrary assume x is rational. By Theorem 0.10 since $\mathbb{Q}$ is closed under multiplication, x^2 is also rational, which is a contradiction. This completes the proof. $\square$

Example 0.4. *Prove that there are no integers between 0 and 1. In other words if an integer n is positive, then $n \geq 1$.*

Solution. Assume n is a positive integer. Then, by definition of $\mathbb{Z}$, we have two cases:
Case I. n is a natural number. We will prove by induction that $n \geq 1$. The basis step $1 \geq 1$ is clear. Suppose $n \geq 1$. Then $n+1 \geq 1+1 > 1+0$ and thus $n+1 \geq 1$, as desired.

Case II. n is the additive inverse of a natural number. Since $-n \in \mathbb{N}$, we have $-n \geq 1$. This implies $-n$ is positive, which contradicts the Positivity Axiom, since $n > 0$. $\square$

Example 0.5. *Prove that for all natural numbers n, we have $n < 2^n$.*

Solution. We will prove this by induction.
Basis Step: For $n = 1$, we know $1 < 2^1 = 2$.
Inductive Step: Suppose $n < 2^n$ for some natural number n. Multiplying by 2 we get $2^{n+1} > 2n = n+n \geq n+1$. Therefore, $n+1 < 2^{n+1}$. $\square$

0.5 More Examples

Example 0.6. *Define a sequence a_n recursively by $a_0 = 0.1$, $a_n = \dfrac{1+a_{n-1}}{4}$ for all $n \geq 1$. Prove that there is a real number M for which $a_n < M$ for all $n \geq 0$.*

Scratch: First we write the first few terms of the sequence: $0.1, \dfrac{1.1}{4}, \dfrac{5.1}{16}$. One might or might not see the terms do not get large very rapidly and might guess what M might work. If you are not able to guess, you could try to see what M would allow us to use induction. We are looking for M for which $a_0 \leq M$ and if $a_{n-1} \leq M$, then $a_n \leq M$. We know $a_n = \dfrac{1+a_{n-1}}{4} \leq \dfrac{1+M}{4}$, which we would like to not exceed M. This means, $1+M \leq 4M$ or $\dfrac{1}{3} \leq M$. We will now turn this into a proof.
Solution. We will prove by induction on n that $a_n \leq 1/3$. $a_0 = 0.1 = 1/10 < 1/3$, since $0 < 3 < 10$, which proves the basis step. Assume $a_{n-1} \leq 1/3$ for some $n \geq 1$. Then $a_n = \dfrac{1+a_{n-1}}{4} \leq \dfrac{1+1/3}{4} = \dfrac{1}{3}$. Thus, by induction $a_n \leq 1/3$ for all $n \geq 0$. This completes the proof. $\square$

Example 0.7. *Given the sequence $a_n = \dfrac{n^2-5}{n^4+1}$, prove there is a real number M for which $|a_n| \leq \dfrac{M}{n^2}$ for all $n \in \mathbb{Z}^+$.*

Solution. We construct the factor of $\frac{1}{n^2}$ as follows:

$$\begin{aligned}
|a_n| &= \left|\frac{n^2(1-\frac{5}{n^2})}{n^4(1+\frac{1}{n^4})}\right| \\
&\le \frac{1}{n^2}\cdot\frac{1+\frac{5}{n^2}}{1+\frac{1}{n^4}} && \text{Tirangle Inequality} \\
&\le \frac{1}{n^2}\cdot\frac{1+5}{1} = \frac{6}{n^2} && \text{Larger numerator and smaller denominator}
\end{aligned}$$

Therefore $M = 6$ works. $\square$

Notice that because of the way the problem is stated, M may NOT depend on n. In other words, it must be a constant.

Example 0.8. *Prove that there is a constant C for which $\left|\frac{1+x}{x^2+1} - \frac{1+y}{y^2+1}\right| \le C|x-y|$ for all $x, y \in [-5, 4]$.*

Solution. Assume $x, y \in [-5, 4]$.

$$\begin{aligned}
\left|\frac{1+x}{x^2+1} - \frac{1+y}{y^2+1}\right| &= \left|\frac{x+y^2+xy^2-y-x^2-yx^2}{1+x^2+y^2+x^2y^2}\right| \\
&= \frac{|(x-y)(1-x-y-xy)|}{1+x^2+y^2+x^2y^2} \\
&\le \frac{|x-y|(1+|x|+|y|+|xy|)}{1} && \text{T.I. and minimizing the denominator.} \\
&\le |x-y|(1+5+5+25) && \text{Maximizing all individual terms}
\end{aligned}$$

Thus, the statement is true for $C = 36$. $\square$

Notice that even though $C = 36$ may not be the *least* C that works, the question is asking you to show *one* such C exists. For the purpose of this problem showing $C = 36$ works is as good as showing $C = 10^{10}$ works!

Example 0.9. *Given the sequence $\{a_n = \frac{n+1}{2+n}\}_{n=1}^{\infty}$, determine if this sequence is increasing, decreasing or neither.*

Solution. Note that $a_1 = \frac{2}{3} < a_2 = \frac{3}{4} < a_3 = \frac{4}{5}$. This makes us believe the sequence may be strictly increasing. We would aim to prove $a_n < a_{n+1}$. This is equivalent to

$$\frac{n+1}{2+n} < \frac{n+2}{3+n} \iff (n+1)(3+n) < (2+n)(n+2) \iff n^2+4n+3 < n^2+4n+4 \iff 3 < 4$$

The above steps are all allowed and are reversible since n and thus all numerators and denominators are positive.

Therefore a_n is strictly increasing. $\square$

Note that in this solution, we made sure all steps can be reversed. Here is a wrong solution:

$$\frac{n+1}{2+n} < \frac{n+2}{3+n} \Rightarrow (n+1)(3+n) < (2+n)(n+2) \Rightarrow n^2+4n+3 < n^2+4n+4 \Rightarrow 3 < 4$$

Assuming the conclusion is a cardinal sin of proof-writing!

Example 0.10. *Prove that if $n > k$ are two integers, then $n - k \in \mathbb{N}$.*

Solution. Note that $n - k > 0$, by properties of inequality. By definition of $\mathbb{Z}$, either $n - k$ is a natural number or is zero or its additive inverse is a natural number. Note that since $n - k > 0$, it is not zero. If its additive inverse $k - n$ were a natural number, then . ☐

Example 0.11. *Complete the proof of Theorem 0.10 by proving that $\mathbb{Z}$ is closed under addition and multiplication.*

Solution. Note that if $m = 0$, then $m + n = n$ and $mn = 0$, which are both integers. So, suppose neither m nor n is zero. By symmetry we may assume $m \leq n$. We have three cases.

Case I: $0 < m \leq n$. In this case both m and n are natural numbers and by Theorem 0.10, $m + n$ and mn are both natural numbers.

Case II: $m \leq n < 0$. In this case $m + n = -((-m) + (-n))$, which is an integer since $(-m) + (-n)$ is a natural number. Similarly $mn = (-m)(-n)$ which is a natural number.

Case III: $m < 0 < n$. We have $mn = -((-m)n)$ which is an integer, since $-m$ and n are natural numbers and $\mathbb{N}$ is closed under multiplication.

If $n \geq (-m)$, then by the previous example $n - (-m) = n + m$ is an integer. Otherwise, $n < -m$ are both natural numbers and thus $-m - n$ is a natural number. Therefore, $-(-m - n) = m + n$ is an integer, as desired. ☐

0.6 Exercises

0.6.1 Practice Problems

Example 0.12. *Let n be an integer. Prove that* $(-1)^n = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$

Exercise 0.1. *Find the largest value of $\dfrac{2n+1}{3n+1}$, where n is a positive integer.*

Exercise 0.2. *Prove that every positive integer n can uniquely be written as a sum of distinct powers of 2.*

Exercise 0.3. *Prove that for every integer n, there is an integer k for which $n = 2k$ or $n = 2k + 1$.*

Exercise 0.4. *Prove that an integer n is even if and only if n^2 is even.*

Exercise 0.5. *Prove that there is no rational number x for which $x^2 = 2$. (i.e. $\sqrt{2}$ is irrational.)*

Exercise 0.6. *Prove that for all real numbers $x_1, \ldots, x_n$ we have*

$$|x_1 + \cdots + x_n| \le |x_1| + \cdots + |x_n|.$$

0.6.2 Challenge Problems

Exercise 0.7. *Prove that every positive integer n can be written as $n = a_1 + a_2 + \cdots + a_k$, where each a_i is an integer of form $2^m 3^\ell$, where m, ℓ are nonnegative integers, and for every two $i \neq j$, $\frac{a_i}{a_j}$ is not an integer.*

0.7 Summary

- You should get yourself familiar with properties of inequalities.
- Difference and sum of n-th powers: $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$ for all positive integers n and $x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1})$, for all positive odd integers n.
- The Geometric Series Formula: $\sum_{k=0}^{n-1} ar^k = \frac{a - ar^n}{1-r}$ if $r \neq 1$.
- The Binomial Theorem states that for every two real numbers a and b, and every positive integer n we have $(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}$.
- The Triangle Inequality states that $|a+b| \le |a| + |b|$, for every two real numbers a, b.
- Make sure you are comfortable with proofs by induction and contradiction and properties of inequalities.

Chapter 1

The Completeness Axiom

Definition 1.1. A subset S of $\mathbb{R}$ is said to be **bounded above** if there is a real number r for which $s \leq r$ for all $s \in S$. This number r is called an **upper bound** for S. The least upper bound of S, if exists, is called the **supremum of** S, and is denoted by $\sup S$.

Similarly, we say S is **bounded below** if there is a real number r for which $r \leq s$ for all $s \in S$. The number r is called a **lower bound** of S. The greatest lower bound of S, if exists, is called the **infimum of** S, and is denoted by $\inf S$.

A set that is both bounded above and bounded below is called **bounded**.

The completeness axiom states the following:

> Suppose S is a non-empty subset of $\mathbb{R}$ that is bounded above. Then S has a least upper bound.

Example 1.1. *Find the infimum and supremum of* $(0, 1)$.

Theorem 1.1. *Every non-empty subset of* $\mathbb{R}$ *that is bounded below has a greatest lower bound.*

Remark 1.1. If $S \subseteq \mathbb{R}$ is not bounded above, we write $\sup S = \infty$, and if S is not bounded below, we write $\inf S = -\infty$.

Example 1.2. *Suppose c is a positive real number. Prove that there is a unique positive real number x for which $x^2 = c$.*

Definition 1.2. Let c be a nonnegative number. The **square root of** c, denoted by $\sqrt{c}$, is the unique nonnegative real number x satisfying $x^2 = c$.

Theorem 1.2 (The Archimedean Property)**.** *Suppose ϵ is a positive number and c is an arbitrary number. Then,*

(a) There is a positive integer n for which $c < n$.

(b) There is a positive integer m for which $\frac{1}{m} < \epsilon$.

Theorem 1.3. *For every integer n, there is no integer in the open interval $(n, n+1)$.*

Theorem 1.4. *For every real number c, there is a unique integer in the interval $[c, c+1)$.*

Theorem 1.5. *If S is a nonempty subset of $\mathbb{Z}$ that is bounded above, then it has a maximum.*

Definition 1.3. A subset S of $\mathbb{R}$ is said to be **dense in** $\mathbb{R}$ if for every two real numbers $a < b$, there is $s \in S$ for which $s \in (a, b)$. Similarly, if $S \subseteq T$ are subsets of $\mathbb{R}$, we say S is **dense in** T if for every two real numbers $a < b$ with $a, b \in T$, there is $s \in S$ for which $s \in (a, b)$.

Theorem 1.6. *$\mathbb{Q}$ is a dense subset of $\mathbb{R}$.*

Example 1.3. *Prove that the set of irrational numbers is dense in $\mathbb{R}$.*

1.1 Warm-ups

Example 1.4. *Prove that a set S is bounded if and only if there is a number r for which $|s| \leq r$ for all $s \in S$.*

Solution. First assume S is bounded, and let x and y be upper and lower bounds of S, respectively. Thus, $x \leq s \leq y$ for all $s \in S$. This means

$$-|x| \leq s \leq |y| \Rightarrow -|x| - |y| \leq s \leq |x| + |y| \Rightarrow |s| \leq |x| + |y|,$$

as desired.

Now, assume there is a real number r for which $|s| \leq r$ for all $s \in S$. This implies $-r \leq s \leq r$, and hence S is both bounded below and above. Therefore, S is bounded. $\square$

1.2 Examples

Example 1.5. *Find the supremum and infimum of each set.*

(i) $A = \{x \in \mathbb{R} \mid 1 < \frac{x-1}{x+1} < 2\}$.

(ii) $B = \{n + \frac{(-1)^n}{n} \mid n \in \mathbb{N}\}$.

Solution. (i) Note that $1 < \frac{x-1}{x+1} \iff 0 < \frac{x-1}{x+1} - 1 \iff 0 < \frac{-2}{x+1} \iff x+1 < 0 \iff x < -1$, by properties of inequality.

Furthermore $\frac{x-1}{x+1} < 2 \iff \frac{x-1}{x+1} - 2 < 0 \iff \frac{-x-3}{x+1} < 0 \iff \frac{x+3}{x+1} > 0$. By properties of inequality either both $x+3$ and $x+1$ are positive or they are both negative. If they are both positive, then $x+3 > 0$ and $x+1 > 0$, which implies $x > -1$. If they are both negative, then $x+3 < 0$ and $x+1 < 0$, which implies $x < -3$.

Therefore, $x \in A$ iff $x < -1$ and ($x > -1$ or $x < -3$), which is equivalent to $x < -3$. Therefore, $A = (-\infty, -3)$. This makes it easier to find $\sup A$ and $\inf A$.

First, note that A is not bounded below, since if A were bounded below by c, then $c < -4$ and thus $c-1 < -3$, which means $c - 1 \in A$, however $c > c - 1$, which means c is not a lower bound for A. Thus $\inf A = -\infty$.

Now, we claim that $\sup A = -3$. Note that by what we found above -3 is an upper bound for A. If $c < -3$, then $(c-3)/2 < (-3-3)/2 = -3$ and $(c-3)/2 > (c+c)/2 = c$. Thus $(c-3)/2 \in A$ and c is not an upper bound for A. Therefore -3 is the smallest upper bound of A, which implies $\sup A = -3$.

(ii) Note that for every positive integer n, we have $n + \dfrac{(-1)^n}{n} \geq 1 - \dfrac{1}{n} \geq 1 - 1 = 0$. Thus 0 is a lower bound for B. Since $0 = 1 + \dfrac{(-1)^1}{1} \in B$, if $b > 0$, then b cannot be a lower bound for B. Therefore $\inf B = 0$. On the other hand, if $c \in \mathbb{R}$, then by the Archimedean Property there is a positive integer m for which $c < m$. Thus, $c < m < 2m + \dfrac{1}{2m} = 2m + \dfrac{(-1)^{2m}}{2m}$. This shows c is not an upper bound for B, which implies B is not bounded above, i.e. $\sup B = +\infty$. $\square$

Example 1.6. *Let $S = \{x \in \mathbb{R} \mid x^3 + x < 1\}$. Prove that S is bounded above and show that if $c = \sup S$, then $c + c^3 = 1$. (Note that finding c requires solving the cubic equation $c + c^3 = 1$, which is not easy!)*

Scratch: Before we can write a complete solution we have to find one! What follows is the thought process behind finding a solution: Showing S is nonempty and bounded above is not difficult (but <u>must</u> be written), but showing $c + c^3 = 1$ needs some work, since we cannot explicitly evaluate c. Suppose on the contrary $c + c^3 \neq 1$. By properties of inequality $c + c^3 < 1$ or $c + c^3 > 1$.

Suppose $c + c^3 < 1$. Intuitively, we need to make c just a tiny bit larger and make sure $(c + \epsilon) + (c + \epsilon)^3 < 1$ and use that to get a contradiction. Let's rewrite the inequality as $c + \epsilon + c^3 + 3c\epsilon^2 + 3c^2\epsilon + \epsilon^3 < 1$. But, this is cubic and solving cubic inequalities is difficult. We can turn this into linear if we select $\epsilon < 1$ and make sure that:

$$c + \epsilon + c^3 + 3c\epsilon^2 + 3c^2\epsilon + \epsilon^3 \leq c + \epsilon + c^3 + 3c^2\epsilon + 3c\epsilon + \epsilon < 1.$$

Solving this for ϵ we obtain $\epsilon < (1 - c - c^3)/(2 + 3c^2 + 3c)$. To make sure both inequalities $\epsilon < 1$ and $\epsilon < (1 - c - c^3)/(2 + 3c^2 + 3c)$ are satisfied we select $\epsilon = \min\{1/2, \dfrac{1 - c - c^3}{2(2 + 3c^2 + 3c)}\}$.

Now, it is very tempting to say "similarly we can show $c + c^3 > 1$ is also impossible." BUT, this is NOT very similar, unfortunately! So, let's get to work and find ϵ such that $c - \epsilon + (c - \epsilon)^3 > 1$ and show $c - \epsilon$ is an upper bound of S. Rewriting the inequality we obtain $c - \epsilon + c^3 - 3c^2\epsilon + 3c\epsilon^2 - \epsilon^3 > 1$. Similarly we drop ϵ^2 and ϵ^3 by assuming $\epsilon < 1$ and using the following chain of inequalities

$$c - \epsilon + c^3 - 3c^2\epsilon + 3c\epsilon^2 - \epsilon^3 \geq c - \epsilon + c^3 - 3c^2\epsilon - \epsilon > 1.$$

Solving this we obtain $\epsilon < \dfrac{c + c^3 - 1}{2 + 3c^2}$. Thus we need to make sure both inequalities $\epsilon < 1$ and $\epsilon < \dfrac{c + c^3 - 1}{2 + 3c^2}$

are satisfied. Select $\epsilon = \min\{1/2, \dfrac{c+c^3-1}{2(2+3c^2)}\}$.

Now, we are at a position to write a complete solution!

Solution. Note that $1/2 + (1/2)^3 = 5/8 < 1$, thus $1/2 \in S$. Therefore S is nonempty.

Now, we will show S is bounded above by 1. If $x > 1$, then $x + x^3 > 1 + 1^3 = 2 > 1$, which implies $x \notin S$. Therefore, if $x \in S$, then $x \leq 1$.

By Completeness Axiom, $\sup S = c$ exists. Assume on the contrary $c + c^3 \neq 1$. By properties of inequality $c + c^3 > 1$ or $c + c^3 < 1$.

Assume $c + c^3 < 1$. Let $\epsilon = \min\{1/2, \dfrac{1-c-c^3}{2(2+3c^2+3c)}\}$. Note that $1/2 > 0$, $c > 0$ (since $1/2 \in S$) and $1 - c - c^3 > 0$, which imply $\epsilon > 0$. Definition of ϵ implies that $\epsilon < 1$ and $\epsilon < \dfrac{1-c-c^3}{(2+3c^2+3c)}$, which implies

$$2\epsilon + 3c\epsilon + 3c^2\epsilon < 1 - c - c^3 \Rightarrow c + c^3 + 2\epsilon + 3c\epsilon + 3c^2\epsilon < 1$$

Note that $\epsilon > \epsilon^2 > \epsilon^3$, since $\epsilon < 1$. Therefore we obtain

$$(c+\epsilon) + (c+\epsilon)^3 = c + \epsilon + c^3 + 3c\epsilon^2 + 3c^2\epsilon + \epsilon^3 < c + c^3 + 2\epsilon + 3c\epsilon + 3c^2\epsilon < 1 \Rightarrow (c+\epsilon) + (c+\epsilon)^3 < 1$$

This shows $c + \epsilon \in S$. However c is an upper bound of S and $c + \epsilon > c$, which is a contradiction.

Now, assume $c + c^3 > 1$ and let $\epsilon = \min\{1/2, \dfrac{c+c^3-1}{2(2+3c^2)}\}$. Since $c > 0, 1/2 > 0$ and $c + c^3 > 1$, we have $\epsilon > 0$. Definition of ϵ implies that $\epsilon < 1$ and $\epsilon < \dfrac{c+c^3-1}{(2+3c^2)}$. Therefore, $c + c^3 - 2\epsilon - 3\epsilon c^2 > 1$. Using this along with the fact that $c, \epsilon > 0$ and $\epsilon < 1$, we obtain:

$$(c-\epsilon) + (c-\epsilon)^3 = c - \epsilon + c^3 - 3c^2\epsilon + 3c\epsilon^2 - \epsilon^3 > c - \epsilon + c^3 - 3c^2\epsilon - \epsilon > 1$$

Therefore, $(c-\epsilon) + (c-\epsilon)^3 > 1$. Since c is the smallest upper bound for S, there must be $x \in S$ for which $x > c - \epsilon$. Therefore $x + x^3 > (c-\epsilon) + (c-\epsilon)^3 > 1$, which contradicts $x \in S$.

Therefore, $c + c^3 = 1$. □

Here is a second solution for the same problem. In this solution we will write the proof backwards. In other words we will start with what we need to prove and see what is required to be proved. We will make sure that in each step the reverse can be proved. Note that we are NOT assuming the conclusion. Do NOT use this method if you are not clear on how its logic works.

Second Solution. Similar to the first solution we show that $c = \sup S \leq 1$ exists. Now we will show $c + c^3$ cannot be more or less than 1.

Assume $c + c^3 < 1$. We will find $c < x$ for which $x + x^3 < 1$. For that to hold we need $x + x^3 - (c + c^3) < 1 - (c + c^3)$. Factoring we obtain $(x - c)(1 + c^2 + cx + x^2) < 1 - c - c^3$. Now we know $c < 1$ and we are going to assume that $x < 1$. Thus, it is enough to have the following:

$$(x - c)(1 + c^2 + cx + x^2) < (x - c)(1 + 1 + 1 + 1) < 1 - c - c^3 \Leftrightarrow x - c < \frac{1 - c - c^3}{4}.$$

Thus, if we select $x = c + \dfrac{1 - c - c^3}{5} = \dfrac{1 + 4c - c^3}{5}$, then $c < x < \dfrac{1 + 4}{5} = 1$. Thus all of the above inequalities are satisfied. Therefore $x \in S$, which contradicts the fact that $x > c$ and c is an upper bound for S.

Assume $c + c^3 > 1$. We will find $x < c$ for which $x + x^3 > 1$. For that we need $c + c^3 - (x + x^3) < c + c^3 - 1$, which is equivalent to $(c - x)(1 + c^2 + cx + x^2) < c + c^3 - 1$. Since $c < 1$ it is enough to have

$$(c - x)(1 + c^2 + cx + x^2) < (c - x)(1 + 1 + 1 + 1) < c + c^3 - 1 \Leftrightarrow x > c - \frac{c + c^3 - 1}{4}$$

as long as we make sure $0 < x$. Take $x = c - \dfrac{c + c^3 - 1}{5}$. Note that $x < c$ and that $x = \dfrac{4c - c^3 + 1}{5} = \dfrac{c(4 - c^2) + 1}{5} > 0$. Thus for this x, all of the above inequalities hold. Since $c = \sup S$, there is $y \in S$ for which $y > x$. This implies $y + y^3 > x + x^3 > 1$, which is a contradiction. $\square$

Example 1.7. *What should we define* $\sup \emptyset$ *and* $\inf \emptyset$*?*

Solution. Note that every real number c is an upper bound for $\emptyset$. That is because the statement "$x \in \emptyset \Rightarrow x < c$" is true by default. Therefore there is no smallest upper bound for $\emptyset$. The best definition would be $\sup \emptyset = -\infty$.

Similarly every real number is a lower bound for $\emptyset$. Thus, there is no largest lower bound for $\emptyset$. The best definition would be $\inf \emptyset = \infty$. $\square$

Note that in this case strangely enough, $\sup \emptyset$ is *less* than $\inf \emptyset$!

Example 1.8. *Suppose A is a nonempty subset of $\mathbb{R}$. Show that if A is bounded above, then its complement A^c is not bounded above.*

Solution. Let c be an upper bound for A. On the contrary assume A^c is bounded above by a. Therefore $a \geq x$ for all $x \in A^c$. Therefore $a + n \in A$ for all $n \in \mathbb{N}$. By Archimedean Property, there is $n \in \mathbb{N}$ for which $n > c - a$. Thus, $a + n > c$, which contradicts the fact that c is an upper bound of A and $a + n \in A$. $\square$

Example 1.9. *Suppose $A \subseteq B$ are nonempty subsets of real numbers that are bounded above. Prove that* $\sup A \leq \sup B$.

Solution. Let $a = \sup A$ and $b = \sup B$. Note that $b \geq x$ for all $x \in B$. Therefore $b \geq x$ for all $x \in A$, since $A \subseteq B$. This implies b is an upper bound for A. Since a is the least upper bound for A, we conclude $b \geq a$, as desired! $\square$

Example 1.10. *Let a be a positive real number and n be a natural number. Prove that there is a unique positive real number c for which $c^n = a$. (This unique c is denoted by $\sqrt[n]{a}$ or $a^{1/n}$.)*

Solution. Consider the set $S = \{x \in \mathbb{R} \mid x^n \leq a\}$. Note that $0 \in S$ and that if $x \geq 1 + a$, then $x^n \geq (1+a)^n \geq 1+na > a$, which implies $x \notin S$. Thus, if $x \in S$, then $x \leq 1+a$. This implies S is nonempty and bounded above by $1 + a$. Therefore, by the Completeness Axiom, $\sup S = c$ exists. Now we will use a similar argument to what we did above to show $c^n = a$.

Suppose $c^n < a$. We will find x such that $c^n < x^n < a$. This is equivalent to $0 < x^n - c^n < a - c^n$. Using the difference of n-th powers we obtain

$$(x-c)(x^{n-1} + x^{n-2}c + \cdots + xc^{n-2} + c^{n-1}) < (x-c)(n(1+a)^{n-1}),$$

since $x^n < a < (1+a)^n$ which implies $x < 1+a$. We also know $c \leq 1+a$ since $1+a$ is an upper bound of S and $c = \sup S$. Thus, it is enough to have $(x-c) < \dfrac{a-c^n}{n(1+a)^{n-1}}$. If we select $x = c + \dfrac{a-c^n}{2n(1+a)^{n-1}}$, then $x > c$ and by what we saw above $x^n - c^n < a - c^n$. Thus $c^n < x^n < a$, which implies $x \in S$ and $c < x$, which is a contradiction.

Suppose $c^n > a$. We will have to find a positive real number x for which $c^n > x^n > a$. Which is equivalent to $c > x$ and $x^n - c^n > a - c^n$. Using the difference of n-th powers we obtain

$$(x-c)(x^{n-1} + x^{n-2}c + \cdots + xc^{n-2} + c^{n-1}) > a - c^n \Leftrightarrow (c-x)(x^{n-1} + x^{n-2}c + \cdots + xc^{n-2} + c^{n-1}) < c^n - a$$

Note that since $x < c$, $(x^{n-1} + \cdots + c^{n-1}) < nc^{n-1}$. Therefore it is enough to have $(c-x)(nc^{n-1}) < c^n - a$, which means $c - x < \dfrac{c^n - a}{nc^{n-1}}$, which implies $c - \dfrac{c^n - a}{nc^{n-1}} < x$. If we select $x = c - \dfrac{c^n - a}{2nc^{n-1}} = \dfrac{(2n-1)c^n + a}{2nc^{n-1}}$, we will have $0 < x < c$ and that $a < x^n$. If $y \in S$, then $y^n < a < x^n$, which implies $y < x$, which shows x is an upper bound for S, which contradicts the fact that $c = \sup S$.

Therefore $c^n = a$. This completes the existence of such c.

Suppose $c^n = d^n = a$, where c, d are both positive. Using the difference of n-th powers identity we obtain $(c-d)(c^{n-1} + \cdots + d^{n-1}) = 0$. Since the second parenthesis is positive, we obtain $c - d = 0$, thus $c = d$, which completes the proof of uniqueness. □

Remark 1.2. This problem will be used in the future.

Example 1.11. *Prove that $\{r^2 \mid r \in \mathbb{Q}\}$ is dense in $[0, \infty)$.*

Solution. Let $S = \{r^2 \mid r \in \mathbb{Q}\}$. Suppose $0 \leq a < b$. Since $\mathbb{Q}$ is rational, there is a rational number r for which $\sqrt{a} < r < \sqrt{b}$. This implies $a < r^2 < b$, which proves the claim. □

Example 1.12. *Suppose a, b are two real numbers for which $a \leq b + \frac{1}{n}$ for all positive integers n. Prove that $a \leq b$.*

Solution. Suppose on the contrary $a > b$. Hence, $a - b > 0$. By the Archimedean Property, there is a positive integer n for which $a - b > \frac{1}{n}$. Therefore, $a > b + \frac{1}{n}$, which contradicts the assumption. Thus, $a \leq b$. □

1.3 Exercises

All students are expected to do all of the exercises listed in the following two sections: Problems for Grading and Problems for Practice. You are only required to submit the ones in the first section for grading.

Challenge Problems are optional.

1.3.1 Problems for Grading

Instructions for submission: To submit your solutions please note the following:

- To submit your homework, go to Elms. Hit "Gradescope" on the left panel. That should allow you to upload a PDF file of your homework.
- Each problem must go on a separate page. **Make sure you assign each page to the appropriate problem number.**
- It is highly recommended (but not required) that you LaTeX your homework.
- If you are not typing your work (which is fine) please make sure your work is legible. You could use the (free) DocScan app to scan and upload your homework.
- Sometime in the next day or two run a test and make sure this all works out so you do not face any issues right before the deadline.
- Homework must be submitted before the class starts on the due date. GradeScope will not allow late submissions.
- You can read more about submitting homework on Gradescope here.
- **All proofs must be complete and solutions must be fully justified.**
- Read and follow the directions carefully.
- Numbered problems are from the textbook.

All submissions must be made on the due dates before the class starts.

The following are due Thursday 6/3/2021.

Exercise 1.1 (15 pts)**.** *Page 10, Problem 2.*

Exercise 1.2 (10 pts)**.** *Page 11, Problem 13.*

Exercise 1.3 (10 pts)**.** *Prove that there is a positive real number r for which $r^3 + 2r^2 = 7$.*

The following are due Friday 6/4/2021.

Exercise 1.4 (10 pts)**.** *Page 16, Problem 6.*

Exercise 1.5 (10 pts)**.** *Page 16, Problem 8.*

Exercise 1.6 (10 pts)**.** *Page 22, Problem 27.*

Exercise 1.7 (10 pts)**.** *Prove the Bernoulli's Inequality: If $b \geq -1$ is a real number, and n is a natural number, then $(b+1)^n \geq 1 + nb$.*

1.3.2 Practice Problems

Exercise 1.8. *Find the supremum and infimum of each set.*

(a) $R = \{2^n + 2^{-n} \mid n \in \mathbb{N}\}$.

(b) $S = \{2^{-n} + 3^{-m} \mid n, m \in \mathbb{N}\}$.

(c) $T = \{x \in \mathbb{R} \mid 3x^2 - 10x + 3 < 0\}$.

(d) $U = \{x \in \mathbb{R} \mid (x-a)(x-b)(x-c) < 0\}$, *where* $a < b < c$ *are constants.*

Exercise 1.9. *Let n be a positive integer and $a_1, \dots a_n$ be real numbers. Prove that there is a real number x for which $x^n > a_1 + a_2 x + \cdots + a_n x^{n-1}$.*

Exercise 1.10. *Assume $A_1, \dots, A_n$ are nonempty subsets of $\mathbb{R}$. Show that* $\sup(\bigcup_{k=1}^{n} A_k) = \max(\sup A_1, \dots, \sup A_n)$. *Prove a similar result for infimum.*

Exercise 1.11. *Suppose $A, B \subseteq \mathbb{R}$ are bounded and nonempty. Define the set $A + B$ as below:*

$$A + B = \{a + b \mid a \in A, \text{ and } b \in B\}.$$

Prove that $\sup(A+B) = \sup A + \sup B$ *and* $\inf(A+B) = \inf A + \inf B$.

Exercise 1.12. *For a nonempty bounded subset A of real numbers let $B = \{|x| \mid x \in A\}$. Prove that* $\sup B = \max(|\inf A|, |\sup A|)$. *Is it true that* $\inf B = \min(|\inf A|, |\sup A|)$*?*

Pages 10-12: 1, 4, 14, 17, 18, 19, 20.

Page 16: 2, 7, 9.

Pages 19-22: 7, 10, 16, 18, 23, 25.

For the following problem you may assume there are infinitely many primes.

Exercise 1.13. *Prove that each of the following sets is dense in* $[0, \infty)$.

(a) $\left\{\dfrac{a^2}{p^2} \mid a \in \mathbb{Z},\ \textit{and } p \textit{ is prime}\right\}$.

(b) $\left\{\dfrac{a^2}{p} \mid a \in \mathbb{Z},\ \textit{and } p \textit{ is prime}\right\}$.

1.3.3 Challenge Problems

For $\alpha \in \mathbb{R}$ and $A, B \subseteq \mathbb{R}$ define $\alpha A = \{\alpha a \mid a \in A\}$, $A + \alpha = \{a + \alpha \mid a \in A\}$, $A \pm B = \{a \pm b \mid a \in A, b \in B\}$ and $AB = \{ab \mid a \in A, b \in B\}$.

Exercise 1.14 (Base k representation)**.** *Let x be a real number and $k > 1$ be an integer. Define a_n recursively as follows:*

- *a_0 is the largest integer satisfying $a_0 \leq x$.*
- *a_n is the largest integer satisfying $a_0 + \dfrac{a_1}{k} + \cdots + \dfrac{a_n}{k^n} \leq x$.*

Prove that

(i) $0 \leq a_n \leq k - 1$ *for all* $n > 0$

(ii) *x is the supremum of* $\{a_0 + \dfrac{a_1}{k} + \cdots + \dfrac{a_n}{k^n} \mid n \in \mathbb{N}\}$.

Exercise 1.15. *Let n be a positive integer, and let $a_1, \ldots a_n, r$ be real numbers for which a_n and r are positive. Prove that, there is a positive real number c for which $a_1 c + \cdots + a_n c^n = r$.*

Exercise 1.16. *Suppose A is an uncountable subset of $[0, \infty)$. Show that the set*

$$\{\sum_{i=1}^{n} a_i \mid a_1, \ldots, a_n \textit{ are distinct elements of } A \textit{ and } n \in \mathbb{N}\}$$

is not bounded above.

Exercise 1.17. *Suppose A and B are bounded nonempty subsets of $\mathbb{R}$. We investigate the relation between* sup *and* inf *of AB and* sup *and* inf *of A and B. Prove that:*

- *If $A, B \subseteq [0, \infty)$, then* $\sup(AB) = (\sup A) \cdot (\sup B)$ *and* $\inf(AB) = (\inf A) \cdot (\inf B)$.
- *If $A, B \subseteq (-\infty, 0]$, then* $\sup(AB) = (\inf A) \cdot (\inf B)$ *and* $\inf(AB) = (\sup A) \cdot (\sup B)$.
- *If $A \subseteq [0, \infty)$ and $B \subseteq (-\infty, 0]$, then* $\sup(AB) = (\sup B) \cdot (\inf A)$ *and* $\inf(AB) = (\inf B) \cdot (\sup A)$.

- *Prove that* $\sup(AB) = \max(\{\sup A,\ \inf A\} \cdot \{\sup B,\ \inf B\})$ *and* $\inf(AB) = \min(\{\sup A,\ \inf A\} \cdot \{\sup B,\ \inf B\})$.

Exercise 1.18. *Let $x_1, \ldots, x_n$ be nonnegative real numbers and let* $A = \frac{\sum_{k=1}^{n} x_k}{n}$ *and* $G = (\prod_{k=1}^{n} x_k)^{1/n}$. *(A is called the Arithmetic Mean and G is called the Geometric Mean of $x_1, \ldots, x_n$.) By following the steps below prove that $A \geq G$ and that $A = G$ iff $x_1 = \cdots = x_n$. (This inequality is called the Arithmetic Mean-Geometric Mean Inequality.)*

- *Prove that $A \geq G$ for $n = 2$ and that equality occurs iff $x_1 = x_2$.*
- *Assume $n = 2^k$ for some integer k. Using induction prove that $A \geq G$ and that $A = G$ iff $x_1 = \cdots = x_n$.*
- *For arbitrary n, select k for which $n < 2^k$ and let $x_{n+1} = \cdots x_{2^k} = A$. Then use the AM-GM inequality for $x_1, \ldots, x_{2^k}$. Simplify to get the AM-GM for $x_1, \ldots, x_n$ and show that equality occurs iff $x_1 = \cdots = x_n$.*

1.3.4 Summary

- The smallest upper bound of a set S is called its supremum and is denoted by $\sup S$.
- To show c is the Supremum of a set S (i.e. $c = \sup S$):
  - (i) Show that $s \leq c$ for all $s \in S$.
  - (ii) Show that if $d < c$, then there is some $x \in S$ for which $d < x$.
- The Completeness Axiom asserts that every nonempty set of real numbers S that is bounded above has the smallest upper bound.
- To use the Completeness Axiom for a set S you must show:
  - (i) $S \neq \emptyset$, and
  - (ii) S is bounded above.
- The Archimedean Property states that for every $\epsilon > 0$ and every real number c, there are natural numbers n, m for which $\frac{1}{n} < \epsilon$ and $c < m$.
- Let $S \subseteq T \subseteq \mathbb{R}$ be nonempty sets. We say S is dense in T if for every $a, b \in T$ with $a < b$, there is an element $s \in S$ for which $a < s < b$.
- $\mathbb{Q}$, $\mathbb{Q}^c$ and the set of all dyadic rationals are dense in $\mathbb{R}$.

Chapter 2

Convergent Sequences

Definition 2.1. A **sequence** is a list of numbers indexed by positive integers as follows:

$$a_1, a_2, \dots .$$

Each of the numbers in this list is called a **term**. The n-th term of this sequence is denoted by a_n.

Example 2.1. *Here are some examples of sequences:*

(a) The sequence of positive odd integers: $1, 3, 5, \dots$. *The n-th term is given by* $2n - 1$.

(b) Any geometric sequence $a, ar, ar^2, \dots$ *is a sequence whose n-th term is given by* ar^{n-1}.

(c) Any arithmetic sequence $a, a+d, a+2d, \dots$ *is a sequence whose n-th term is given by* $a + (n-1)d$.

(d) The sequence of partial sums of a sequence a_n*:* $a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots$.

Definition 2.2. We say a sequence a_n **converges** to a number a, written as $\lim\limits_{n\to\infty} a_n = a$ or $a_n \to a$, if the following holds:

$$\forall\ \epsilon > 0\ \exists\ N \in \mathbb{N} \text{ such that } \forall\ n \in \mathbb{N},\ n \geq N \Rightarrow |a_n - a| < \epsilon.$$

If no such real number a exists we say a_n **diverges**.

Definition 2.3. We say a sequence a_n **diverges to** ∞, written as $\lim\limits_{n\to\infty} a_n = \infty$ or $a_n \to \infty$, if the following holds:

$$\forall\ M > 0\ \exists\ N \in \mathbb{N} \text{ such that } \forall\ n \in \mathbb{N},\ n \geq N \Rightarrow a_n > M.$$

Similarly we say a_n **diverges to** $-\infty$, written as $\lim\limits_{n\to\infty} a_n = -\infty$ or $a_n \to -\infty$, if the following holds:

$$\forall\ M < 0\ \exists\ N \in \mathbb{N} \text{ such that } \forall\ n \in \mathbb{N},\ n \geq N \Rightarrow a_n < M.$$

Example 2.2. *Using the definition of limit, find the limit of each the following sequences:*

(a) $\dfrac{1}{n}$.

(b) $\dfrac{n^2-2n}{n^2+1}$

(c) n^2-3n.

Example 2.3. *Prove that the sequence* $(-1)^n$ *does not converge.*

Definition 2.4. A sequence is said to be **bounded** if the set consisting of all of its terms is a bounded set.

Theorem 2.1. *Every convergent sequence is bounded.*

Lemma 2.1 (The Comparison Lemma)**.** *Suppose a_n, b_n are two sequences, a, b, c are three real numbers, and N is a natural number for which*

$$\forall\ n \geq N, |a_n - a| \leq c|b_n - b|.$$

If $\lim_{n\to\infty} b_n = b$*, then* $\lim_{n\to\infty} a_n = a$.

Note that by the Comparison Lemma if two sequences differ by only finitely many terms, their limits are the same. Therefore, in finding limits we can ignore the first few terms of the sequence.

Theorem 2.2 (Properties of Limits)**.** *Suppose a_n, b_n are two sequences that converge to a, b, respectively, and α, β are two real numbers. Then:*

(a) (Linearity of Limit) $\lim_{n\to\infty}(\alpha a_n + \beta b_n) = \alpha a + \beta b$.

(b) $\lim_{n\to\infty} a_n b_n = ab$.

(c) If $b_n \neq 0$ for all n and $b \neq 0$, then $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$.

Definition 2.5. A **polynomial** is a function $p : \mathbb{R} \to \mathbb{R}$ given by

$$p(x) = c_k x^k + \cdots + c_1 x + c_0,$$

where $c_0, c_1, \ldots, c_k$ are all constants. The polynomial with all c_j's zero is called the **zero polynomial**. If $c_k \neq 0$ we say the **degree** of p is k, and we write $\deg p = k$. The numbers $c_0, \ldots, c_k$ are called **coefficients** of p. The number c_k is called the **leading coefficient** of p.

Theorem 2.3. *Let $p(x)$ be a polynomial with real coefficient and assume a_n is a sequence that converges to a. Then, $p(a_n)$ converges to $p(a)$.*

Definition 2.6. We say a sequence a_n is **in the set** S if each term of the sequence a_n is in S.

Definition 2.7. A subset S of $\mathbb{R}$ is said to be **sequentially dense** if every real number is the limit of a sequence in S.

Theorem 2.4. *A subset of $\mathbb{R}$ is dense if and only if it is sequentially dense.*

Example 2.4. *Every real number is the limit of a sequence of rationals. Also, every real number is the limit of a sequence of irrationals.*

Definition 2.8. A subset S of $\mathbb{R}$ is said to be **closed** if the limit of every convergent sequence in S belongs to S.

Theorem 2.5. *Suppose a_n is a sequence that converges to a. Let b be a number and N be a positive integer for which $a_n \geq b$ for all $n \geq N$. Then $a \geq b$. Similarly if $a_n \leq b$ for all $n \geq N$, then $a \leq b$.*

Theorem 2.6. *Every interval of the form $[a, b]$ is a closed subset of $\mathbb{R}$.*

Example 2.5. *$\mathbb{Q}$ is not closed.*

Solution. Choose an irrational number such as $\sqrt{2}$. Since $\mathbb{Q}$ is dense, it is sequentially dense. Therefore, there is a sequence of rational numbers that approaches $\sqrt{2}$. However, since $\sqrt{2}$ is not rational, $\mathbb{Q}$ is not closed, by definition. $\square$

Example 2.6. *Every finite subset of $\mathbb{R}$ is closed.*

Solution. Suppose $A = \{x_1, \ldots, x_n\}$ is a set of n distinct real numbers. Assume a_n is a sequence in A that converges to number a. Let ϵ be the minimum of all $|x_i - x_j|/2$ with $i \neq j$. There is $N \in \mathbb{N}$ for which $|a_n - a| < \epsilon$ for all $n \geq N$. Using the Triangle Inequality we obtain:

$$|a_n - a| + |a_N - a| = |a_n - a| + |a - a_N| \leq |a_n - a_N| < \epsilon.$$

However, ϵ is smaller than the distance between each two elements of A. Thus, $a_n = a_N$. Therefore, the sequence a_n is constant after some point and thus its limit must be a_N, which is an element of A. $\square$

2.1 Warm-ups

Example 2.7. *Let c be a constant. Using the definition of limit, prove that $\lim_{n\to\infty} c = c$.*

Solution. For every $\epsilon > 0$ we have $|c - c| = 0 < \epsilon$. Therefore, $\lim_{n\to\infty} c = c$. $\square$

Example 2.8. *Prove that if a_n and b_n are bounded sequences, then the sequences $a_n + b_n$ and $a_n b_n$ are both bounded, but the sequence $\dfrac{a_n}{b_n}$ may not be bounded, even if $b_n \neq 0$ for all n.*

Solution. By definition, there are real numbers M_1, M_2 for which

$$|a_n| \leq M_1, \text{ and } |b_n| \leq M_2.$$

By the triangle inequality and properties of inequalities we have

$$|a_n + b_n| \leq |a_n| + |b_n| \leq M_1 + M_2, \text{ and } |a_n b_n| \leq M_1 M_2.$$

Therefore, $a_n + b_n$ and $a_n b_n$ are bounded.

Let $a_n = 1$ and $b_n = \frac{1}{n}$. Both sequences are bounded below by zero and bounded above by 1. We have $\frac{a_n}{b_n} = n$ is unbounded by the Archimedean property. $\square$

2.2 More Examples

Example 2.9. *Let a_n be a sequence of real numbers. Show that if $|a_n|$ converges to 0, then so does a_n.*

Solution. Let $\epsilon > 0$, by definition of limit, there is a natural number N for which $||a_n| - 0| < \epsilon$ for all $n \geq N$. Thus $|a_n| < \epsilon$, which implies $|a_n - 0| < \epsilon$. Therefore, a_n converges to 0. $\square$

Example 2.10. *Using the definition of limit, find the limit of each sequence.*

(a) $a_n = \frac{2n+1}{n-1} + \frac{1}{n}$.

(b) $b_n = n^2 - 5n$.

(c) $c_n = \sqrt{n^2 + n} - n$

Solution. (a) First, note that for large n, $\frac{2n+1}{n-1} + \frac{1}{n} \approx 2 + 0$, so we suspect the limit is 2. We need to show for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $|\frac{2n+1}{n-1} + \frac{1}{n} - 2| < \epsilon$. Simplifying the last inequality we obtain $|\frac{3}{n-1} + \frac{1}{n}| < \epsilon$. Note that $\frac{3}{n-1} + \frac{1}{n} < \frac{3}{n-1} + \frac{1}{n-1} = \frac{4}{n-1} \leq \frac{4}{N-1}$. Therefore it is enough to select N in such a way that $\frac{4}{N-1} < \epsilon$. Solving this we get $\frac{4}{\epsilon} + 1 < N$. This can be done by the Archimedean Property. [*If you include all of the above explanations clearly and completely this could be the end of your proof, however, I would strongly recommend that after you obtain the appropriate N, rewrite the proof in the correct manner.*]
Here is the solution: Let $\epsilon > 0$. By the Archimedean Property, there is $N \in \mathbb{N}$ for which $N > 1 + \frac{4}{\epsilon}$. Therefore $N - 1 > \frac{4}{\epsilon}$, which implies $\frac{4}{N-1} < \epsilon$. If $n \geq N$, then

$$|\frac{2n+1}{n-1} + \frac{1}{n} - 2| = \frac{3}{n-1} + \frac{1}{n} < \frac{4}{n-1} \leq \frac{4}{N-1} < \epsilon$$

Therefore, by the definition of limit, $\lim_{n\to\infty} a_n = 2$.

(b) Similar to part (a), we will estimate b_n for large n: $n^2 - 5n \approx n^2$, which diverges to infinity. We prove $\lim_{n\to\infty} b_n = \infty$ using the definition of limit. Let $M > 0$. We would like to make sure $b_n > M$ for all $n \geq N$. Note that if we make sure $N \geq 6$, then $n^2 - 5n = n(n-5) \geq N(N-5) \geq N$ and then we can make sure $N \geq M$. That leads to the following solution:

Let $M > 0$. By the Archimedean Property, there is $N \in \mathbb{N}$ for which $N > \max(M, 5)$. If $n \geq N$, then $n^2 - 5n = n(n-5) \geq N(N-5) \geq N > M$. Thus by definition $\lim_{n\to\infty} b_n = \infty$.

(c) To be able to estimate c_n we multiply by the conjugate to obtain

$$\sqrt{n^2+n}-n=\frac{n^2+n-n^2}{\sqrt{n^2+n}+n}=\frac{1}{\sqrt{1+1/n}+1}\approx 1/2.$$

Thus, we guess that the limit is 1/2.

$$|\sqrt{n^2+n}-n-1/2|=|\frac{n^2+n-(n+1/2)^2}{\sqrt{n^2+n}+n+1/2}|=\frac{1/4}{\sqrt{n^2+n}+n+1/2}<\frac{1/4}{\sqrt{n^2}}<\frac{1}{n}\le\frac{1}{N}$$

Thus it is enough to use the Archimedean Property and take $N>1/\epsilon$. Then you would have to write the proof in the correct direction. □

Example 2.11. *Suppose a_n converges to a, where $a\neq 0$. Prove that $(-1)^n a_n$ does not converge.*

Solution. On the contrary assume $(-1)^n a_n$ converges to b. If $b\neq a$, then by taking $\epsilon=|b-a|/2$ one can select $N_1,N_2\in\mathbb{N}$ for which $|a_n-a|<\epsilon$ for all $n\ge N_1$ and $|(-1)^n a_n-b|<\epsilon$ for all $n\ge N_2$. Let n be an even integer more than $\max(N_1,N_2)$. Thus

$$\epsilon>|(-1)^n a_n-b|=|a_n-b|\ge|a-b|-|a-a_n|>|a-b|-\epsilon.$$

Therefore $2\epsilon>|a-b|$, which is a contradiction. This contradiction shows that $a=b$. Applying a similar argument and selecting n to be odd, one can show that $b=-a$ as follows. Assume $b\neq -a$ and take $\epsilon=|b+a|/2$. If n is odd and $n>\max(N_1,N_2)$, then

$$\epsilon>|(-1)^n a_n-b|=|-a_n-b|=|a_n+b|\ge|a+b|-|a-a_n|>|a+b|-\epsilon.$$

This implies $2\epsilon>|a+b|$, which is a contradiction. Thus, $b=-a$. Therefore, $a=-a$, which implies $a=0$, a contradiction. □

Example 2.12. *Let $a\ge b>0$ be two real numbers. Prove that*

(a) $\lim_{n\to\infty} a^{1/n}=1$

(b) $\lim_{n\to\infty}(a^n+b^n)^{1/n}=a$

Solution. (a) First suppose $a\ge 1$. Using the difference of n-th powers identity, we can write

$$|a^{1/n}-1|=\frac{|a-1|}{a^{(n-1)/n}+a^{(n-2)/n}+\cdots+a^{1/n}+1}\le\frac{|a-1|}{n}=|a-1|\cdot|\frac{1}{n}-0|,$$

since there are n terms in the denominator that each is at least 1. Note that $1/n$ approaches zero as n approaches infinity. Therefore, by the Comparison Lemma, $\lim_{n\to\infty} a^{1/n}=1$.

If $a<1$ then by properties of limit we have $\lim_{n\to\infty} a^{1/n}=(\lim_{n\to\infty}(1/a)^{1/n})^{-1}=1^{-1}=1$.

(b) Note that

$$|(a^n+b^n)^{1/n}-a|=a|(1+(b/a)^n)^{1/n}-1|\le a|2^{1/n}-1|$$

By part (a), $\lim_{n\to\infty} 2^{1/n} = 1$. Thus, by Comparison Lemma, $\lim_{n\to\infty}(a^n + b^n)^{1/n} = a$. □

Example 2.13. *Suppose a_n is a sequence that diverges to ∞ ($-\infty$, resp.) Prove that every subsequence of a_n converges to ∞ ($-\infty$, resp.)*

Solution. Suppose $a_n \to \infty$.

Let a_{n_k} be a subsequence of a_n. Let M be a positive real number. By definition of limits, there is $N \in \mathbb{N}$ for which for every $k \geq N$ we have $a_k > M$. Since $n_1 < n_1 < \cdots < n_k$ are positive integers, we have $n_k \geq k$. If $k \geq N$ then $n_k \geq N$. Therefore, $a_{n_k} > M$, and thus $\lim_{k\to\infty} a_{n_k} = \infty$. □

Example 2.14. *Prove that the only subset of $\mathbb{R}$ that is both dense and closed is $\mathbb{R}$.*

Solution. Let S be a subset of $\mathbb{R}$ and let r be a real number. Since S is dense, it is sequentially dense. Therefore, there is a sequence in S whose limit is r. Since S is closed, r must be in S. Therefore, every real number is in S and thus $S = \mathbb{R}$. □

Example 2.15. *Suppose a_n is a bounded sequence, and b_n diverges to ∞. Prove that $a_n + b_n$ diverges to ∞.*

Solution. By definition there is a real number M for which $|a_n| \leq M$ for all $n \in \mathbb{N}$. Let A be a positive real number. By definition, there is $N \in \mathbb{N}$ for which $a_n + b_n > A + M$ for all $n \geq N$. Therefore,

$$b_n > A + M - a_n \geq A + M - |a_n| \geq A \Rightarrow b_n > A.$$

This means $b_n \to \infty$, as desired. □

Example 2.16. *Suppose a_n converges to a real number $a \in (-1, 1)$. Prove that a_n^n converges to zero.*

Since a_n approaches a and a is in the interval $(-1, 1)$, a_n would also lie in the interval $(-1, 1)$ after some point. This means

$$a_n \approx a^n \to 0.$$

To make this rigorous we need to make sure we are avoiding -1 and 1.

Solution. Let $\epsilon = \dfrac{1-|a|}{2}$ in the definition of limit. We obtain conclude that there is $N \in \mathbb{N}$ for which

$$\forall\ n \geq N\ |a_n - a| < \frac{1-|a|}{2} \Rightarrow |a_n| \leq |a_n - a| + |a| < \frac{1-|a|}{2} + |a| = \frac{1+|a|}{2}.$$

Therefore,

$$|a_n^n| < \left(\frac{1+|a|}{2}\right)^n.$$

Since $|a| < 1$ we have $\dfrac{1+|a|}{2} < 1$. Therefore, $\left(\dfrac{1+|a|}{2}\right)^n \to 0$. By Comparison Lemma $a_n^n \to 0$. □

2.3 Exercises

All students are expected to do all of the exercises listed in the following two sections: Problems for Grading and Problems for Practice. You are only required to submit the ones in the first section for grading.

Challenge Problems are optional.

2.3.1 Problems for Grading

For submission please follow the same instructions as before.

All submissions must be made on the due dates before the class starts.

The following are due Monday 6/7/2021.

Exercise 2.1 (20 pts)**.** *Page 32, Problem 1.*

Exercise 2.2 (10 pts)**.** *Page 33, Problem 6.*

Exercise 2.3 (10 pts)**.** *Page 34, Problem 13.*

Exercise 2.4 (15 pts)**.** *Using the definition of limit determine each limit or show it does not exist.*

(a) $\lim_{n\to\infty} \frac{n^2-1}{2n+1}$.

(b) $\lim_{n\to\infty} \sqrt{n+1}-\sqrt{n}$.

(c) $\lim_{n\to\infty} \frac{(-2)^n}{n}$.

Exercise 2.5 (25 pts)**.** *Page 37, Problem 1.*

2.3.2 Practice Problems

Exercise 2.6. *Suppose a_n is a sequence such that $a_n^2 \to 0$. Prove that $a_n \to 0$.*

Exercise 2.7. *Suppose a_n, b_n are two sequences for which a_n converges but b_n diverges. Prove that $a_n + b_n$ diverges.*

Exercise 2.8. *Suppose a_n is a sequence with the following property:*

If b_n diverges, then so does $a_n + b_n$.

Prove that a_n must be a convergent sequence.

Pages 33-35: 7, 9, 10, 14, 15, 16, 17.

Page 37: 2, 4.

2.3.3 Challenge Problems

Exercise 2.9. *Let m be a positive integer and $c_0, c_1, \ldots, c_m$ be real constants and let*

$$a_n = c_0 + c_1 n + \cdots + c_m n^m.$$

Prove that

(a) a_n diverges to ∞ if $c_m > 0$.

(b) a_n diverges to $-\infty$ if $c_m < 0$.

Exercise 2.10. *Define a sequence a_n by $a_1 = a_2 = 1$ and $a_{n+1} = \sqrt{1 + a_n a_{n-1}}$ for all $n \geq 2$. Prove that a_n is unbounded.*

Exercise 2.11. *Suppose a_n is a sequence with the following property:*

If b_n diverges, then so does $a_n b_n$.

Prove that a_n must be a convergent sequence.

2.4 Summary

- To prove a_n converges to a, we need to show that for every $\epsilon > 0$, there is $N \in \mathbb{N}$ for which $|a_n - a| < \epsilon$ for all integers $n \geq N$.
- To prove a_n diverges to ∞ we need to show that for all $M > 0$, there is $N \in \mathbb{N}$ for which $a_n > M$ for all $n \geq N$. (Similar for when a_n diverges to $-\infty$.)
- The Comparison Lemma states that if for sequences a_n and b_n and for constants a, b, c, N we have $|b_n - b| \leq c|a_n - a|$ for all $n \geq N$ and that $\lim_{n\to\infty} a_n = a$. Then $\lim_{n\to\infty} b_n = b$.
- Every convergent sequence is bounded, but the converse is false.
- A set S is dense in $\mathbb{R}$ iff every real number is the limit of a sequence in S.
- If $a \leq c_n \leq b$ and c_n is convergent, then $a \leq \lim_{n\to\infty} c_n \leq b$.

Chapter 3

The Monotone Convergence Theorem

Definition 3.1. A sequence a_n is said to be **increasing** if $a_n \leq a_{n+1}$ for all positive integers n. It is called **decreasing** if $a_n \geq a_{n+1}$ for all positive integers n. A sequence is called **monotone** if it is either increasing or decreasing. a_n is called **strictly increasing** if $a_n < a_{n+1}$ for all n. We say a_n is **strictly decreasing** if $a_{n+1} < a_n$ for all n. A sequence is said to be **strictly monotone** if it is strictly increasing or strictly decreasing.

Theorem 3.1 (The Monotone Convergence Theorem)**.** *A monotone sequence converges if and only if it is bounded. Furthermore,*

(a) If a_n is increasing then $\lim_{n\to\infty} a_n = \sup a_n$.

(b) If a_n is decreasing then $\lim_{n\to\infty} a_n = \inf a_n$.

Example 3.1. *Show that the sequence $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ diverges.*

Example 3.2. *Let a_n be a sequence of real numbers satisfying*

$$a_1 = 0, \ a_{n+1} = a_n^2 + \frac{1}{4}, \ \textit{for all } n \geq 1.$$

Prove that a_n converges and find its limit.

Theorem 3.2 (The Nested Interval Theorem)**.** *Suppose $I_n = [a_n, b_n]$ is a sequence of intervals for which $I_{n+1} \subseteq I_n$ for all n. Then, the intersection of these intervals is a nonempty closed interval:*

$$\bigcap_{n=1}^{\infty} I_n = [a, b],$$

where $a = \sup a_n$ and $b = \inf b_n$. Furthermore, if $\lim_{n\to\infty} (a_n - b_n) = 0$, then $\bigcap_{n=1}^{\infty} I_n$ contains a single point.

Note that the intervals in the Nested Interval Theorem must be closed. For example

$$\bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset,$$

since by the Archemidean Property if $x > 0$ then there is a natural number n for which $x \notin (0, 1/n)$.

Definition 3.2. Let a_n be a sequence and $n_1 < n_2 < \cdots$ be a strictly increasing sequence of positive integers. Then, the sequence a_{n_k} given below

$$a_{n_1}, a_{n_2}, a_{n_3}, \ldots$$

is called a **subsequence** of a_n.

Example 3.3. *The sequence of odd positive integers $b_n = 2n-1$ is a subsequence of positive integers $a_n = n$.*

Theorem 3.3. *Assume a sequence a_n converges to a real number a. Then every subsequence of a_n also converges to a.*

Theorem 3.4. *Every sequence of real numbers has a monotone subsequence.*

Theorem 3.5. *Every bounded sequence has a convergent subsequence.*

Definition 3.3. A subset S of $\mathbb{R}$ is said to be **compact** (or sequentially compact) if every sequence in S has a convergent subsequence that converges to an element of S.

Example 3.4. *The following are two examples of sets that are not compact:*

(a) $[1, \infty)$ *is not compact.*

(b) $(0, 1)$ *is not compact.*

Theorem 3.6 (The Sequential Compactness Theorem)**.** *Every interval of the form $[a, b]$, where a, b are real numbers is compact.*

3.1 Warm-ups

Example 3.5. *Suppose a_n is a strictly increasing sequence of positive integers. Prove that $a_n \geq n$.*

Solution. We will prove the statement by induction on n.
Basis step. Since a_1 is a positive integer, we have $a_1 \geq 1$.
Inductive step. Suppose $a_n \geq n$. We know $a_{n+1} > a_n$. Since there are no integers in the interval (a_n, a_n+1) we have $a_{n+1} \geq a_n + 1$. Combing this with the indutive hypothesis we obtain $a_{n+1} \geq n + 1$, as desired. $\square$

3.2 More Examples

Example 3.6. *Check if each sequence is increasing or decreasing.*

(a) $n^2 - n$

(b) $n + \dfrac{3}{n}$

Solution. (a) By writing the first few terms (i.e., $0, 2, 6, \ldots$) we suspect the sequence is increasing. To prove that, we consider the difference of two consecutive terms: $(n+1)^2 - (n+1) - (n^2 - n)$ and wish to see if it is nonnegative, nonpositive or neither.

$$(n+1)^2 - (n+1) - (n^2 - n) = n^2 + 2n + 1 - n - 1 - n^2 + n = 2n > 0$$

Thus the sequence is increasing.
(b) Similar to part (a) we write the first few terms: $4, 3.5, 4$. We see $a_1 > a_2 < a_3$. Thus the sequence is neither increasing nor decreasing. $\square$

Example 3.7. *Show that the sequence* $a_n = \sum_{k=1}^{n} 2^{-k^2}$ *converges.*

Solution. Note that for any natural number k, we have $k^2 \geq k$. Thus, $2^{-k^2} \leq 2^{-k}$. Therefore,

$$a_n \leq \sum_{k=1}^{n} 2^{-k} = \frac{2^{-1} - 2^{-n-1}}{1 - 2^{-1}} = 1 - 2^{-n} < 1,$$

which shows a_n is bounded. Note that $a_{n+1} - a_n = 2^{-(n+1)^2} > 0$. Thus, a_n is increasing. Therefore by MCT, a_n converges. $\square$

Example 3.8. *Define a sequence a_n by $a_1 = 1$, $a_{n+1} = \sqrt{3a_n}$ for all $n \geq 1$. Prove that a_n converges and find its limit.*

The idea of solving this problem is to use MCT and show a_n has a limit. Then take the limit of both sides of the recursion and find the limit. By writing a few of the terms one can see that the sequence seems to be increasing. To show it is bounded we would like to show $a_n \leq M$. Since we are dealing with recursion we will use induction. For the induction to work we need $a_{n+1} = \sqrt{3a_n} \leq \sqrt{3M}$. So we need to make sure $\sqrt{3M} \leq M$. We pick $M = 3$. Now we are at a position to write down a complete solution.

Solution. First we will show a_n is increasing. For that we will prove by induction that $a_n < a_{n+1}$ for all $n \geq 1$. The inequality $a_2 = \sqrt{3} > a_1 = 1$ proves our basis step of the induction. Assuming $a_n < a_{n+1}$ we get $3a_n < 3a_{n+1}$, which implies $\sqrt{3a_n} < \sqrt{3a_{n+1}}$. Therefore $a_{n+1} < a_{n+2}$.

Now, by induction we will prove that $a_n < 3$ for all $n \geq 1$. We see that $a_1 = 1 < 3$, which proves the basis step. Assume $a_n < 3$. Then $a_{n+1} = \sqrt{3a_n} < \sqrt{3 \times 3} = 3$. Therefore $a_{n+1} < 3$.

Since a_n is bounded and increasing, $\lim_{n\to\infty} a_n = c$ exists. Squaring and then taking limit of both sides of the recursion we obtain:

$$\lim_{n\to\infty} a_{n+1}^2 = \lim_{n\to\infty} 3a_n \Rightarrow c^2 = 3c \Rightarrow c = 0, 3$$

Note that since a_n is increasing and $a_1 = 1$, $c \geq 1$, which means $c = 3$. Therefore $\lim_{n\to\infty} a_n = 3$ $\square$

Note that we still have not shown that square root of limit equals limit of square root. That is why we squared the recursion first and then we took the limit.

Example 3.9. *Let a_n be a sequence defined by*

$$a_1 = 1, a_{n+1} = \sqrt{2a_n + 1}.$$

Find $\lim\limits_{n\to\infty} a_n$ or show the limit does not exist.

Solution. We will prove that $a_n < a_{n+1}$ and $a_n < 3$ by induction on n.
Basis step: $a_1 = 1 < \sqrt{5} = a_2$, and $a_1 = 1 < 3$.
Inductive step: Suppose $a_n < a_{n+1}$ and $a_n < 3$. We have $2a_n + 1 < 2a_{n+1} + 1$ and thus

$$\sqrt{2a_n + 1} < \sqrt{2a_{n+1} + 1}$$

by properties of inequalities. Therefore, $a_{n+1} < a_{n+2}$. Since $a_n < 3$ we have

$$a_{n+1} = \sqrt{2a_n + 1} < \sqrt{2 \times 3 + 1} = \sqrt{7} < \sqrt{9} = 3.$$

Therefore, a_n is bounded and increasing. By the Monotone Convergence Theorem, a_n converges. Let $\lim\limits_{n\to\infty} a_n = a$. Since $1 \le a_n < 3$ we have $1 \le a \le 3$. By assumption $a_{n+1}^2 = 2a_n + 1$ $(*)$. The sequence a_{n+1} is a subsequence of a_n and thus converges to a. By taking the limit of both sides of $(*)$ we obtain $a^2 = 2a + 1$. This implies $a^2 - 2a + 1 = 2$ or $(a-1)^2 = 2$. Since $a - 1$ is nonnegative we obtain $a - 1 = \sqrt{2}$ or $a = 1 + \sqrt{2}$. Thus,

$$\lim_{n\to\infty} a_n = 1 + \sqrt{2}.$$

□

Example 3.10. *Prove that the union of every two compact subsets of $\mathbb{R}$ is compact.*

Solution. Suppose S and T are compact subsets of $\mathbb{R}$. Let a_n be a sequence in $S \cup T$. Since we only have two subsets S and T and infinitely many terms $a_1, a_2, \ldots$, there is a subsequence a_{n_k} of a_n whose terms all lie in S or all lie in T. Suppose a_{n_k} all lie in S. Since S is compact, there is a subsequence a_{m_k} of a_{n_k} that converges to an element of S. This subsequence a_{m_k} is a subsequence of a_n that converges to an element of $S \cup T$, and thus $S \cup T$ is compact. □

3.3 Exercises

All students are expected to do all of the exercises listed in the following two sections: Problems for Grading and Problems for Practice. You are only required to submit the ones in the first section for grading.

Challenge Problems are optional.

3.3.1 Problems for Grading

For submission please follow the same instructions as before.

All submissions must be made on the due dates before the class starts.

The following problems are due Tuesday 6/8/2021 before the class starts.

Exercise 3.1 (10 pts)**.** *Page 42, Problem 2.*

Exercise 3.2 (10 pts)**.** *Page 42, Problem 6.*

Exercise 3.3 (25 pts)**.** *Page 46, Problem 2.*

Exercise 3.4 (10 pts)**.** *Page 47, Problem 10.*

Exercise 3.5 (10 pts)**.** *Page 47, Problem 12.*

3.3.2 Problems for Practice

Exercise 3.6. *Let a_n be a sequence defined by*

$$a_1 = 1, \text{ and } a_{n+1} = 1 + \frac{1}{1 + \frac{1}{a_n}} \text{ for all } n \geq 1.$$

(a) Show that a_n is convergent.

(b) Find $\lim\limits_{n\to\infty} a_n$.

Page 42-43: 1, 4, 5, 7, 8, 9, 11.

Page 47: 2, 3,9

3.3.3 Challenge Problems

Exercise 3.7. *Let $a_n = \sum\limits_{k=1}^{n} \frac{1}{10^{k!}}$. Prove that a_n is convergent and the limit is irrational.*

Exercise 3.8. *Define a sequence $a_n = \sum\limits_{k=1}^{n} \frac{1}{k^2}$.*

- *Show that a_n converges and let $\lim\limits_{n\to\infty} a_n = c$.*
- *In terms of c, find the limit of the sequences $x_n = \sum\limits_{k=1}^{n} \frac{1}{(2k)^2}$ and $y_n = \sum\limits_{k=1}^{n} \frac{1}{(2k+1)^2}$.*

Exercise 3.9. *Show that the sequence $(1 + 1/n)^n$ converges.*

Exercise 3.10. *Evaluate the $\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \sqrt{1 + 4\sqrt{1 + \cdots}}}}}$. (Note that you need to first make sense of this "infinite radical".)*

3.4 Summary

- The Monotone Convergence Theorem (MCT) can be used to show a monotone bounded sequence converges.
- When dealing with sequences that are defined recursively consider using induction.
- The Nested Interval Theorem can be used to show a decreasing sequence of closed intervals share a point.
- A set S is called sequentially compact if every sequence a_n in S has a subsequence that converges to an element of S.
- The Sequential Compactness Theorem states that every close and bounded interval $[a, b]$ is sequentially compact.

Chapter 4

Continuous Functions

Throughout this chapter, unless otherwise stated, the domain of all functions are subsets of $\mathbb{R}$. These domains are typically denoted by D, D_1, D_2.

4.1 Basic Properties

Definition 4.1. Let x_0 be a point in D. A function $f : D \to \mathbb{R}$ is said to be **continuous at** x_0, provided whenever x_n is a sequence in D that approaches x_0, the sequence $f(x_n)$ approaches $f(x_0)$, otherwise we say f **has a discontinuity at** x_0 or f is **discontinuous at** x_0. We say f is **continuous** over D if it is continuous at every point in D.

Example 4.1. *Every polynomial is continuous over* $\mathbb{R}$*.*

Example 4.2. *The Dirichlet's function*

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous everywhere.

Example 4.3. *Consider the function* $f : [0,1] \cup [2,3] \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ x+1 & \text{if } 2 \le x \le 3 \end{cases}$$

Determine all points where f *is continuous.*

Definition 4.2. Given two functions $f, g : D \to \mathbb{R}$ we define the **sum** $f + g : D \to \mathbb{R}$ and the **product** $fg : D \to \mathbb{R}$ of f and g as follows:

$$(f+g)(x) = f(x) + g(x), \text{ and } (fg)(x) = f(x)g(x).$$

Furthermore, if $g(x) \neq 0$ for all $x \in D$, then the **quotient** $f/g : D \to \mathbb{R}$ of f by g is defined by

$$(f/g)(x) = \frac{f(x)}{g(x)}.$$

Similarly we define sums and products of any number of functions.

Theorem 4.1 (Properties of Continuity)**.** *Suppose x_0 is a point in D and $f, g : D \to \mathbb{R}$ are continuous at x_0. Then, so are $f+g$ and fg. Furthermore, if $g(x) \neq 0$ for all $x \in D$, then f/g is continuous at x_0.*

Corollary 4.1. *Suppose p, q are polynomials. Let*

$$D = \{x \in \mathbb{R} \mid q(x) \neq 0\}.$$

Then, $p/q : D \to \mathbb{R}$ is continuous over D.

Definition 4.3. A function of the form $\dfrac{p(x)}{q(x)}$, where p, q are polynomials is called a **rational function**. The domain of such a function is the set given below:

$$D = \{x \in \mathbb{R} \mid q(x) \neq 0\}.$$

Theorem 4.2. *Let $f : D_1 \to \mathbb{R}$ and $g : D_2 \to \mathbb{R}$ be two functions for which $f(D_1) \subseteq D_2$. Suppose $x_0 \in D_1$ and f is continuous at x_0, and g is continuous at $f(x_0)$. Then, $g \circ f$ is continuous at x_0.*

4.1.1 Warm-ups

Example 4.4. *Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{x+1}{x^2+1}$ is continuous over $\mathbb{R}$.*

Solution. Note that by the trivial inequality (See Theorem 0.9) $x^2 + 1 \geq 1 > 0$ for every $x \in \mathbb{R}$ and thus $x^2+1 \neq 0$. Both $x+1$ and x^2+1 are polynomials. Therefore, by Corollary 4.1 this function is continuous. $\square$

4.1.2 More Examples

Example 4.5. *Prove that every function $f : \mathbb{Z} \to \mathbb{R}$ is continuous.*

Solution. Let x be an integer, and let a_n be a sequence of integers that approaches x. Letting $\epsilon = 1$ in the definition of limit, we obtain the following:

$$\exists\ N \in \mathbb{N} \text{ such that } |a_n - x| < 1 \text{ for all } n \geq N.$$

This implies $x - 1 < a_n < x + 1$. Since a_n and x are both integers, we must have $x = a_n$. Therefore, $f(a_n) = f(x)$ for all $n \geq N$. This implies $\lim_{n\to\infty} f(a_n) = f(x)$. Therefore, f is continuous at x for all x and thus f is continuous. $\square$

Example 4.6. *Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous nowhere, but $|f|$ is continuous everywhere.*

Solution. The following function is one example of such function:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{Q}^c \end{cases}$$

Suppose on the contrary that f is continuous at a point $x \in \mathbb{R}$. Since $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense, by Theorem 2.4, there are sequences a_n of rational numbers and b_n of irrational numbers that approach x. Since f is continuous at x we have

$$\lim_{n\to\infty} f(a_n) = f(x), \text{ and } \lim_{n\to\infty} f(b_n) = f(x).$$

Therefore, $f(x) = 1$, and $f(x) = -1$, which implies $1 = -1$, a contradiction.

Note that $|f(x)| = 1$ is a polynomial which is continuous. □

Example 4.7. *Suppose $f : D \to \mathbb{R}$ is continuous at a point $x_0 \in D$. Prove that $|f| : D \to \mathbb{R}$ given by $|f|(x) = |f(x)|$ is also continuous at x_0.*

Solution. Let a_n be a sequence in D that approaches x. By triangle inequality

$$|f(a_n)| \leq |f(x)| + |f(a_n) - f(x)|, \text{ and } |f(x)| \leq |f(a_n)| + |f(x) - f(a_n)|.$$

This implies

$$-|f(a_n) - f(x)| \leq |f(a_n)| - |f(x)| \leq |f(a_n) - f(x)| \Rightarrow ||f(a_n)| - |f(x)|| \leq |f(a_n) - f(x)|.$$

Since f is continuous, $f(a_n) \to f(x)$. Therefore, by Comparison Lemma, $|f(a_n)| \to |f(x)|$, as desired. □

Example 4.8. *Suppose $f_1, f_2, \dots, f_n : D \to \mathbb{R}$ are continuous at a point x_0 in D. Prove that $f_1 + \cdots + f_n$ and $f_1 \cdots f_n$ are also continuous at x_0.*

Solution. We will prove the claim by induction on n.

Basis step. For $n = 1$, the claim is obvious, since f_1 is continuous at x_0.

Inductive step. Suppose the sum and product of any n functions that are continuous at x_0 are also continuous at x_0. Suppose $f_1, \dots, f_{n+1} : D \to \mathbb{R}$ are continuous at x_0. By inductive hypothesis, $f_1 + \cdots + f_n$ and $f_1 \cdots f_n$ are continuous at x_0. Therefore, by Theorem 4.1 the following functions are also continuous at x_0:

$$(f_1 + \cdots + f_n) + f_{n+1}, \text{ and } (f_1 \cdots f_n) f_{n+1},$$

as desired. □

Example 4.9. *Find all points x_0 for which the following function is continuous at x_0.*

$$f(x) = \begin{cases} 0 & \textit{if } x \textit{ is dyadic} \\ x & \textit{otherwise} \end{cases}$$

Solution. Suppose f is continuous at x_0. By a problem, the set of dyadic numbers is dense. By a Theorem, there is a sequence s_n of dyadics for which $s_n \to x_0$. Since f is continuous at x_0, we must have $f(s_n) \to f(x_0)$. Since $f(s_n) = 0$, we must have $f(x_0) = 0$.

Similarly, since $\mathbb{Q}^c$ is dense, there is a sequence b_n of irrationals for which $b_n \to x_0$. Since f is continuous at x_0, we have $f(b_n) \to f(x_0)$. Thus, $b_n \to f(x_0)$, which implies $f(x_0) = x_0$.

Therefore $f(x_0) = x_0$ and $f(x_0) = 0$, which proves $x_0 = 0$. Therefore, f is not continuous at x_0, if $x_0 \neq 0$.

Now, we will show that f is continuous at 0 and discontinuous everywhere else. Suppose x_n is a sequence that approaches 0. Note that $|f(x_n)| = 0$ or $|f(x_n)| = |x_n|$. Therefore,

$$|f(x_n) - 0| = |f(x_n)| \leq |x_n| = |x_n - 0|.$$

Therefore, by the Comparison Lemma, $\lim_{n\to\infty} f(x_n) = 0 = f(0)$. Thus, f is continuous at zero. □

Example 4.10. *Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous for which $f(r) = g(r)$ for all $r \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$.*

Solution. Note that the function $h = f - g$ is continuous and $h(r) = 0$ for all $r \in \mathbb{Q}$. Given $x \in \mathbb{R}$, using the fact that $\mathbb{Q}$ is dense, we can find a sequence of rationals r_n for which $r_n \to x$. Since h is continuous we must have $h(r_n) \to h(x)$. Since $h(r_n) = 0$, we obtain $h(x) = 0$, which shows $f(x) = g(x)$. □

Example 4.11. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Prove that if f is continuous, then*

$$\lim_{h\to 0}[f(x+h) - f(x-h)] = 0, \text{ for all } x.$$

Is the converse true?

Note that both $x-h$ and $x+h$ approach x, and thus if f is continous, $f(x-h)$ and $f(x+h)$ both approach $f(x)$. For the converse, the given assumption would be satisfied if $f(x-h)$ and $f(x+h)$ approach the same number. However the limit does not have to be $f(x)$. This motivates a counterexample for the converse.
Solution. Suppose f is continuous $\lim_{h\to 0}[f(x-h) - f(x+h)] = f(x) - f(x) = 0$.

The converse is false. Let $f(x) = 0$ for all $x \neq 0$ and $f(0) = 1$. We will show that this function is a counterexample.

Suppose $x \neq 0$. Let h_n be a sequence that approaches zero. Thus, there is N such that $|h_n| < |x|/2$ for all $n \geq N$, which means $|h_n| \neq |x|$. Therefore, $x - h_n \neq 0$ and $x + h_n \neq 0$. Thus $f(x+h_n) = f(x-h_n) = 0$ for all $n \geq N$. Therefore $\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$.

If $x = 0$, then $f(-h) = f(h) = 1$, for all $h \neq 0$, which imply $\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$.

However f is not continuous at 0, as $1/n \to 0$ but $f(1/n) \to 0 \neq f(0)$. □

4.2 The Extreme Value Theorem

Definition 4.4. We say $f : D \to \mathbb{R}$ attains its **maximum** value at $x_0 \in D$ if $f(x) \leq f(x_0)$ for all $x \in D$. We say x_0 is a **maximizer** of f. Similarly we define **minimum** and **minimizer**.

Example 4.12. *The function $f : (0,1) \to \mathbb{R}$ defined by $f(x) = 1/x$ has no maximum or minimum value.*

Theorem 4.3 (The Extreme Value Theorem)**.** *A continuous function on a closed and bounded interval attains both a maximum and minimum value.*

Example 4.13. *Find the maximum and minimum of the function $f : [0,2] \to \mathbb{R}$ defined by $f(x) = x^2 - 2x$.*

4.2.1 Warm-ups

Example 4.14. *In the Extreme Value Theorem we proved that a continuous function $f : [a,b] \to \mathbb{R}$ attains both a maximum and a minimum value. Is it true that if $f : [a,b] \to \mathbb{R}$ attains a maximum and minimum value, then f must be continuous?*

Solution. The answer is no. For example $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q}^c \end{cases}$$

By definition, 0 is the minimum value of f and 1 is the maximum value of f. This function is not continuous anywhere by an argument similar to that of Example 4.2. □

Example 4.15. *Suppose $f, g : D \to \mathbb{R}$ are both bounded. Prove that fg and $f+g$ are also bounded.*

Solution. By definition, there are real numbers M_1, M_2 for which

$$|f(x)| \leq M_1, \text{ and } |g(x)| \leq M_2 \text{ for all } x \in D.$$

By the triangle inequality and properties of inequalities we have

$$|(f+g)(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq M_1 + M_2, \text{ and } |(fg)(x)| = |f(x)||g(x)| \leq M_1 M_2.$$

Therefore, $f+g$ and fg are bounded. □

4.2.2 More Examples

Example 4.16. *Prove that if D is unbounded above, then there is a continuous function $f : D \to \mathbb{R}$ that does not attain a maximum value.*

Solution. $f(x) = x$ is such an example. Since it is a polynomial it is continuous. Since D is unbounded above for every $M \in \mathbb{R}$ there is $x \in D$ for which $x > M$. Thus, $f(x) > M$, which means f has no maximum value. □

Example 4.17. *Prove that if D is not closed, then, there is a continuous function $f : D \to \mathbb{R}$ for which f does not attain its maximum value.*

Solution. By definition, there must be a convergent sequence a_n in D whose limit a is not in D. Define $f : D \to \mathbb{R}$ by $f(x) = \dfrac{1}{|x-a|}$. Since $x \neq a$ for all $x \in D$ and absolute value function and polynomials are continuous, f is continuous. Let M be a positive number. By definition of limit, there is $N \in \mathbb{N}$ for which $|a_n - a| < 1/M$ and thus $f(a_n) > M$. Therefore, f does not have a maximum value. □

Example 4.18. *Find the maximum and minimum of the function $f(x) = x^4(1+x)^2$ over the interval $[-1, 2]$.*

Solution. Note that since 4 and 2 are even $x^4(1+x)^2 \geq 0$. Since $f(0) = 0$, the minimum of f is zero.

Note that when $-1 \leq x \leq 2$, we have $0 \leq x^4 \leq 16$ and $0 \leq (1+x)^2 \leq 9$. Therefore $f(x) \leq 16 \cdot 9 = 144$. Since $f(2) = 144$, the maximum of f is 144. □

4.3 Exercises

All students are expected to do all of the exercises listed in the following two sections: Problems for Grading and Problems for Practice. You are only required to submit the ones in the first section for grading.

Challenge Problems are optional.

4.3.1 Problems for Grading

For submission please follow the same instructions as before.

All submissions must be made on the due dates before the class starts.

The following problems are due Thursday 6/10/2021 before the class starts.

Exercise 4.1 (10 pts)**.** *Page 57, Problem 4.*

Exercise 4.2 (10 pts)**.** *Page 58, Problem 9.*

Exercise 4.3 (10 pts)**.** *Page 58, Problem 13.*

Exercise 4.4 (15 pts)**.** *Page 61, Problem 2.*

4.3.2 Problems for Practice

Exercise 4.5. *Suppose* $f : [0, \infty) \to \mathbb{R}$ *and* $g : (-\infty, 0] \to \mathbb{R}$ *are continuous and* $f(0) = g(0)$*. Prove that* $h : \mathbb{R} \to \mathbb{R}$ *defined by*

$$h(x) = \begin{cases} f(x) & \text{if } x \geq 0 \\ g(x) & \text{if } x < 0 \end{cases}$$

is continuous.

Exercise 4.6. *Give an example of a function* $f : \mathbb{R} \to \mathbb{R}$ *that is continuous nowhere but* $f \circ f$ *is continuous everywhere.*

Pages 57-58: 1, 2, 6, 7, 8, 11, 12.

Page 61-62: 1, 3, 5

4.3.3 Challenge Problems

Exercise 4.7. *Prove that every polynomial of even degree attains a minimum value.*

Exercise 4.8. *Prove that if all values of a polynomial* $p(x)$ *are non-negative, then this polynomial can be written as a sum of squares of polynomials.*

Exercise 4.9. *Let* $p(x)$ *be a polynomial. Prove that* $|p(x)|$ *attains its minimum over* $\mathbb{R}$*.*

Exercise 4.10. *Suppose* $f, g : \mathbb{R} \to \mathbb{R}$ *are continuous functions for which* $(f(x))^2 = (g(x))^2$ *for all* $x \in \mathbb{R}$*. Suppose also that* $f(x) \neq 0$ *for every* $x \in \mathbb{R}$*. Prove that either for all* x *we have* $f(x) = g(x)$ *or for all* x *we have* $f(x) = -g(x)$*.*

Exercise 4.11. *Find all points for which the following function is continuous.*

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ 1/m & \text{if } x = n/m \text{ is in simplest form with} m > 0 \end{cases}$$

Exercise 4.12. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *satisfies* $f(x + y) = f(x) + f(y)$*. Suppose* f *is continuous at* 0*. Prove that* $f(x) = cx$ *for a constant* c*.*

Exercise 4.13. *Let* n *be a positive integer.*

(a) Prove that if n *is odd, then there is a continuous function* $f : \mathbb{R} \to \mathbb{R}$ *that takes on each value exactly* n *times.*

(b) Prove that if n *is even, then there is no continuous function* $f : \mathbb{R} \to \mathbb{R}$ *that takes on each value exactly* n *times.*

4.3.4 Summary

- To prove a function f is continuous at x_0: Assume x_n is a sequence in the domain that approaches x_0. Prove that $f(x_n) \to f(x_0)$.
- To prove f is not continuous at x_0 we often assume it is and find multiple sequences that approach x_0, use the definition and get a contradiction.
- The Extreme Value Theorem states that any continuous function $f : [a, b] \to \mathbb{R}$ attains its maximum and minimum values.
- The Extreme Value Theorem can be used to show the existence of maximum and minimum of continuous functions over closed and bounded intervals, but other tools (such as inequalities) must be used to find the maximum and minimum values.

4.4 The Intermediate Value Theorem

Theorem 4.4. *Suppose $f : [a, b] \to \mathbb{R}$ is continuous and c is a number between $f(a)$ and $f(b)$. Then, there is $x \in [a, b]$ for which $f(x) = c$.*

Definition 4.5. We say a function $f : D \to \mathbb{R}$ satisfies the **intermediate value property** if for any $a, b \in D$ and c that lies strictly between $f(a)$ and $f(b)$, there is $x_0 \in D$ that is strictly between a and b for which $f(x_0) = c$.

Example 4.19. *Given every positive real number c, prove that there is a positive real number x for which $x^2 = c$.*

Example 4.20. *Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is onto.*

Theorem 4.5. *Suppose I is an interval and $f : I \to \mathbb{R}$ is continuous. Then, $f(I)$ is also an interval.*

Example 4.21. *Recall that the function $f : [0, 1] \cup [2, 3] \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ x + 1 & \text{if } 2 \le x \le 3 \end{cases}$$

is continuous. However this function does not satisfy the intermediate value property.

4.4.1 Warm-ups

Example 4.22. *Prove that there is a real number x for which $x^3 - 2x = 3$.*

Solution. Note that the function $p(x) = x^3 - 2x$ is a polynomial and thus it is continuous everywhere. We note that $p(0) = 0$ and $p(2) = 4$. Since 3 is between 0 and 4, by the Intermediate Value Theorem, there is c between 0 and 2 for which $p(c) = 3$, as desired. $\square$

4.4.2 More Examples

Example 4.23. *Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - 3x + 2$ is onto.*

Given a real number c, we would like to show $f(x) = c$ has a solution. For that we will use the IVT. We want to find an integer n for which $n^3 - 3n + 2 > c$. It is enough to have the following:

$$n^3 - 3n + 2 > 4n - 3n = n > c, \text{ if we assume } n \geq 2.$$

For the other direction we will find a natural number n for which $(-n)^3 - 3(-n) + 2 < c$, which is equivalent to $n^3 - 3n - 2 > -c$. If we select $n \geq 2$ we will have $n^3 - 3n - 2 \geq 4n - 3n - 2 = n - 2$, thus it is enough to make sure $n > -c + 2$.

Solution. Let c be a real number. By A.P. there is a natural number n for which $n > \max(c, 2 - c, 2)$. $f(n) = n^3 - 3n + 2 > 4n - 3n + 2 = n + 2 > n > c$.

On the other hand, we have

$$f(-n) = -n^3 + 3n + 2 < -4n + 3n + 2 = -n + 2 < -(2 - c) + 2 = c.$$

Therefore, $f(-n) < c < f(n)$. Since $f(x)$ is continuous, by the IVT, there is $x \in (-n, n)$ for which $f(x) = c$. Therefore, f is onto. □

Example 4.24. *Show that there is a real number x for which $\dfrac{1}{\sqrt{1+x}} = x^2 - 1$.*

Solution. Note that we have not yet shown the square root function is continuous, so we will eliminate the square root by squaring both sides. Consider the function $f(x) = 1 - (1 + x)(x^2 - 1)^2$. Note that f, as a polynomial, is continuous, $f(1) = 1 > 0$ and $f(2) = -26 < 0$. By the IVT, there is $c \in (1, 2)$ for which $f(c) = 0$. Therefore $\dfrac{1}{1+c} = (c^2 - 1)^2$. Note that since $c^2 > 1$ and $1 + c > 0$, taking square root of both sides yields $\dfrac{1}{\sqrt{1+c}} = c^2 - 1$, which completes the proof. □

Example 4.25. *Prove that for every real number $c \geq 1$, there is a real number x for which*

$$\sqrt{x} + \sqrt{x+1} = c.$$

Sketch. Since we have not yet proved that the square root function is continuous, we will have to eliminate all square roots. Squaring both sides we obtain

$$x + x + 1 + 2\sqrt{x^2 + x} = c^2 \Rightarrow 2\sqrt{x^2 + x} = c^2 - 2x - 1 \Rightarrow 4(x^2 + x) = (c^2 - 2x - 1)^2.$$

Solution. Consider the polynomial $f(x) = 4(x^2 + x) - (c^2 - 2x - 1)^2$. Since f is a polynomial it is continuous. We have $f(0) = -(c^2 - 1)^2 \leq 0$. Also,

$$f\left(\frac{c^2 - 1}{2}\right) = 4\left(\frac{c^2 - 1}{2}\right)^2 + 4\frac{c^2 - 1}{2} \geq 0, \text{ since } c \geq 1.$$

Therefore, by the Intermediate Value Theorem, there is $z \in [0, (c^2-1)/2]$ for which $f(z) = 0$. This implies:

$$4(z^2+z) = (c^2-2z-1)^2 \Rightarrow 2\sqrt{z^2+z} = c^2-2z-1, \text{ since } z \leq \frac{c^2-1}{2} \text{ i.e. } c^2-2z-1 \geq 0.$$

This implies

$$2z+1+2\sqrt{z^2+z} = c^2 \Rightarrow z+z+1+2\sqrt{z(z+1)} = c^2 \Rightarrow (\sqrt{z}+\sqrt{z+1})^2 = c^2 \Rightarrow \sqrt{z}+\sqrt{z+1} = c,$$

since c is positive. □

4.5 Uniform Continuity

Definition 4.6. A function $f : D \to \mathbb{R}$ is called **uniformly continuous** if every two sequences u_n, v_n in D satisfy the following:

$$\text{if } \lim_{n\to\infty}(u_n - v_n) = 0, \text{ then } \lim_{n\to\infty}(f(u_n) - f(v_n)) = 0.$$

Example 4.26. *The following functions are uniformly continuous:*

(a) $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x$.

(b) $f : [0,1] \to \mathbb{R}$ *defined by* $f(x) = x^2$.

Example 4.27. *The following functions are not uniformly continuous:*

(a) $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x^2$.

(b) $f : (0,1) \to \mathbb{R}$ *defined by* $f(x) = \dfrac{1}{x}$.

Theorem 4.6. *If* $f : D \to \mathbb{R}$ *is uniformly continuous, then it is also continuous.*

Theorem 4.7 (Heine-Cantor Theorem)**.** *Every continuous function* $f : [a,b] \to \mathbb{R}$ *over a closed and bounded interval is uniformly continuous.*

4.5.1 Warm-ups

Example 4.28. *Prove that every linear function* $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = ax + b$*, where* $a, b \in \mathbb{R}$ *are constants, is uniformly continuous.*

Solution. Suppose u_n, v_n are sequences of real numbers for which $u_n - v_n \to 0$. Then,

$$f(u_n) - f(v_n) = au_n + b - av_n - b = a(u_n - v_n) \to a \times 0 = 0.$$

Therefore, f is uniformly continuous. □

4.5.2 More Examples

Example 4.29. *Using the definition prove that $f : [1,3] \to \mathbb{R}$ defined by $f(x) = x^2 - 2x$ is uniformly continuous.*

Solution. Suppose u_n and v_n are two sequences in $[1,3]$ with $\lim\limits_{n\to\infty}(u_n - v_n) = 0$. We have:

$$|f(u_n) - f(v_n)| = |u_n^2 - v_n^2 - 2(u_n - v_n)| = |u_n - v_n| \cdot |u_n + v_n - 2| \leq |u_n - v_n| \cdot (3 + 3 + 2) = 8|u_n - v_n|.$$

Therefore, by the Comparison Lemma, $\lim\limits_{n\to\infty}(f(u_n) - f(v_n)) = 0$. Thus, f is uniformly continuous over $[1,3]$. □

Example 4.30. *Prove that every quadratic polynomial $f : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous.*

Solution. Suppose a, b, c are constants, and $f(x) = ax^2 + bx + c$ with $a \neq 0$. Consider the sequences $u_n = n$ and $v_n = n + \frac{1}{n}$. $u_n - v_n = -\frac{1}{n} \to 0$ by an example. However

$$f(u_n) - f(v_n) = an^2 + bn + c - an^2 - 2a - \frac{a}{n^2} - bn - \frac{b}{n} - c = -2a - \frac{a}{n^2} - \frac{b}{n} \to -2a - 0 - 0 = -2a \neq 0.$$

Here we use the fact that $\frac{1}{n} \to 0$. Therefore, f is not uniformly continuous. □

Example 4.31. *Suppose $f : (a,b) \to \mathbb{R}$ is uniformly continuous, where $a, b \in \mathbb{R}$. Prove that the image of f is bounded.*

Solution. Suppose the image of f is unbounded above. There is u_1 for which $f(u_1) > 1$, and u_2 for which $f(u_2) > f(u_1) + 1$, and u_3 for which $f(u_3) > f(u_2) + 1$, and so on. Recursively, given $u_1, \dots, u_n$ we can find u_{n+1} for which $f(u_{n+1}) > f(u_n) + 1$. This means $f(u_{n+1}) - f(u_n) > 1$ for all n. By Sequential Compactness Theorem, the sequence u_n has a subsequence that converges to an element of $[a,b]$. Let this subsequence be v_n. We have $v_{n+1} - v_n \to 0$ but $f(v_{n+1}) - f(v_n) > 1$ for all n, which is a contradiction. Therefore, f is bounded above. Similarly we can show f is bounded below. □

4.6 Excercises

All students are expected to do all of the exercises listed in the following two sections: Problems for Grading and Problems for Practice. You are only required to submit the ones in the first section for grading.

Challenge Problems are optional.

4.6.1 Problems for Grading

The following problems are due Friday 6/11/2021 before the class starts.

Exercise 4.14 (10 pts). *Page 65, Problem 3.*

Exercise 4.15 (10 pts). *Page 66, Problem 7.*

Exercise 4.16 (20 pts). *Page 69, Problem 1.*

Exercise 4.17 (10 pts). *Page 69, Problem 6.*

4.6.2 Problems for Practice

Exercise 4.18. *Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function for which $f(a) < f(b)$. Prove that there are real numbers c, d satisfying $a \le c < d \le b$ for which $f(a) = f(c), f(d) = f(b)$ and for every $x \in (c, d)$ we have $f(a) < f(x) < f(b)$.*

Page 65-66: 2, 4, 5, 6, 8, 10

Page 69: 3, 4, 5, 7, 10.

4.6.3 Challenge Problems

Exercise 4.19. *Suppose $f(x)$ is a continuous function and n be a positive integer for which*

$$\lim_{x\to\infty} \frac{f(x)}{x^n} = \lim_{x\to-\infty} \frac{f(x)}{x^n} = 0.$$

(a) Prove that if n is odd, then there if a real number x for which $x^n + f(x) = 0$.

(b) Prove that if n is even, then the function $f(x) + x^n$ attains a minimum value.

Exercise 4.20. *Suppose $f : [0, 1] \to \mathbb{R}$ is continuous for which $f(0) = f(1)$. Prove that for every positive integer n there is some $x \in [0, 1 - \frac{1}{n}]$ for which $f(x) = f(x + \frac{1}{n})$.*

4.6.4 Summary

- To prove $f(x) = c$ has a solution using the Intermediate Value Theorem:
  - Find two numbers a, b for which f is continuous over $[a, b]$.
  - Make sure you prove f is continuous.
  - Show that c is between $f(a)$ and $f(b)$.
- To prove a function f is uniformly continuous, prove that if u_n, v_n are sequences for which $u_n - v_n \to 0$ then $f(u_n) - f(v_n) \to 0$.
- To prove f is not uniformly continuous, we need to find sequences u_n, v_n for which $u_n - v_n \to 0$ but $f(u_n) - f(v_n)$ does not approach zero. This can often be achieved by looking for u_n, v_n where the slope of the graph of f is unbounded.
- Every uniformly continuous function is continuous.
- Every continuous function whose domain is a closes and bounded interval $[a, b]$ is uniformly continuous.

4.7 The ϵ-δ Criterion

Definition 4.7. Let $f : D \to \mathbb{R}$ be a function and x_0 be a point in D. We say f **satisfies the** $\epsilon - \delta$ **criterion at** x_0 if the following holds:

$$\forall\ \epsilon > 0\ \exists\ \delta > 0 \text{ such that } (x \in D \text{ and } |x - x_0| < \delta) \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

We say f **satisfies the** $\epsilon - \delta$ **criterion on** D if the following holds:

$$\forall\ \epsilon > 0\ \exists\ \delta > 0\ \text{ such that } (u, v \in D \text{ and } |u - v| < \delta) \Rightarrow |f(u) - f(v)| < \epsilon.$$

Example 4.32. *Using the definition, prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ satisfies the $\epsilon - \delta$ criterion at any real number, but it does not satisfy the $\epsilon - \delta$ criterion on $\mathbb{R}$.*

Example 4.33. *Using the definition, prove that $f : [0, 1] \to \mathbb{R}$ given by $f(x) = \dfrac{x}{x + 1}$ satisfies the $\epsilon - \delta$ criterion on $[0, 1]$.*

Theorem 4.8. *Let $f : D \to \mathbb{R}$ be a function and x_0 be a point in D.*

(a) f is continuous at x_0 if and only if it satisfies the $\epsilon - \delta$ criterion at x_0.

(b) f is uniformly continuous if and only if f satisfies the $\epsilon - \delta$ criterion on D.

4.7.1 Warm-ups

Example 4.34. *Find two positive values of δ that satisfy each of the following:*

(a) If $|x - 2| < \delta$, then $|2x - 4| < 0.1$.

(b) If $|x - 1| < \delta$, then $\left|\dfrac{3x - 1}{2x - 1} - 2\right| < 2$.

Solution.

(a) We note that $|2x - 4| < 0.1$ is the same as $|x - 2| < 0.1/2 = 0.05$. So, $\delta = 0.05$ and $\delta = 0.04$ (and anything less than 0.05) work. Here is the proof:

Suppose $|x - 2| < 0.05$. Multiplying by 2 we obtain $|2x - 4| < 0.1$, which means any $\delta \leq 0.05$ works.

(b) Note that $\left|\dfrac{3x - 1}{2x - 1} - 2\right| = \left|\dfrac{-x + 1}{2x - 1}\right|$. We need to control the denominator. To keep $2x - 1$ away from zero, we need to make sure sure x is away from $1/2$. Since x is near 1, we can make sure $\delta < 1/3$ ($1/3$ is some arbitrary positive number less than $1/2$.) In that case $|x - 1| < 1/3$ implies $2/3 < x < 4/3$, which implies $1/3 < 2x - 1 < 5/3$, which means $|2x - 1| > 1/3$ or $\left|\dfrac{-x + 1}{2x - 1}\right| < \dfrac{\delta}{1/3} = 3\delta$. To make this less than 2 we need $3\delta \leq 2$, however we need to combine that with $\delta < 1/3$. Thus we can select any $\delta \in (0, 1/3)$. Here is the proof:

Suppose δ is a number in $(0, 1/3)$, and suppose $|x - 1| < \delta$. We have $|x - 1| < \delta < 1/3$, which implies $2/3 < x < 4/3$, or $1/3 < 2x - 1 < 5/3$. Therefore, $|2x - 1| > 1/3$. Note that $\left|\dfrac{3x - 1}{2x - 1} - 2\right| = \left|\dfrac{-x + 1}{2x - 1}\right| < \dfrac{\delta}{1/3} = 3\delta < 1 < 2$, since $\delta < 1/3$. This completes the proof.

□

4.7.2 More Examples

Example 4.35. *Consider the function* $f(x) = \dfrac{1-x}{x+2}$.

(a) Prove that f *satisfies the* $\epsilon - \delta$ *criterion at* x_0 *for every* $x_0 \neq -2$.

(b) Prove that f *satisfies the* $\epsilon - \delta$ *criterion over the interval* $[1, \infty)$.

For the first part, Suppose $x_0 \neq -2$. We would like to prove $\left|\dfrac{1-x}{x+2} - \dfrac{1-x_0}{x_0+2}\right| < \epsilon$. It is enough to have

$$\left|\frac{1-x}{x+2} - \frac{1-x_0}{x_0+2}\right| = \left|\frac{(1-x)(x_0+2)-(1-x_0)(x+2)}{(x+2)(x_0+2)}\right| = \left|\frac{3x_0-3x}{(x+2)(x_0+2)}\right| < \epsilon$$

We want to make sure $|x+2|$ is not too close to zero. So, we take the distance between x_0 and -2 and divide that by 2 and make sure x is at least this distance away from -2. In other words, we will make sure $|x - x_0| < |x_0 + 2|/2$. This guarantees $|x + 2| \geq |x_0 + 2| - |x_0 - x|$, by the Triangle Inequality. Since $|x - x_0| < |x_0 + 2|/2$, we obtain $|x + 2| > |x_0 + 2|/2$.

So, in order to show the claim we will make sure

$$\left|\frac{3x_0-3x}{(x+2)(x_0+2)}\right| < \frac{3\delta}{|x_0+2|/2 \cdot |x_0+2|} < \epsilon.$$

Therefore, we will select $\delta = \min(|x_0 + 2|/2, \epsilon|x_0 + 2|^2/6)$.

For the second part, we would like to show

$$\left|\frac{3x-3y}{(x+2)(y+2)}\right| < \epsilon,$$

for all $x, y \in [1, \infty)$. Note that the denominator is at least $(1 + 2)(1 + 2) = 9$. Thus it is enough to have $3|x - y|/9 < \epsilon$. Thus, we select $\delta = 3\epsilon$.

Solution.

(a) Suppose $\epsilon > 0$ and let $\delta = \min(|x_0 + 2|/2, \epsilon|x_0 + 2|^2/6)$. Since $x_0 \neq -2$, $\delta > 0$. If $|x - x_0| < \delta$, then

$$|\frac{1-x}{x+2} - \frac{1-x_0}{x_0+2}| = |\frac{(1-x)(x_0+2)-(1-x_0)(x+2)}{(x+2)(x_0+2)}| = |\frac{3x_0-3x}{(x+2)(x_0+2)}| < \frac{3\delta}{|x+2| \cdot |x_0+2|}.$$

By Triangle Inequality, we have $|x+2| \geq |2+x_0|-|x_0-x| > |2+x_0|-\delta \geq |2+x_0|-|2+x_0|/2 = |2+x_0|/2$. This implies

$$\frac{3\delta}{|x+2| \cdot |x_0+2|} < \frac{3\delta}{|x_0+2|^2/2} \leq \epsilon$$

Therefore

$$\left|\frac{1-x}{x+2} - \frac{1-x_0}{x_0+2}\right| < \epsilon,$$

as desired.

(b) Suppose $\epsilon > 0$ and let $\delta = 3\epsilon$. If $x, y \in [1, \infty)$ and $|x - y| < \delta$, then

$$\left|\frac{1-x}{x+2} - \frac{1-y}{y+2}\right| = \frac{3|x-y|}{(x+2)(y+2)} < \frac{3\delta}{(1+2)(1+2)} = \epsilon.$$

This prove that f satisfied the $\epsilon - \delta$ criterion over $[1, \infty)$.

$\square$

Example 4.36. *For every $\epsilon > 0$, find the largest δ that makes the statement of the $\epsilon - \delta$ criterion for $\lim_{x \to 2} x^2 = 4$ true.*

Solution. We need to have the following:

$$|x-2| < \delta \Rightarrow |x^2 - 4| < \epsilon \qquad (*)$$

Note that $|x^2 - 4| = |x-2||x+2| < \delta|x+2|$. By the Triangle Inequality we have

$$|x+2| \leq 4 + |x-2| < 4 + \delta \Rightarrow |x^2 - 4| < 4\delta + \delta^2.$$

Therefore, setting $\epsilon = 4\delta + \delta^2$ we see that $\delta = \sqrt{\epsilon + 4} - 2$ works.

Now, suppose δ satisfies $(*)$. Let a_n be a sequence of real numbers for which $a_n \to \delta$ and $0 < a_n < \delta$ (e.g. $a_n = \delta - \delta/(n+1)$). Then, setting $x_n = 2 + a_n$ we have $0 < x_n - 2 < \delta$. Also,

$$|x_n^2 - 4| = |a_n^2 + 4a_n| < \epsilon \Rightarrow \delta^2 + 4\delta \leq \epsilon \Rightarrow (\delta + 2)^2 \leq \epsilon + 4 \Rightarrow \delta \leq \sqrt{\epsilon + 4} - 2.$$

This means $\delta = \sqrt{\epsilon + 4} - 2$ is the largest δ. $\square$

4.7.3 Summary

- Note the difference between the $\epsilon - \delta$ criterion at one point x_0, and the $\epsilon - \delta$ criterion over a set. In the former, δ may depend on x_0 but in the latter, δ may only depend on ϵ.

- The $\epsilon - \delta$ criterion at x_0 is equivalent to continuity at x_0, and the $\epsilon - \delta$ criterion over a domain is equivalent to uniform continuity over that domain.

- To find δ in terms of ϵ start with the inequality $|f(x) - f(x_0)| < \epsilon$. Simplify and use the inequality $|x - x_0| < \delta$ to get a relation of the form

  $$\text{an expression of } \delta < \epsilon.$$

  Solve this for δ. This gives you the idea of how to solve the problem. To write down a full solution, start with

  $$\text{“Let } \epsilon > 0 \text{ and set } \delta = \cdots\text{”},$$

  where $\cdots$ is replaced by what you found in your scratch. Then start with $|x - x_0| < \delta$ and re-write the scratch work in the appropriate order to obtain the inequality $|f(x) - f(x_0)| < \epsilon$.

4.8 Inverse Functions

Definition 4.8. $f : D \to \mathbb{R}$ is said to be **one-to-one** if whenever $f(x_1) = f(x_2)$ for $x_1, x_2 \in D$, we have $x_1 = x_2$. When f is one-to-one, its **inverse**, denoted by f^{-1}, is the function from $f(D)$ to D satisfying $f(x) = y$ if and only if $f^{-1}(y) = x$.

Definition 4.9. Consider a function $f : D \to \mathbb{R}$.

(a) We say f is **increasing** if $x < y$ implies $f(x) \leq f(y)$ for all $x, y \in D$.

(b) We say f is **decreasing** if $x < y$ implies $f(x) \geq f(y)$ for all $x, y \in D$.

(c) We say f is **strictly increasing** if $x < y$ implies $f(x) < f(y)$ for all $x, y \in D$.

(d) We say f is **strictly decrasing** if $x < y$ implies $f(x) > f(y)$ for all $x, y \in D$.

A function that is either increasing or decreasing is called **monotone**, and a function that is either strictly increasing or strictly decreasing is called **strictly monotone**.

Theorem 4.9. *Any strictly monotone function $f : D \to \mathbb{R}$ is one-to-one and thus has an inverse.*

Example 4.37. *Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is continuous and bijective. Prove its inverse is continuous.*

Theorem 4.10. *Suppose $f : D \to \mathbb{R}$ is monotone. If its image is an interval, then f is continuous.*

Corollary 4.2. *Suppose I is an interval and $f : I \to \mathbb{R}$ is a monotone function. Then, f is continuous if and only if its image is an interval.*

Theorem 4.11. *Let I be an interval, and let $f : I \to \mathbb{R}$ be a strictly monotone function. Then, its inverse $f^{-1} : f(I) \to \mathbb{R}$ is continuous.*

Example 4.38. *For every positive integer n, the function $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = x^n$ is strictly increasing. Therefore, its inverse is continuous.*

Definition 4.10. Given a rational number $r = \dfrac{m}{n}$ with $m, n \in \mathbb{Z}$ and $n > 0$, and a positive number x we define

$$x^r = (x^m)^{1/n},$$

where $g(x) = x^{1/n}$ is the inverse of the function $f(x) = x^n$.We also define $0^r = 0$ if $r \neq 0$.

Notation. $x^{1/n}$ is also denoted by $\sqrt[n]{x}$.

Theorem 4.12. *For every two rationals r, s and nonnegative real numbers x, y we have*

$$x^r x^s = x^{r+s},\ (x^r)^s = x^{rs},\ \textit{and}\ (xy)^r = x^r y^r.$$

Theorem 4.13. *For every rational number $r \neq 0$ the function*

$$f : [0, \infty) \to \mathbb{R},\ \textit{given by}\ f(x) = x^r\ \textit{for all}\ x \geq 0$$

is continuous.

4.8.1 More Examples

Example 4.39. *Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x + x^3$.*

(a) Prove that f has an inverse and its inverse is continuous.

(b) Evaluate $\lim_{x\to 2} f^{-1}(x)$.

Solution. (a) Note that if $x < y$, then $x^3 + x < y + y^3$, by properties of inequality. Thus, $f(x) < f(y)$, which implies $f(x) \neq f(y)$. Therefore, f is one-to-one, which proves f has an inverse. By a Theorem since f is continuous and its domain is an interval, f^{-1} is also continuous.
(b) Since f^{-1} is continuous, $\lim_{x\to 2} f^{-1}(x) = f^{-1}(2)$. Now, note that $f(1) = 1 + 1^3 = 2$ implies $f^{-1}(2) = 1$, which shows $\lim_{x\to 2} f^{-1}(x) = 1$. □

Example 4.40. *Suppose n is an odd positive integer. Prove that the inverse function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^n$ is continuous.*

Solution. First, note that by properties of inequalities if $x < y$ then $x^n < y^n$ and thus f is strictly increasing. Furthermore, f as a polynomial is continuous. We will now show f is onto. Let $c \in \mathbb{R}$ and choose $N \in \mathbb{N}$ for which $N > c$. Thus, $N^n \geq N > c$ and thus $f(N) > c$. Now assume $M \in \mathbb{N}$ with $M > -c$. This implies $M^n > -c$ and thus $(-M)^n < c$ or $f(-M) < c$. Therefore,

$$f(-M) < c < f(N).$$

By the Intermediate Value Theorem, there is $x \in \mathbb{R}$ for which $x^n = c$ or f, i.e. onto. This means the image of f is an interval. Thus, by Theorem 4.11, f^{-1} is continuous. □

Example 4.41. *Suppose I is an interval and $f : I \to \mathbb{R}$ is monotone. Assume in addition that for every $c \in \mathbb{R}$ the equation $f(x) = c$ has finitely many solutions. Prove that f is strictly monotone.*

Solution. Suppose f is increasing, and on the contrary assume f is not strictly increasing. Therefore, there are $a < b$ for which $f(a) = f(b)$. If $x \in (a, b)$, then we have

$$f(a) \leq f(x) \leq f(b).$$

Since $f(a) = f(b)$ we have $f(x) = f(a)$. Therefore the equation $f(x) = f(a)$ has infinitely many solutions for x, which is a contradiction. The proof for when f is decreasing is similar. □

Example 4.42. *Find the image of each function:*

(a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \dfrac{1}{x^2 + 1}$.

(b) $f : (1, 2) \to \mathbb{R}$ given by $f(x) = \dfrac{1}{x^2 - 3x}$

Solution. (a) We note that $x^2 \geq 0$. Thus $x^2 + 1 \geq 1$, i.e. $f(x) \leq 1$. Note that $f(0) = 1$.
If $\epsilon \in (0, 1]$ then there is $n \in \mathbb{N}$ for which $\epsilon > 1/n$. This implies

$$\epsilon > \frac{1}{n} \geq \frac{1}{n^2} > \frac{1}{n^2+1} = f(n).$$

Therefore,

$$f(n) < f(\epsilon) \leq f(0).$$

By the Intermediate Value Theorem $f(x) = \epsilon$ for some x, and hence, the image of f contains the interval $(0, 1]$. Since $1/(x^2 + 1) > 0$, the image is $(0, 1]$.
(b) $x^2 - 3x = (x - 3/2)^2 - 9/4$. Using the given assumption and properties of inequalities we obtain the following:

$$1 < x < 2 \Rightarrow -\frac{1}{2} < x - \frac{3}{2} < \frac{1}{2} \Rightarrow 0 \leq (x - 3/2)^2 < \frac{1}{4} \Rightarrow -\frac{9}{4} \leq x^2 - 3x < -2 \Rightarrow -\frac{1}{2} < f(x) \leq -\frac{4}{9}.$$

Therefore, the image of this function is a subset of $(-1/2, -4/9]$. Note that $f(3/2) = -4/9$. Furthermore, if a_n is a sequence approaching 1, $f(a_n)$ approaches $f(1) = -1/2$. Thus, by the Intermediate Value Theorem the image is precisely $(-1/2, -4/9]$. □

Example 4.43. *Suppose $F : (0, \infty) \to \mathbb{R}$ is a function that satisfies*

$$F(ab) = F(a) + F(b), \text{ for all } a, b > 0.$$

Prove that for every $a > 0$ and every rational number r we have

$$F(a^r) = rF(a).$$

Solution. First note that $F(1) = F(1) + F(1)$ and hence $F(1) = 0$. Thus, $F(a^0) = F(1) = 0 = 0F(a)$, which proves the statement for $r = 0$.

Also note that

$$F(1) = F(a^r a^{-r}) = F(a^r) + F(a^{-r}) \Rightarrow 0 = F(a^r) + F(a^{-r}) \Rightarrow F(a^{-r}) = -F(a^r).$$

Thus, it is enough to prove the statement for when r is a positive rational number.

We will now prove the statement by induction, when r is a positive integer.
Basis step. $F(a^1) = F(a) = 1F(a)$.
Inductive step. Suppose $F(a^r) = rF(a)$ for some positive integer r and all $a > 0$. We have

$$F(a^{r+1}) = F(a^r a) = F(a^r) + F(a) = rF(a) + F(a) = (r+1)F(a),$$

by inductive hypothesis and the assumption.

If $r = m/n$, where m and n are positive integers, then we have

$$nF(a^r) = F((a^r)^n) = F(a^m) = mF(a) \Rightarrow F(a^r) = mF(a)/n \Rightarrow F(a^r) = rF(a),$$

as desired. □

4.9 Exercises

All students are expected to do all of the exercises listed in the following two sections: Problems for Grading and Problems for Practice. You are only required to submit the ones in the first section for grading.

Challenge Problems are optional.

4.9.1 Problems for Grading

The following problems are due Monday 6/14/2021 before the class starts.

Exercise 4.21 (10 pts)**.** *Page 73, Problem 2.*

Exercise 4.22 (15 pts)**.** *Page 74, Problem 7.*

Exercise 4.23 (10 pts)**.** *Page 74, Problem 8.*

Exercise 4.24 (10 pts)**.** *Page 81, Problem 13.*

4.9.2 Problems for Practice

Page 73: 5, 6.

Pages 80-81: 1, 3, 6, 14, 15.

4.9.3 Summary

- If the domain of a strictly monotone function is an interval, then its inverse will be continuous.
- Often, to show a continuous function has an inverse we show it is strictly monotone.

4.10 Limits

Definition 4.11. We say a real number x_0 is a **limit point** of a set D if there is a sequence x_n in $D \backslash \{x_0\}$ that converges to x_0.

Example 4.44. 0 *and* 1 *are limit points of* $(0, 1)$.

Definition 4.12. Assume $f : D \to \mathbb{R}$ is a function, ℓ is a real number, and x_0 is a limit point of D. We say $\lim_{x \to x_0} f(x) = \ell$ if whenever x_n is a sequence in $D \backslash \{x_0\}$ that converges to x_0, the sequence $f(x_n)$ converges to ℓ.

We say $\lim_{x \to x_0} f(x) = \infty$ if whenever x_n is a sequence in $D \backslash \{x_0\}$ that converges to x_0, then $f(x_n) \to \infty$.

If D is unbounded above, then we say $\lim_{x \to \infty} f(x) = \ell$ if whenever x_n is a sequence in D that diverges to ∞, the sequence $f(x_n)$ converges to ℓ.

Similarly we can define

$$\lim_{x \to x_0} f(x) = -\infty, \lim_{x \to -\infty} f(x) = \ell, \lim_{x \to \infty} f(x) = \infty,$$

$$\lim_{x \to \infty} f(x) = -\infty, \lim_{x \to -\infty} f(x) = \infty, \text{ and } \lim_{x \to -\infty} f(x) = -\infty.$$

In each case if such a real number ℓ does not exist and the limit is not $\pm\infty$ we say the limit does not exist. In each case, when the limit is a real number ℓ, we say the function **converges** to ℓ. Otherwise, we say the function **diverges**.

Remark 4.1. It is not difficult to see f is continuous at x_0 if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.

Remark 4.2. Suppose $f, g : D \to \mathbb{R}$ are two functions and x_0 is a limit point of D. If $f(x) = g(x)$ for all $x \in D \backslash \{x_0\}$, then

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x), \text{ if } \lim_{x \to x_0} f(x) \text{ exists.}$$

Example 4.45. *Prove both of the following:*

(a) $\lim_{x \to 3} \sqrt{\dfrac{x^2+1}{x+1}} = \sqrt{\dfrac{5}{2}}$.

(b) $\lim_{x \to 2} \dfrac{x^2-4}{x-2} = 4$.

Theorem 4.14. *Assume $f, g : D \to \mathbb{R}$ are two functions and x_0 is a limit point of D. Assume*

$$\lim_{x \to x_0} f(x) = A, \text{ and } \lim_{x \to x_0} g(x) = B.$$

Then,

(a) $\lim_{x \to x_0} (f+g)(x) = A + B$.

(b) $\lim_{x \to x_0} (fg)(x) = AB$.

(c) $\lim_{x \to x_0} (f/g)(x) = A/B$, *if* $B \neq 0$.

Theorem 4.15. *Suppose D, U are two subsets of $\mathbb{R}$. Let x_0 be a limit point of D, and y_0 be a limit point of U. Suppose $f : D \to \mathbb{R}$ and $g : U \to \mathbb{R}$ are such that*

$$\lim_{x \to x_0} f(x) = y_0, \text{and} \lim_{y \to y_0} g(y) = \ell.$$

Assume

$$f(D\backslash\{x_0\}) \subseteq U\backslash\{y_0\}.$$

Then, $\lim_{x\to x_0} g \circ f(x) = \ell$.

Example 4.46. *Assume* $f : \mathbb{R} \to \mathbb{R}$ *is continuous at* 1 *and that* $f(1) = 3$. *Evaluate*

$$\lim_{x\to 0} f(x^2 + 1).$$

4.10.1 More Examples

Example 4.47. *Evaluate*

$$\lim_{x\to 1} \frac{x^3 - 3x + 2}{x - 1}.$$

Solution. Note that the numerator can be written as

$$x^3 - 3x + 2 = (x - 1)(x^2 + x - 2).$$

Therefore, for every $x \neq 1$ we have

$$\frac{x^3 - 3x + 2}{x - 1} = x^2 + x - 2.$$

By Remark 4.2

$$\lim_{x\to 1} \frac{x^3 - 3x + 2}{x - 1} = \lim_{x\to 1} x^2 + x - 2.$$

Since x^2+x-2 is a polynomial, it is continuous and thus the above limit is equal to $1^2+1-2 = 0$. Therefore, the answer is 0. □

Example 4.48. *Let* D *be a subset of* $\mathbb{R}$, *and* U *be the set consisting of all limit points of* D. *Prove that* U *is a closed subset of* $\mathbb{R}$.

Solution. Suppose x_n is a sequence in U that converges to a real number x_0. We need to show $x_0 \in U$. By definition of limit, for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that if $n \geq N$ then $|x_n - x_0| < \epsilon$. If $x_N = x_0$ then $x_0 \in U$, as desired. Otherwise, choose

$$\delta = \min(|x_N - x_0|, \epsilon - |x_N - x_0|).$$

Note that $\delta > 0$. Since x_N is a limit point of D, there is an element $z \in D$ with $|x_N - z| < \delta$. We see that $z \neq x_0$ since $|x_N - z| < |x_N - x_0|$. Also,

$$|x_0 - z| \leq |x_0 - x_N| + |x_N - z| < |x_0 - x_N| + \delta \leq |x_0 - x_N| + \epsilon - |x_0 - x_N| = \epsilon.$$

Thus, $|x_0 - z| < \epsilon$. Therefore, every neighborhood of x_0 contains an element of $D\backslash\{x_0\}$. Thus, x_0 is a limit point of D, i. e. $x_0 \in U$. □

4.11 Exercises

All students are expected to do all of the exercises listed in the following two sections: Problems for Grading and Problems for Practice. You are only required to submit the ones in the first section for grading.

Challenge Problems are optional.

4.11.1 Problems for Grading

The following problems are due Tuesday 6/15/2021 before the class starts.

Exercise 4.25 (10 pts)**.** *Find each of the following limits:*

(a) $\displaystyle\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x-1}$.

(b) $\displaystyle\lim_{x\to 0} \frac{x+\sqrt{x}}{\sqrt{x+1}}$.

Exercise 4.26 (10 pts)**.** *Page 85, Problem 5.*

Exercise 4.27 (10 pts)**.** *Page 86, Problem 9.*

4.11.2 Problems for Practice

Page 86: 10, 11, 12.

Definition 4.13. We say a function $f(x)$ approaches a real number a as $x \to \infty$, written as

$$\lim_{x\to\infty} f(x) = a,$$

if whenever a sequence a_n approaches ∞, the sequence $f(a_n)$ approaches a.

Exercise 4.28. *Let $f : D \to \mathbb{R}$ be a function and x_0 be a limit point of D. Prove that if $\lim_{x\to x_0} |f(x_0)| = 0$, then $\lim_{x\to x_0} f(x_0) = 0$.*

Exercise 4.29. *Suppose $f, g : D \to \mathbb{R}$ are two functions and x_0 is a limit point of D. Suppose f is bounded and $\lim_{x\to x_0} g(x) = 0$. Prove that*

$$\lim_{x\to x_0} (fg)(x) = 0.$$

Exercise 4.30. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function for which $f(x) > 0$ for all x. Assume*

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0.$$

Prove that f attains a maximum value.

4.11.3 Challenge Problems

Exercise 4.31. *Suppose r, s are nonzero rational numbers. Prove that*

$$\lim_{x\to 1} \frac{x^r - 1}{x^s - 1} = \frac{r}{s}.$$

4.11.4 Summary

- To show $\lim_{x \to a} f(x) = \ell$, start with a sequence x_n that converges to a, and that $x_n \neq a$ for all n, and then prove $\lim_{n \to \infty} f(x_n) = \ell$.
- Typical properties of limits hold: Limit of sum is sum of limits; limit of product is product of limits; and limit of ratio of two functions in ratio of their limits.

Chapter 5

Differentiation

5.1 Basic Properties

Definition 5.1. A **neighborhood** of a real number x_0 is an open interval containing x_0.

Remark. If $f(x) = g(x)$ for all $x \neq x_0$ that lie in a neighborhood of x_0, then

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x).$$

Definition 5.2. Let I be a neighborhood of a real number x_0, and let $f : I \to \mathbb{R}$ be a function. We say f is **differentiable at** x_0 if the following limit converges:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

This limit is denoted by $f'(x_0)$. We say f is differentiable over I if it is differentiable at all points of I.

Example 5.1. *Find the derivative of each of the following functions on its domain:*

(a) $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = ax + b$, *where* $a, b \in \mathbb{R}$ *are constants.*

(b) $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = x^n$, *where* n *is a nonzero integer.*

(c) $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = |x|$.

Theorem 5.1. *Let* I *be a neighborhood of* x_0 *and* $f : I \to \mathbb{R}$ *be differentiable at* x_0. *Then,* f *is continuous at* x_0.

Theorem 5.2 (Properties of Derivatives)**.** *Let* I *be a neighborhood of* x_0 *and* $f, g : I \to \mathbb{R}$ *be differentiable at* x_0. *Then,* $f + g$ *and* fg *are both differentiable at* x_0 *and*

$$(f + g)'(x_0) = f'(x_0) + g'(x_0), \text{ and } (fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

If $g(x) \neq 0$ *for all* $x \in I$, *then*

$$(f/g)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}.$$

Corollary 5.1. *All rational functions are differentiable over their domains.*

5.1.1 More Examples

Example 5.2. *Find the derivative of* $f(x) = \dfrac{1+x}{x^2+1}$ *using the definition.*

Solution. By definition,

$$
\begin{aligned}
f'(x_0) &= \lim_{x\to x_0} \frac{\dfrac{1+x}{x^2+1} - \dfrac{1+x_0}{x_0^2+1}}{x - x_0} \\
&= \lim_{x\to x_0} \frac{1 + x + x_0^2 + xx_0^2 - (1 + x_0 + x^2 + x_0x^2)}{(x-x_0)(x^2+1)(x_0^2+1)} \\
&= \lim_{x\to x_0} \frac{(x-x_0)(1 - x - x_0 - xx_0)}{(x-x_0)(x^2+1)(x_0^2+1)} \\
&= \lim_{x\to x_0} \frac{1 - x - x_0 - xx_0}{(x^2+1)(x_0^2+1)} = \frac{1 - 2x_0 - x_0^2}{(x_0^2+1)^2},
\end{aligned}
$$

since rational functions are continuous. Therefore, $f'(x) = \dfrac{1-2x-x^2}{(x^2+1)^2}$. $\square$

Example 5.3. *Suppose* $f, g : \mathbb{R} \to \mathbb{R}$ *are functions such that both* f *and* fg *are differentiable everywhere. Is it true that* g *must be differentiable?*

Sketch. If f were nonzero we could write $g = fg/f$, and thus g would be differentiable. However it is possible that f has some roots. Perhaps we could choose $g(x) = |x|$ which is not differentiable and find a counterexample. That yields the following solution:

Solution. This is false. Let $f(x) = x$ and $g(x) = |x|$. Note that f is continuous everywhere and g is not continuous at $x = 0$. Note also that $(fg)(x) = x|x| = x^2$ if $x > 0$ and thus it is differentiable. Similarly $(fg)(x) = -x^2$ if $x < 0$ and thus it is differentiable at x. At $x = 0$ we have

$$(fg)'(0) = \lim_{x\to 0} \frac{x|x| - 0}{x - 0} = \lim_{x\to 0} |x| = 0,$$

since x and $|x|$ are continuous. $\square$

5.2 Summary

- To evaluate $f'(x_0)$ using the definition, we need to calculate $\displaystyle\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$.
- Differentiability implies continuity but not the other way around!
- Sums, products, and quotient rules for derivatives are proved.

5.3 Chain Rule and Derivative of Inverse Functions

Example 5.4. *Let x_0 be a real number. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0^3. Prove that the derivative of $f(x^3)$ at x_0 is $3x_0^2 f'(x_0^3)$.*

Theorem 5.3 (The Chain Rule)**.** *Let I be a neighborhood of a real number x_0 and $f : I \to \mathbb{R}$ be differentiable at x_0. Let J be an open interval for which $f(I) \subseteq J$, and suppose $g : J \to \mathbb{R}$ is differentiable at $f(x_0)$. Then, $g \circ f : I \to \mathbb{R}$ is differentiable at x_0, and*

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

Example 5.5. *Consider the function $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = x^2$. Find the derivative of the inverse function $f^{-1} : (0, \infty) \to (0, \infty)$ given by $f^{-1}(x) = \sqrt{x}$.*

Theorem 5.4 (Derivative of Inverse)**.** *Let I be a neighborhood of a number x_0 and $f : I \to \mathbb{R}$ be a strictly monotone and continuous function. Suppose f is differentiable at x_0, and that $f'(x_0) \neq 0$. Then, $f^{-1} : f(I) \to \mathbb{R}$ is differentiable at $y_0 = f(x_0)$ and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Theorem 5.5. *Let r be a rational number. Then, the derivative of the function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = x^r$ is $f'(x) = rx^{r-1}$.*

Example 5.6. *Let $f : (1, 3) \to \mathbb{R}$ be a function defined by $f(x) = x^3 - x$. Prove that f has a inverse, and evaluate $(f^{-1})'(6)$.*

5.3.1 More Examples

Example 5.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x^3 + 2x$. Show that*

(a) f has an inverse.

(b) The domain and range of f^{-1} are both $\mathbb{R}$

(c) f^{-1} is differentiable over $\mathbb{R}$.

(d) Evaluate $(f^{-1})'(3)$.

Solution. (a) Note that by properties of inequalities if $x < y$, then $x^3 < y^3$. Therefore $x^3 + 2x < y^3 + 2y$, which shows f is strictly increasing. Therefore f has an inverse.
(b) We will show that f is onto. For any $c \in \mathbb{R}$, by the A.P there is $n \in \mathbb{N}$ for which $c < n$. Thus $c < n(n^2 + 2) = f(n)$. Similarly, there is $m \in \mathbb{N}$ for which $m > -c$. Thus $c > -m > -m(m^2 + 2) = f(-m)$. By the IVT, there is x between $-m$ and n for which $f(x) = c$. Therefore f is onto. This implies the domain of f and range of f are both $\mathbb{R}$. Thus, the range and the domain of f^{-1} are also both $\mathbb{R}$.
(c) Since f is differentiable over $\mathbb{R}$ and $f'(x) = 3x^2 + 2 \neq 0$, by a theorem, f^{-1} is differentiable everywhere.
(d) Note that $f(1) = 3$, which implies $f^{-1}(3) = 1$. By a theorem $(f^{-1})'(3) = \dfrac{1}{f'(f^{-1}(3))} = \dfrac{1}{f'(1)} = \dfrac{1}{5}$. □

Example 5.8. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function that has a differentiable inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$. Prove that $f'(x) \neq 0$ for all $x \in \mathbb{R}$.*

Solution. We know $f^{-1}(f(x)) = x$. Differentiating both sides we obtain

$$(f^{-1})'(f(x))f'(x) = 1 \Rightarrow f'(x) \neq 0.$$

□

5.4 Exercises

All students are expected to do all of the exercises listed in the following two sections: Problems for Grading and Problems for Practice. You are only required to submit the ones in the first section for grading.

Challenge Problems are optional.

5.4.1 Problems for Grading

The following problems are due Thursday 6/17/2021 before the class starts.

Exercise 5.1 (15 pts)**.** *Page 93, Problem 1.*

Exercise 5.2 (10 pts)**.** *Using the definition of derivative evaluate the derivative of $\sqrt{x+1}$ for all $x > -1$.*

Exercise 5.3 (10 pts)**.** *Page 95, Problem 10.*

Exercise 5.4 (10 pts)**.** *Page 100, Problem 2.*

Exercise 5.5 (10 pts)**.** *Page 101, Problem 8.*

5.4.2 Practice Problems

Exercise 5.6. *Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are functions that satisfy both of the following:*

1. *$f + g$ and fg are differentiable, and*

2. *$f(x) < g(x)$ for all $x \in \mathbb{R}$.*

Prove that both f and g must be differentiable over $\mathbb{R}$.

Pages 94-96: 4, 5, 7, 8, 9, 11, 13, 16, 19.

Pages 100-101: 1, 3, 6, 9.

5.4.3 Summary

- The Chain Rule states that $(f \circ g)'(x) = f'(g(x))g'(x)$.
- If $y_0 = f(x_0)$ and $f'(x_0) \neq 0$. Then $(f^{-1})'(y_0) = \dfrac{1}{f'(x_0)}$.

5.5 The Mean Value Theorem

Definition 5.3. A number x_0 in the domain of a function $f : D \to \mathbb{R}$ is said to be a **local minimizer** if there is a neighborhood I of x_0 for which

$$f(x_0) \leq f(x) \text{ for all } x \in I \cap D.$$

Similarly we say x_0 is a **local maximizer** if there is a neighborhood I of x_0 for which

$$f(x_0) \geq f(x) \text{ for all } x \in I \cap D.$$

We say x_0 is a **local extreme point** if it is either a local maximizer or a local minimizer.

Theorem 5.6. *Suppose I is a neighborhood of a real number x_0, and x_0 is a local extreme point of $f : I \to \mathbb{R}$. If f is differentiable at x_0, then $f'(x_0) = 0$.*

Theorem 5.7 (The Rolle's Theorem)**.** *Suppose $f : [a, b] \to \mathbb{R}$ is continuous, and its restriction to (a, b) is differentiable. If $f(a) = f(b)$, then there is $c \in (a, b)$ for which $f'(c) = 0$.*

Example 5.9. *Prove that the equation $x^3 + 3x + 1 = 0$ has a unique solution.*

Theorem 5.8 (The Mean Value Theorem)**.** *Suppose $f : [a, b] \to \mathbb{R}$ is continuous, and its restriction to (a, b) is differentiable. Then there is $c \in (a, b)$ for which*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 5.9 (The Identity Criterion)**.** *Let $f, g : I \to \mathbb{R}$ be differentiable functions, where I is an open interval. Then, $f'(x) = g'(x)$ for all $x \in I$ if and only if $f(x) - g(x) = C$ is a constant function over I.*

Example 5.10. *Suppose $f : [0, 1] \to \mathbb{R}$ is a continuous function for which $f'(x) = x^2 - 2x$ for all $x \in (0, 1)$. Find $f(x)$.*

Theorem 5.10 (The Monotonicity Criterion)**.** *Suppose $f : I \to \mathbb{R}$ is a differentiable function over an open interval I. Then,*

(a) f is increasing if and only if $f'(x) \geq 0$ for all $x \in I$.

(b) f is decreasing if and only if $f'(x) \leq 0$ for all $x \in I$.

(c) If $f'(x) > 0$ for all $x \in I$, then f is strictly increasing.

(d) If $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing.

The following example shows the converses of parts (c) and (d) of the above theorem are not valid.

Example 5.11. *The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is strictly increasing but $f'(0) = 0$.*

Definition 5.4. Let I be an open neighborhood of x_0. The n-th derivative of f at x_0 is defined recursively as follows:

(a) The 0-th derivative of f at x_0 is $f(x_0)$.

(b) The n-th derivative of f is defined to be the derivative of the $(n-1)$-th derivative of f

The n-th derivative of f is denoted by $f^{(n)}$. A function whose n-th derivative is continuous is called n **times continuously differentiable**.

Theorem 5.11 (Second Derivative Test). *Suppose I is a neighborhood of a real number x_0 and assume $f : I \to \mathbb{R}$ is twice differentiable. Assume $f'(x_0) = 0$. Then,*

(a) If $f''(x_0) > 0$, then x_0 is a local minimizer for f.

(b) If $f''(x_0) < 0$, then x_0 is a local maximizer for f.

Note that if $f''(x_0) = 0$, the above test is inconclusive, i.e. x_0 could be a maximizer, a minimizer or neither.

Example 5.12. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x^4 + 4x^3 + 1$.*

(a) Find and classify all local extreme points of f.

(b) Find the range of f.

5.5.1 Warm-ups

Example 5.13. *Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x^3 + 4x$ is one-to-one.*

Solution. We have:

$$f'(x) = 6x^2 + 4 \geq 4 > 0, \text{ by the trivial inequality}$$

Therefore, by the Monotonicity Criterion f is strictly increasing and hence, it is one-to-one. □

Example 5.14. *Find all values of c that satisfy the conclusion of the Mean Value Theorem for $f : [0, 1] \to \mathbb{R}$ given by $f(x) = x^3 - 2x$ with $a = 0, b = 1$.*

Solution. $f'(x) = 3x^2 - 2$. We are looking for all $c \in (0, 1)$ for which

$$3c^2 - 2 = \frac{f(0) - f(1)}{0 - 1} = \frac{0 - (-1)}{-1} = -1 \Rightarrow c^2 = 1/3 \Rightarrow c = 1/\sqrt{3}.$$

□

5.5.2 More Examples

Example 5.15. *Suppose $f : [a, b] \to \mathbb{R}$ is differentiable over (a, b) and continuous over $[a, b]$. Suppose in addition that $f'(x) \geq 0$ for all $x \in (a, b)$. Suppose the set $\{x \in (a, b) \mid f'(x) = 0\}$ is finite. Prove that f is strictly increasing.*

Solution. By the Monotonicity Criterion, f is increasing over (a, b). Consider $x \in (a, b)$. Note that since f is continuous, $f(a + 1/n) \to f(a)$ and since after some point $a + 1/n < x$, we have $f(a + 1/n) \leq f(x)$, and hence $f(a) \leq f(x)$. Similarly $f(x) \leq f(b)$. Thus, f is increasing over $[a, b]$.

We will now show f is strictly increasing. Suppose on the contrary $f(x_1) = f(x_2)$ for some $x_1 < x_2$. For every $z \in [x_1, x_2]$, we have $f(x_1) \leq f(z) \leq f(x_2) = f(x_1)$. Therefore, f is constant over $[x_1, x_2]$, which by the Identity Criterion it implies $f'(z) = 0$ for all $z \in (x_1, x_2)$. Therefore, $\{x \in (a, b) \mid f'(x) = 0\}$ contains (x_1, x_2) which shows this set is not finite. This contradiction shows f is strictly increasing. □

Example 5.16. *Let a, b, c be three real numbers for which $a^2 - 3b < 0$. Prove that there is a unique real number x for which $x^3 + ax^2 + bx + c = 0$.*

Solution. Let $f(x) = x^3 + ax^2 + bx + c$. By the Archemidean property there is a natural number n for which $n > \max(3|a|, 3|b|, 3|c|)$. Therefore,

$$f(n) = n^3 + an^2 + bn + c = n^3(1 + \frac{a}{n} + \frac{b}{n^2} + \frac{c}{n^3}) \geq n^3(1 - \frac{|a|}{n} - \frac{|b|}{n^2} - \frac{|c|}{n^3}) > n^3(1 - \frac{1}{3} - \frac{1}{3} - \frac{1}{3}) = 0.$$

Similarly, there is a natural number m for which $m^3 - am^2 + bm - c > 0$. Therefore,

$$f(-m) = (-m)^3 + a(-m)^2 + b(-m) + c) = -(m^3 - am^2 + bm - c) < 0.$$

By the Intermediate Value Theorem, there is $x \in \mathbb{R}$ for which $f(x) = 0$. This proves the existenc of a root.

For the uniqueness, suppose there are two roots for $f(x) = 0$. By the Rolle's Theorem, there must exist z between the roots for which $f'(z) = 0$. Therefore, $3z^2 + 2az + b = 0$, which implies the discriminant of this quadratic is non-negative. Therefore, $(2a)^2 - 12b \geq 0$, which contradicts $a^2 - 3b < 0$. □

Example 5.17. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Let ϵ be a positive real number for which $f'(x) \geq \epsilon$ for all $x \in \mathbb{R}$. Prove that $\lim_{x\to\infty} f(x) = \infty$ and that $\lim_{x\to-\infty} f(x) = -\infty$.*

Solution. Let $x > 0$. By the Mean Value Theorem, there is $c \in (0, x)$ for which $f'(c) = \dfrac{f(x) - f(0)}{x}$. Thus, $f(x) - f(0) \geq \epsilon x$, which implies $f(x) \geq f(0) + \epsilon x$.

Let $M > 0$. If $x > \max(0, \dfrac{M - f(0)}{\epsilon})$, then $f(x) \geq f(0) + \epsilon \cdot \dfrac{M - f(0)}{\epsilon} = M$. This proves $\lim_{x\to\infty} f(x) = \infty$.

The other part is similar. □

Example 5.18. *Suppose $I = (a, b)$ is a neighborhood of a real number x_0. Let $f : I \to \mathbb{R}$ be a continuous function for which f is differentiable over (a, x_0) and also over (x_0, b). Prove that if $\lim_{x\to x_0} f'(x) = L$ exists, then f is differentiable at x_0 and $f'(x_0) = L$.*

Solution. By definition $f'(x_0) = \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$. Let x_n be a sequence in $I \backslash \{x_0\}$ that converges to x_0. By the Mean Value Theorem, there is z_n strictly between x_0 and x_n for which $\dfrac{f(x_n) - f(x_0)}{x_n - x_0} = f'(z_n)$. Note that $|x_n - x_0| > |z_n - x_0|$ since z_n is strictly between x_n and x_0. By the Comparison Lemma, $z_n \to x_0$. This implies $\lim_{n \to \infty} f'(z_n) = L$. Therefore $\lim_{n \to \infty} \dfrac{f(x_n) - f(x_0)}{x_n - x_0} = L$. Therefore by the definition of limit, $\lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = L$, which proves $f'(x_0) = L$. $\square$

Example 5.19. *Suppose there is a constant C for which the function $f : \mathbb{R} \to \mathbb{R}$ satisfies*

$$f(x) - f(y) \leq C(x - y)^2, \text{ for all } x, y \in \mathbb{R}.$$

Prove that f is a constant function.

Solution. Note that

$$|f(x) - f(y)| = f(x) - f(y) \text{ or } f(y) - f(x) \leq C(x - y)^2 \text{ or } C(y - x)^2 = C(x - y)^2 \Rightarrow |f(x) - f(y)| \leq C(x - y)^2.$$

Dividing both sides by $|x - y|$, where $x \neq y$ we obtain the following:

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq C|x - y|.$$

Now, let y_n be a sequence in $\mathbb{R} \backslash \{y\}$ that converges to y. We have

$$\left| \frac{f(y_n) - f(y)}{y_n - y} \right| \leq C|y_n - y|.$$

By the Comparison Lemma, we obtain

$$\lim_{n \to \infty} \frac{f(y_n) - f(y)}{y_n - y} = 0.$$

Therefore, by definition of limit, $f'(y) = 0$. Since this holds for all $y \in \mathbb{R}$,the function f is constant by the Identity Criterion. $\square$

Example 5.20. *Let $n \geq 1$ be an integer. Prove that every polynomial of degree n has at most n distinct roots.*

Solution. We will prove this by induction on n.

Basis step. Any polynomial of degree 1 is of the form $p(x) = ax + b$ with $a \neq 0$ and has precisely one root $x = -b/a$.

Inductive step. Let $p(x)$ be a polynomial of degree n with $n \geq 2$. Suppose on the contrary $p(x)$ has more than n roots: $x_1 < x_2 < \cdots < x_m$ with $m > n$. Applying the Rolle's Theorem to the intervals (x_i, x_{i+1}) we obtain a real number $y_i \in (x_i, x_{i+1})$ for which $p'(y_i) = 0$. Therefore, $p'(x)$ has at least $m - 1$ roots. Note that the degree of $p'(x)$ is $n - 1$ and thus, by inductive hypothesis $m - 1 \leq n - 1$ and thus $m \leq n$. This completes the proof. $\square$

Example 5.21. *Is the following modification of the Mean Value Theorem true?*

If $f : [a, b] \to \mathbb{R}$ is continuous and its restriction to (a, b) is differentiable, then for every $c \in (a, b)$ there are distinct $x, y \in (a, b)$ for which

$$f'(c) = \frac{f(x) - f(y)}{x - y}.$$

Solution. This is false. Consider $f : [-1, 1]$ defined by $f(x) = x^3$. We see that f is differentiable over $\mathbb{R}$ and that $f'(0) = 0$. However, for $x \neq y$ we have

$$\frac{f(x) - f(y)}{x - y} = \frac{x^3 - y^3}{x - y} = 0 \Rightarrow x^3 = y^3 \Rightarrow x = y.$$

This is a contradiction, since $x \neq y$. □

Example 5.22. *By three examples show that when $f''(x_0) = 0$, the second derivative test is truly inconclusive.*

Solution. Let $f(x) = x^3, g(x) = x^4, h(x) = -x^4$. Note that

$$f'(0) = g'(0) = h'(0) = 0, \text{ and that } f''(0) = g''(0) = h''(0) = 0.$$

However, 0 is a minimzer for g, a maximizer for h and neither for f, since every neighborhood of 0 contains some negative values of $f(x)$ and some positive values for $f(x)$. □

5.6 Exercises

All students are expected to do all of the exercises listed in the following two sections: Problems for Grading and Problems for Practice. You are only required to submit the ones in the first section for grading.

Challenge Problems are optional.

5.6.1 Problems for Grading

The following problems are due Tuesday 6/22/2021 before the class starts.

Exercise 5.7 (10 pts)**.** *Page 108, Problem 6.*

Exercise 5.8 (10 pts)**.** *Page 109, Problem 15.*

Exercise 5.9 (10 pts)**.** *Page 110, Problem 20.*

Exercise 5.10 (20 pts)**.** *Let a, b be real numbers. Prove that the cubic equation*

$$x^3 + ax + b = 0$$

has precisely three solutions if and only if $4a^3 + 27b^2 < 0$.

Hint: First assume there are three solutions x_1, x_2, x_3. Then, apply the Rolle's Theorem twice. Use the fact that the cubic diverges to ∞ as $x \to \infty$ and it diverges to $-\infty$ as $x \to -\infty$. (This must be proved!)

5.6.2 Problems for Practice

Exercise 5.11. *Suppose I is an open interval and $f : I \to \mathbb{R}$ is a differentiable function for which $f' : I \to \mathbb{R}$ is bounded. Prove that f is uniformly continuous.*

Exercise 5.12. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that*

$$\lim_{x\to\infty} f'(x) = 0.$$

Prove that

$$\lim_{x\to\infty} (f(x+1) - f(x)) = 0.$$

Exercise 5.13. *Prove that the equation $x^{10} + 3x^2 - 12 = 0$ has exactly two real solutions.*

Exercise 5.14. *Prove that if $f : (a, b) \to \mathbb{R}$ has a bounded derivative, then f is also bounded.*

Pages 108-110: 1, 7, 10, 11, 14, 16, 24.

5.6.3 Challenge Problems

Exercise 5.15. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Let $a < b$ be real numbers. Prove that for every $\epsilon > 0$ there is $\delta > 0$ for which the following holds:*

$$\forall x, y \in [a, b],\ \text{If } |x - y| < \delta,\ \text{then } \left|\frac{f(x) - f(y)}{x - y} - f'(x)\right| < \epsilon.$$

Exercise 5.16. *Assume $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and that $f''(x) > 0$ for all $x \geq 0$. Suppose $f'(0) > 0$. Prove $\lim_{n\to\infty} f(n) = \infty$.*

Exercise 5.17. *Let I be an open interval and $f : I \to \mathbb{R}$ be differentiable. Prove that $f' : I \to \mathbb{R}$ satisfies the intermediate value property.*

5.6.4 Summary

- We proved the Rolle's and the Mean Value Theorems.
- The Identity Criterion is often used to show two functions differ by a constant when we have some information about their derivatives.
- To Show an equation has precisely n solution:
  - Use the IVT to show the existence of n solutions.
  - Then on the contrary assume $f(x_1) = \cdots = f(x_{n+1}) = 0$ and repeatedly apply the Rolle's Theorem to f, f', etc. to obtain a contradiction.
- $f'(x) \geq$ iff f is increasing. Similar for decreasing functions.
- If $f'(x) > 0$, then f is strictly increasing but the converse does not hold. Similar for decreasing functions.

5.7 The Cauchy Mean Value Theorem

Theorem 5.12 (The Cauchy Mean Value Theorem)**.** *Suppose the functions $f, g : [a, b] \to \mathbb{R}$ are continuous and their restrictions to the open interval (a, b) are differentiable. Moreover, assume*

$$g'(x) \neq 0 \text{ for all } x \in (a, b).$$

Then, there is a point $c \in (a, b)$ for which

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(c)}{g'(c)}.$$

Theorem 5.13. *Let I be an open interval, n be a natural number, and $f : I \to \mathbb{R}$ have n derivatives. Suppose at some point $x_0 \in I$ we know*

$$f(x_0) = f'(x_0) = \cdots = f^{(n-1)}(x_0) = 0.$$

Then, for each point $x \in I$ with $x \neq x_0$, there is a point z that lies strictly between x and x_0 for which

$$f(x) = \frac{f^{(n)}(z)}{n!}(x - x_0)^n.$$

Example 5.23. *Consider the function $f : [0, \infty) \to \mathbb{R}$ defined by*

$$f(x) = \sqrt{x+1} - 1 - \frac{1}{2}x.$$

Show that for all $x > 0$, there is $c \in (0, x)$ for which

$$f(x) = \frac{-1}{8}(c+1)^{-3/2}x^2.$$

Use this to prove

$$\sqrt{x+1} > 1 + \frac{1}{2}x - \frac{x^2}{8}, \text{ for all } x > 0.$$

5.7.1 More Examples

Example 5.24. *Find c in the Cauchy Mean Value Theorem for the functions $f, g : [0, 1] \to \mathbb{R}$ given by $f(x) = x^3$ and $g(x) = x^2 + x$.*

Solution. We are looking for $c \in (0, 1)$ for which

$$\frac{f(1) - f(0)}{g(1) - g(0)} = \frac{f'(c)}{g'(c)} \Rightarrow \frac{1 - 0}{2 - 0} = \frac{3c^2}{2c + 1} \Rightarrow 6c^2 = 2c + 1 \Rightarrow c = \frac{1 + \sqrt{7}}{6}.$$

□

Example 5.25. *Suppose $a > b > 0$ are real numbers and $f : [a, b] \to \mathbb{R}$ is continuous and its restriction to (a, b) is differentiable. Prove that there is a real number $c \in (a, b)$ for which*

$$\frac{af(b) - bf(a)}{a - b} = f(c) - cf'(c).$$

Sketch. In order to write the left hand side in the form of the Cauchy Mean Value Theorem, we divide the numerator and denominator by ab. That yields:

$$\frac{af(b) - bf(a)}{a - b} = \frac{\frac{f(b)}{b} - \frac{f(a)}{a}}{\frac{1}{b} - \frac{1}{a}}.$$

Solution. Consider the function $F(x) = \dfrac{f(x)}{x}$, and $G(x) = \dfrac{1}{x}$. Since the interval $[a, b]$ does not contain 0, both F and G are continuous over $[a, b]$ and differentiable over (a, b). Furthermore, $G'(x) = -1/x^2 \neq 0$. By the Cauchy Mean Value Theorem, there is $c \in (a, b)$ for which

$$\frac{F(a) - F(b)}{G(a) - G(b)} = \frac{F'(c)}{G'(c)} \Rightarrow \frac{\frac{f(a)}{a} - \frac{f(b)}{b}}{\frac{1}{a} - \frac{1}{b}} = \frac{\frac{cf'(c) - f(c)}{c^2}}{-\frac{1}{c^2}} \Rightarrow \frac{bf(a) - af(b)}{b - a} = f(c) - cf'(c).$$

Negating the numerator and denominator of the left hand side yields the results. ☐

Example 5.26. *Is the following stronger version of the Cauchy Mean Value Theorem true?*

"Suppose the functions $f, g : [a, b] \to \mathbb{R}$ are continuous and their restrictions to the open interval (a, b) are differentiable. Moreover, assume $g'(x) \neq 0$ for all $x \in (a, b)$. Then, there is a point $c \in (a, b)$ for which

$$\frac{f(a) - f(b)}{a - b} = f'(c), \ \text{and} \ \frac{g(a) - g(b)}{a - b} = g'(c)."$$

Solution. This is not true. Define $f, g : [0, 1] \to \mathbb{R}$ by $f(x) = x^2$, $g(x) = x^3$. Suppose there is such a $c \in (0, 1)$. We should have

$$\frac{f(0) - f(1)}{0 - 1} = 2c, \ \text{and} \ \frac{g(0) - g(1)}{0 - 1} = 3c^2 \Rightarrow 1 = 2c, \ \text{and} \ 1 = 3c^2 \Rightarrow c = 1/2, \ \text{and} \ c^2 = 1/3.$$

This is a contradiction. ☐

5.8 Exercises

All students are expected to do all of the exercises listed in the following two sections: Problems for Grading and Problems for Practice. You are only required to submit the ones in the first section for grading.

Challenge Problems are optional.

5.8.1 Problems for Grading

The following problems are due Thursday 6/24/2021 before the class starts.

Exercise 5.18 (10 pts)**.** *Page 112, Problem 1.*

Exercise 5.19 (10 pts)**.** *Page 113, Problem 5.*

Exercise 5.20 (10 pts)**.** *Page 113, Problem 6.*

Exercise 5.21 (10 pts)**.** *Suppose I is a neighborhood of a real number x_0 for which the functions $f, g : I \to \mathbb{R}$ are n times differentiable at x_0. Prove that fg is also n times differentiable at x_0 and that*

$$(fg)^{(n)}(x_0) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x_0) g^{(n-k)}(x_0).$$

Hint: Use induction on n.

5.8.2 Problems for Practice

Exercise 5.22. *Suppose a function $f : \mathbb{R} \to \mathbb{R}$ has two derivatives with $f(0) = f'(0) = 0$ and $|f''(x)| \leq 1$ whenever $|x| < 1$. Prove that $|f(\frac{1}{2})| \leq \frac{1}{8}$.*

Pages 112-113: 3, 4, 7.

5.8.3 Summary

- Given two continuous functions $f, g : [a, b] \to \mathbb{R}$ that are differentiable over (a, b) for which $g'(x) \neq 0$, there is $c \in (a, b)$ for which
$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(c)}{g'(c)}.$$
- Assume f is defined over a neighborhood I of x_0 and that
$$f(x_0) = f'(x_0) = \cdots = f^{(n-1)}(x_0) = 0.$$
Then, for every point x with $x \neq x_0$, there is a point z that lies strictly between x and x_0 for which
$$f(x) = \frac{f^{(n)}(z)}{n!}(x - x_0)^n.$$
We will later see some consequences of this theorem when we talk about the Lagrange Remainder Theorem.

Chapter 6

Elementary Functions

In this section we will discuss properties of natural logarithm and exponential functions. First, we will assume the existence of the natural logarithm function in the following theorem.

Theorem 6.1. *There is a unique function* $F : (0, \infty) \to \mathbb{R}$ *for which*

$$F'(x) = \frac{1}{x}, \text{ for all } x > 0, \text{ and } F(1) = 0.$$

The uniqueness in the above theorem follows from the Identity Criterion. The existence will be proved in the next chapter.

Theorem 6.2. *Suppose* $F : (0, \infty) \to \mathbb{R}$ *is the function satisfying*

$$F'(x) = \frac{1}{x}, \text{ for all } x > 0, \text{ and } F(1) = 0.$$

Then, for all positive real numbers a, b *and every rational number* r *we have:*

(a) F *is strictly increasing, and thus one-to-one.*

(b) $F(ab) = F(a) + F(b)$.

(c) $F(a^r) = rF(a)$.

(d) For every $c \in \mathbb{R}$, *there is a unique* $x > 0$ *for which* $F(x) = c$.

Definition 6.1. The unique function F satisfying

$$F'(x) = \frac{1}{x}, \text{ for all } x > 0, \text{ and } F(1) = 0$$

is called the **natural logarithm function** and is denoted by $\ln$. We denote the inverse of this function by $\exp$. The unique positive number a that satisfies $\ln a = 1$ is denoted by e and is sometimes called the **Euler's number**.

Note that since $\frac{d(\ln x)}{dx} = \frac{1}{x} \neq 0$, the derivative of its inverse is given by

$$\frac{d(\exp x)}{dx} = \frac{1}{(\ln)'(\exp x)} = \exp x.$$

Theorem 6.3. *For every rational number r we have* $\exp(r) = e^r$.

The above theorem motivates us to define e^x as $\exp x$ for all real numbers x. Furthermore, since exp and ln are inverses of each other we have $a = \exp \ln a = e^{\ln a}$. Thus, we may want to write

$$a^x = (e^{\ln a})^x = e^{x \ln a}.$$

Note that the above will have to be turned into a definition. In order to make sure this definition is compatible with exponentiation when x is rational we need the following:

Theorem 6.4. *If a is a positive number and r is a rational number, then* $a^r = \exp(r \ln a)$.

Definition 6.2. For every positive real number a and every real number x we define a^x as $\exp(x \ln a)$.

Theorem 6.5. *For every positive constant a, and every constant b we have the following:*

(a) $\dfrac{d(a^x)}{dx} = a^x \ln a.$

(b) $\dfrac{d(x^b)}{dx} = bx^{b-1}.$

Similar to the natural logarithmic function we can classify the exponential function exp using differential equations.

Theorem 6.6. *Let c, k be two real numbers. Then, the only function $f : \mathbb{R} \to \mathbb{R}$ that satisfies*

$$f'(x) = kf(x), \text{ for all } x \in \mathbb{R}, \text{ and } f(0) = c$$

is $f(x) = ce^{kx}$.

In a similar manner one can define sine and cosine functions using differential equations. We will skip that, and assume without proof, the following properties of sine and cosine functions:

Theorem 6.7. *There are periodic differentiable functions* $\sin, \cos : \mathbb{R} \to [-1, 1]$ *that satisfy the following:*

(a) $\dfrac{d}{dx}(\sin x) = \cos x.$

(b) $\dfrac{d}{dx}(\cos x) = -\sin x.$

(c) $\sin(a + b) = \sin a \cos b + \cos a \sin b$, *for all* $a, b \in \mathbb{R}$.

(d) $\cos(a + b) = \cos a \cos b - \sin a \sin b$, *for all* $a, b \in \mathbb{R}$.

(e) $\sin(-x) = -\sin x$ *and* $\cos(-x) = \cos x$ *for all* $x \in \mathbb{R}$.

(f) $\sin 0 = 0$, *and* $\cos 0 = 1$.

6.1 Warm-ups

Example 6.1. *Solve the equation* $\ln x + \ln 2 = \ln 4$.

Solution. By properties of ln the equation is equivalent to $\ln(2x) = \ln 4$. Since ln is one-to-one the latter is equivalent to $2x = 4$, which implies $x = 2$ is the only solution. □

6.2 Examples

Example 6.2. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a differentiable function. Prove that there is a unique function* $f(x)$ *that satisfies*

$$\begin{cases} f'(x) = g'(x)f(x) & \text{for all } x \in \mathbb{R} \\ f(0) = 1 \end{cases}$$

Note: $f(x) = e^{g(x)}$ satisfies $f'(x) = g'(x)f(x)$, and so does $f(x) = Ce^{g(x)}$. This can be written as $f(x)e^{-g(x)} = C$.

Solution. Let $h(x) = f(x)e^{-g(x)}$. By properties of derivative, we obtain

$$h'(x) = f'(x)e^{-g(x)} - f(x)g'(x)e^{-g(x)} = e^{-g(x)}(f'(x) - g'(x)f(x)) = 0.$$

Therefore, by the IC, $h(x)$ is a constant. We have $h(0) = f(0)e^{-g(0)} = e^{-g(0)}$. Thus, $h(x) = e^{-g(0)}$, which implies $f(x) = e^{g(x)-g(0)}$. □

Example 6.3. *Given a positive real number* $a \neq 1$ *and two real numbers* b, c, *prove that* $a^b = c$ *if and only if* $\dfrac{\ln c}{\ln a} = b$.

Solution. By definition of the exponential function a^x we have $a^b = c$ iff, $e^{b \ln a} = c$ iff $b \ln a = \ln c$ iff $b = \dfrac{\ln c}{\ln a}$. □

Example 6.4. *Let* c *be a positive constant. Find the number of real solutions of* $x^c = c^x$ *with* $x > 0$.

Solution. We have $c^x = x^c$ iff $x \ln c = c \ln x$ iff $\dfrac{\ln c}{c} = \dfrac{\ln x}{x}$. Let $f(x) = \dfrac{\ln x}{x}$. We have $f'(x) = \dfrac{1 - \ln x}{x^2}$. Therefore if $x > e$, then $f'(x) < 0$ and if $x < e$, then $f'(x) > 0$ and $f'(e) = 0$. Thus the function $f(x)$ is strictly decreasing over $[e, \infty)$ and strictly increasing over $(0, e]$ by a previous example. Therefore $f(x) = \dfrac{\ln c}{c}$ has at most two solutions, one with $x < e$ and one with $x \geq e$. Furthermore e is the only maximizer for f.

Clearly $x = c$ is one solution to the equation. We consider three cases:

Case I: $c = e$. In that case, the only solution is $x = e$, because if $x < e$, then $f(x) > f(e)$ and if $x > e$, then $f(x) < f(e)$.

Case II: $c \leq 1$. In that case $f(x) = f(c)$ implies $f(x) \leq 0$, which can only happen if $x \leq 1$. Since f is strictly increasing over $(0, e]$, the equation has at most one solution in this case. Taking a positive integer n, we get $f(2^{-n}) = \dfrac{-n\ln 2}{2^{-n}} = -n\ 2^n \ln 2$. Selecting $n > \dfrac{-\ln c}{c\ln 2}$, we obtain $f(2^{-n}) < 2^n \dfrac{\ln c}{c} \leq \dfrac{\ln c}{c} = f(c)$. Note that $f(1) = 0 \geq f(c)$. Therefore, by the IVT the equation has at least one solution in $(2^{-n}, 1]$. Thus, the equation has precisely one solution.

Case III: $c > 1$ but $c \neq e$. Since e is the only maximizer of f we have $f(1) = 0 < f(c) < f(e)$. Furthermore $f(2^n) = \dfrac{n\ln 2}{2^n} \leq \dfrac{n\ln 2}{n^2}$ for every integer $n \geq 4$ ($n^2 \leq 2^n$ should be proved by induction). Thus, $f(2^n) < \dfrac{\ln 2}{n}$ which can be made less than $f(c)$ by the AP. Therefore the equation has two solutions one in the interval $(1, e)$ and one in the interval (e, ∞).

To summarize:

- The equation $c^x = x^c$ has precisely one solution if $c = e$ or $c \leq 1$.
- The equation $c^x = x^c$ has precisely two solutions if $c \neq e$ and $c > 1$.

$\square$

Example 6.5. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which $f'(x) = xe^x$ for all $x \in \mathbb{R}$ and $f(0) = 1$.*

Sketch. We know from single variable calculus that $f(x) = \int xe^x\ dx$. Applying the integration by parts yields $f(x) = xe^x - e^x + C$. Using $f(0) = 1$ we obtain $C = 2$. To justify this fully we will have to completely rewrite the proof backwards!

Solution. First note that if $g(x) = xe^x - e^x$, then by properties of derivatives $g'(x) = e^x + xe^x - e^x = xe^x$. Therefore, $f'(x) = g'(x)$ for all $x \in \mathbb{R}$. By the Identity Criterion $f(x) = xe^x - e^x + C$ for some constant C. Using $f(0) = 1$ we conclude $C = 2$ and thus $f(x) = xe^x - e^x + 2$. $\square$

Example 6.6. *Let a, b be two positive real numbers and x, y be two real numbers. Prove that:*

(a) $a^x a^y = a^{x+y}$.

(b) $(a^x)^y = a^{xy}$.

(c) $a^x b^x = (ab)^x$.

(d) If $b > 0$, then $(a/b)^x = a^x/b^x$.

(e) $\ln(a^x) = x \ln a$.

Solution. (a) By definition of a^x, the fact that ln is one-to-one, and the properties of ln, the given equality is equivalent to

$$\exp(x\ln a)\exp(y\ln a) = \exp((x+y)\ln a) \Leftrightarrow \ln(\exp(x\ln a)) + \ln(\exp(y\ln a)) = \ln(\exp((x+y)\ln a)).$$

Since ln and exp are inverses the latter is equivalent to

$$x \ln a + y \ln x = (x + y) \ln a.$$

This follows from the distributive property.

(b) By definition, the given equality is equivalent to

$$\exp(y \ln(a^x)) = \exp(xy \ln a) \Leftrightarrow \exp(y \ln(\exp(x \ln a))) = \exp(xy \ln a).$$

Since ln and exp are inverses of each other, the latter is equivalent to

$$\exp(yx \ln a) = \exp(xy \ln a),$$

which holds.

(c) Similar to the previous parts, the given equality is equivalent to

$$\exp(x \ln a) \exp(x \ln b) = \exp(x \ln(ab)).$$

By part (a) the left hand side is equal to

$$\exp(x \ln a + x \ln b) = \exp(x(\ln a + \ln b)) = \exp(x \ln(ab)),$$

as desired.

(d) By part (c) we have

$$(a/b)^x b^x = a^x \Rightarrow (a/b)^x = a^x/b^x.$$

(e) $\ln(a^x) = \ln(\exp(x \ln a)) = x \ln a$, since ln and exp are inverses. □

Example 6.7. *Prove that the sequence* $\dfrac{\ln n}{\ln(n+1)}$ *converges to* 1.

6.3 Exercises

6.3.1 Problems for Grading

The following problems are due Friday 6/25/2021 before the class starts.

Exercise 6.1 (10 pts)**.** *Page 124, Problem 5.*

Exercise 6.2 (10 pts)**.** *Page 124, Problem 11.*

Exercise 6.3 (10 pts)**.** *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is a function that is not identically zero and for which*

$$f(a + b) = f(a)f(b), \text{ for all } a, b \in \mathbb{R}.$$

Assume f *is differentiable at* 0.

(a) Prove that f is differentiable everywhere and $f'(x) = f'(0)f(x)$ for all $x \in \mathbb{R}$.

(b) Show that $f(x) = e^{cx}$ for some constant c.

Exercise 6.4 (15 pts)**.** *Prove that for every positive integer n, there is a positive constant M_n for which*

$$e^x \geq M_n x^n \text{ for all } x \in [0, \infty).$$

What is the largest possible M_n?

Hint: Use the fact that $\ln x$ is strictly increasing. Take ln of both sides and consider the function $x - n \ln x$.

6.3.2 Practice Problems

Page 123-124: 1, 2, 3, 6, 8, 12.

6.3.3 Challenge Problems

Exercise 6.5. *Give an example of a function $f : (0, \infty) \to \mathbb{R}$ that is unbounded and for every positive integer n every function f^n is uniformly continuous.*

6.3.4 Summary

- ln and exp are defined and usual properties are proved.
- a^x is defined as $\exp(x \ln a)$, for $a > 0$ and $x \in \mathbb{R}$.

Chapter 7

Integration

In this chapter, unless otherwise stated, all intervals of the form $[a, b]$ are so that $a < b$ are two real numbers, and $f : [a, b] \to \mathbb{R}$ is a bounded function.

7.1 Upper and Lower Integrals

Definition 7.1. Given an interval $[a, b]$ with $a < b$, let

$$x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b$$

be a list of real numbers. Then, the set

$$\mathcal{P} = \{x_0, \ldots, x_n\}$$

is called a **partition** of $[a, b]$. In every partition we always list the elements in increasing order.

The **gap** of a partition is defined to be the largest of $x_i - x_{i-1}$:

$$\text{gap } \mathcal{P} = \max\{x_i - x_{i-1} \mid i = 1, \ldots, n\}.$$

Given a bounded function $f : [a, b] \to \mathbb{R}$ we let

$$m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}, \text{ and } M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}, \text{ for } i = 1, \ldots, n.$$

We define and denote the **lower sum of f determined by $\mathcal{P}$** by

$$L(f, \mathcal{P}) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

Similarly the **upper sum of f determined by $\mathcal{P}$** is given by

$$U(f, \mathcal{P}) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

The above notation will be used throughout this chapter.

Note that since $m_i \le M_i$, we have $L(f, \mathcal{P}) \le U(f, \mathcal{P})$.

Example 7.1. *Define a function* $f : [a, b] \to \mathbb{R}$ *by*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q}^c \end{cases}$$

Determine $U(f, \mathcal{P})$ *and* $L(f, \mathcal{P})$ *for each partition* $\mathcal{P}$ *of* $[a, b]$.

Definition 7.2. Suppose $\mathcal{P}, \mathcal{Q}$ are two partitions of $[a, b]$. We say $\mathcal{Q}$ is a **refinement** of $\mathcal{P}$ if $\mathcal{P} \subseteq \mathcal{Q}$.

Lemma 7.1 (The Refinement Lemma)**.** *Let* $f : [a, b] \to \mathbb{R}$ *be a bounded function, and* $\mathcal{P}, \mathcal{P}^\star$ *be partitions of* $[a, b]$ *for which* $\mathcal{P}^\star$ *is a refinement of* $\mathcal{P}$. *Then,*

$$L(f, \mathcal{P}) \leq L(f, \mathcal{P}^\star) \leq U(f, \mathcal{P}^\star) \leq U(f, \mathcal{P}).$$

Corollary 7.1. *Let* $\mathcal{P}, \mathcal{Q}$ *be two partitions of an interval* $[a, b]$. *Then*

$$L(f, \mathcal{P}) \leq U(f, \mathcal{Q}).$$

The above corollary shows that the set of all lower sums is bounded above and the set of all upper sums is bounded below. This motivates the following definition.

Definition 7.3. The lower integral of f over $[a, b]$ is defined by

$$\underline{\int_a^b} f = \sup\{L(f, \mathcal{P}) \mid \mathcal{P} \text{ is a partition of } [a, b]\}.$$

Similarly the upper integral of f over $[a, b]$ is defined by

$$\overline{\int_a^b} f = \inf\{U(f, \mathcal{P}) \mid \mathcal{P} \text{ is a partition of } [a, b]\}.$$

Example 7.2. *Find the lower and upper integral of* $f : [0, 1] \to \mathbb{R}$ *given by*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q}^c \end{cases}$$

Theorem 7.1. *For every bounded function* $f : [a, b] \to \mathbb{R}$ *we have*

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Example 7.3. *Let* $m \leq M$ *be two real numbers. Prove that for every* $a < b$, *there is a function* $f : [a, b] \to \mathbb{R}$ *for which*

$$\underline{\int_a^b} f = m, \text{ and } \overline{\int_a^b} f = M.$$

7.2 Warm-ups

Example 7.4. *Prove that for every closed interval* $[a, b]$ *(with* $a < b$*), every positive real number* ϵ, *and every positive integer* n, *there is a partition of* $[a, b]$ *whose gap is less than* ϵ.

Solution. By the Archemidean property there is a positive integer n for which $n > \dfrac{b-a}{\epsilon}$. Consider the partition of $[a, b]$ into n subintervals of equal width. The width of each subinterval is $\dfrac{b-a}{n} < \epsilon$. Therefore, the gap of this partition is less than ϵ. □

Example 7.5. *Evaluate $U(f, P)$, where $f : [0, 1] \to \mathbb{R}$ is a function given by $f(x) = x^2 - x$, and $P = \{0, 0.25, 0.5, 0.75\}$.*

Solution. Note that

$$f(x) = (x - 1/2)^2 - 1/4 = (1/2 - x)^2 - 1/4.$$

If $x \leq 1/2$, then $x - 1/2$ is negative and increasing and thus $f(x)$ is decreasing over $[0, 1/2]$. Similarly $f(x)$ is increasing over $[1/2, 1]$. Therefore,

$$M_1 = 0, M_2 = 1/16 - 1/4 = -3/16, M_3 = 9/16 - 3/4 = -3/16, M_4 = 1 - 1 = 0.$$

Therefore,

$$U(f, \mathcal{P}) = 0(1/4 - 0) + (-3/16)(1/2 - 1/4) + (-3/16)(3/4 - 1/2) + 0(1 - 3/4) = -3/32.$$

□

Example 7.6. *In this section, why do we assume f is bounded?*

Solution. This is because, otherwise, M_i could be ∞ and m_i could be $-\infty$. This would mean the upper and lower sums may not be finite real numbers. □

7.3 More Examples

Example 7.7. *Suppose $f : [0, 1] \to \mathbb{R}$ is defined by*

$$f(x) = 0, \text{ for all } x \neq 0.5, \text{ and } f(0.5) = 1.$$

Prove that

$$\underline{\int_0^1} f = \overline{\int_0^1} f = 0.$$

Solution. Using the standard notations, $m_i = 0$ for each i since all functional values are zero except for one that is positive. This means all lower sums are zero. Thus, the lower integral is zero.

Consider the partition

$$0 < \frac{1}{2^n} < \frac{2}{2^n} < \cdots < \frac{2^n - 1}{2^n} < 1.$$

All intervals except for two do not contain 0.5. Thus, $M_i = 0$ for all i except for two. Thus, the upper sum associated to this partition is

$$\frac{1}{2^n} \cdot 1 + \frac{1}{2^n} \cdot 1 = \frac{1}{2^{n-1}} \to 0.$$

Since this sequence of upper sums approaches zero, their lower bound cannot be positive. Thus the upper integral cannot be positive. Since the upper integral is not less than the lower integral, it must be non-negative. Thus, the upper integral of f is also zero. $\square$

Example 7.8. *Suppose a continuous function $f : [a, b] \to \mathbb{R}$ has the property that all lower sums determined by all partitions are equal. Prove that f is a constant function.*

Solution. We will use the standard notation for evaluating lower sums. First, note that since f is continuous, by the Extreme Value Theorem, the infimum over each interval $[x_{i-1}, x_i]$ is also the minimum over the same interval.

Let m be the minimum of f over $[a, b]$, and let $\mathcal{P}$ be the partition a, b. Then,

$$L(f, \mathcal{P}) = m(b - a).$$

Given any other partition

$$\mathcal{Q} : a = x_0 < \cdots < x_n = b,$$

we have

$$L(f, \mathcal{Q}) = \sum m_i(x_i - x_{i-1}) \leq \sum m(x_i - x_{i-1}) = m(b - a) = L(f, \mathcal{P}).$$

Since by assumption $L(f, \mathcal{P}) = L(f, \mathcal{Q})$ we must have $m_i = m$, i.e. the mimimum of f is the same as the minimum of f over each subinterval $[x_{i-1}, x_i]$ of $[a, b]$. Given a point $x_0 \in [a, b]$, we can then find a sequence of closed intervals I_n that contain x_0 and their length approaches zero (e.g. $[x_0, x_0 + (b - x_0)/n]$ or $[x_0 - (x_0 - a)/n, x_0]$). Let z_n be the minimizer in the n-th interval. By the Comparison Lemma, know $z_n \to x_0$ since

$$|z_n - x_0| \leq \text{ the length of } I_n.$$

Since f is continuous, $f(z_n) \to f(x_0)$. Since $f(z_n) = m$, we obtain $f(x_0) = m$, i.e. f is a constant function. $\square$

7.4 Exercises

7.4.1 Problems for Grading

The following problems are due Monday 6/28/2021 before the class starts.

Exercise 7.1 (10 pts)**.** *Page 141, Problem 3.*

Exercise 7.2 (10 pts)**.** *Page 141, Problem 5.*

Exercise 7.3 (10 pts)**.** *Page 142, Problem 6.*

7.4.2 Practice Problems

Pages 141: 1, 2, 4.

7.4.3 Summary

- Given a partition $\mathcal{P}$ of an interval $[a,b]$, to evaluate the lower sum $L(f,\mathcal{P})$:
  - Find $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$.
  - Evaluate $\sum m_i(x_i - x_{i-1})$.
- $\underline{\int}_a^b f$ is the supremum of $L(f,\mathcal{P})$, where $\mathcal{P}$ ranges over all possible partitions of $[a,b]$.
- $\overline{\int}_a^b f$ is the infimum of $U(f,\mathcal{P})$, where $\mathcal{P}$ ranges over all possible partitions of $[a,b]$.

7.5 The Archimedes-Riemann Theorem

Definition 7.4. We say a bounded function $f : [a,b] \to \mathbb{R}$ is **integrable on** $[a,b]$ if

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

When this is the case we denote this common integral by $\displaystyle\int_a^b f$.

Theorem 7.2 (Archimedes-Riemann Theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then, f is integrable on $[a,b]$ if and only if there is a sequence $\mathcal{P}_n$ of partitions of $[a,b]$ for which*

$$\lim_{n\to\infty} [U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n)] = 0.$$

Moreover, for any such sequence

$$\int_a^b f = \lim_{n\to\infty} U(f,\mathcal{P}_n) = \lim_{n\to\infty} L(f,\mathcal{P}_n).$$

Definition 7.5. Let f be an integrable function on $[a,b]$. A sequence $\mathcal{P}_n$ of partitions of $[a,b]$ is called an **Archimedean sequence of partitions** for f on $[a,b]$ if

$$\lim_{n\to\infty} [U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n)] = 0.$$

Definition 7.6. For every positive integer n the partition of $[a,b]$ of the form:

$$a < a + \frac{b-a}{n} < a + \frac{2(b-a)}{n} < \cdots < a + \frac{(n-1)(b-a)}{n} < b$$

is called the **regular partition of** $[a,b]$ **into** n **subintervals**.

Example 7.9. *Prove that $f(x) = x$ is integrable over $[0,1]$ and find its integral.*

Theorem 7.3. *Every monotone function $f : [a,b] \to \mathbb{R}$ is integrable. Furthermore, the sequence of regular partitions of $[a,b]$ is an Archimedean sequence of partitions for f.*

Definition 7.7. A function $f : [a, b] \to \mathbb{R}$ is called a **step function** if there is a partition

$$z_0 = a < z_1 < \cdots < z_{n-1} < z_n = b$$

of $[a, b]$ for which the function f is constant over each of the open intervals (z_{i-1}, z_i) for all $i = 1, \ldots, n$.

Theorem 7.4. *Every step function f over an interval $[a, b]$ is integrable. Furthermore, the sequence of regular partitions of $[a, b]$ is an Archimedean sequence of partitions for f.*

Example 7.10. *Define a function $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \textit{ The largest integer not exceeding } x.$$

Prove that f is integrable over each interval $[0, n]$ for every natural number n, and find its integral.

7.5.1 Warm-ups

Example 7.11. *Prove that if f is integrable on $[a, b]$, then for every positive number ϵ there is a partition $\mathcal{P}$ of $[a, b]$ for which*

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon.$$

Solution. By the Archimedes-Riemann Theorem, there is a sequence $\mathcal{P}_n$ of partitions of $[a, b]$ for which

$$\lim_{n\to\infty} [U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n)] = 0.$$

By definition of limit, there is $N \in \mathbb{N}$ for which

$$U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) < \epsilon, \text{ for all } n \geq N.$$

This completes the proof. □

7.5.2 More Examples

Example 7.12. *Suppose $f : [0, 1] \to \mathbb{R}$ is a bounded function for which f^2 is integrable. Is it true that f must be integrable?*

Solution. This is false. Consider the function $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{Q}^c \end{cases}$$

$f^2(x) = 1$ is monotone, and thus integrable. For every partition $\mathcal{P}$ of $[0, 1]$ we have $m_i = -1$ and $M_i = 1$ since $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense. This implies

$$L(f, \mathcal{P}) = -(1 - 0) = -1, \text{ and } U(f, \mathcal{P}) = 1(1 - 0) = 1.$$

Therfore,

$$\underline{\int_0^1} f = -1 \neq \overline{\int_0^1} f = 1.$$

This means f is not integrable. $\square$

Example 7.13. *Let f be an integrable function on $[a, b]$, and let $g : [m, M] \to \mathbb{R}$ be a Lipschitz function, where m and M are the infimum and supremum of the image of f, respectively. Prove that $g \circ f$ is integrable on $[a, b]$.*

Solution. Using the standard notation, let m_i, M_i be infimum and supremum of f over an interval $[x_{i-1}, x_i]$. Since g is Lipschitz, there is a constant C for which

$$|g(x) - g(y)| \leq C|x - y|, \text{ for all } x, y \in [m, M].$$

For every two $x, y \in [m_i, M_i]$ we have

$$m_i \leq f(x), f(y) \leq M_i \Rightarrow |f(x) - f(y)| \leq M_i - m_i.$$

This implies

$$|g \circ f(x) - g \circ f(y)| \leq C|f(x) - f(y)| \leq C(M_i - m_i) \Rightarrow g \circ f(x) \leq C(M_i - m_i) + g \circ f(y).$$

Since this is true for all $x \in [x_{i-1}, x_i]$ we obtain

$$M'_i \leq C(M_i - m_i) + g \circ f(y),$$

where M'_i is the supremum of $g \circ f$ over $[x_{i-1}, x_i]$. Since the above is true for all $y \in [x_{i-1}, x_i]$ we obtain

$$M'_i - C(M_i - m_i) \leq m'_i,$$

where m'_i is the infimum of $g \circ f$ over $[x_{i-1}, x_i]$. This implies

$$M'_i - m'_i \leq C(M_i - m_i).$$

Multiplying these inequalities for $i = 1, \ldots, n$ by $x_i - x_{i-1}$ and adding these up we obtain the following inequality:

$$0 \leq U(g \circ f, \mathcal{P}) - L(g \circ f, \mathcal{P}) \leq C[U(f, \mathcal{P}) - L(f, \mathcal{P})]$$

If $\mathcal{P}_n$ is an Archimedean sequence of partitions of $[a, b]$ for f, by the Comparison Lemma we conclude that $\mathcal{P}_n$ is also an Archimedean sequence of partitions for $g \circ f$, and thus $g \circ f$ is integrable on $[a, b]$. $\square$

Example 7.14. *Prove that the function $f : [0, 2] \to \mathbb{R}$ given by*

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2 & \text{if } 1 < x \leq 2 \end{cases}$$

is integrable and find $\displaystyle\int_0^2 f$.

Solution. For every natural number n let $\mathcal{P}_n$ be the partition of $[0,2]$ obtained from taking the union of the regular partitions of $[0,1]$ and $[1,2]$ into n subintervals each. We can see that

$$U(f,\mathcal{P}_n) = \sum_{i=1}^{n} \frac{i}{n}\frac{1}{n} + \sum_{i=1}^{n} 2\frac{1}{n} = \frac{1}{n^2}\left(\frac{n(n+1)}{2} + 2n^2\right) = \frac{1}{2} + \frac{1}{2n} + 2,$$

and

$$L(f,\mathcal{P}_n) = \sum_{i=1}^{n} \frac{i-1}{n}\frac{1}{n} + \frac{1}{n} + \sum_{i=2}^{n} 2\frac{1}{n} = \frac{1}{n^2}\left(\frac{n(n-1)}{2} + n + 2n(n-1)\right) = \frac{1}{2} - \frac{3}{2n} + 2.$$

The difference of the upper and lower sums above is $2/n$ which approaches zero. Thus, $\mathcal{P}_n$ is an Archimedean sequence of partitions for f on $[0,2]$. This means f is integrable and

$$\int_0^2 f = \lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{2n} + 2\right) = 2.5.$$

□

7.6 Exercises

7.6.1 Problems for Grading

The following problems are due Tuesday 6/29/2021 before the class starts.

Exercise 7.4 (10 pts)**.** *Page 149, Problem 3.*

Exercise 7.5 (10 pts)**.** *Page 149, Problem 6.*

Exercise 7.6 (10 pts)**.** *Page 150, Problem 9.*

Exercise 7.7 (10 pts)**.** *Define a function* $f:[0,1]\to\mathbb{R}$ *by*

$$f(x) = \begin{cases} x & \text{if } x\in\mathbb{Q} \\ 0 & \text{if } x\in\mathbb{Q}^c \end{cases}$$

Evaluate $\underline{\int_0^1} f$ *and* $\overline{\int_0^1} f$.

Hint: For the upper integral note that the upper sums are the same as the upper sums for the function x.

7.6.2 Practice Problems

Pages 149-150: 1, 5, 7, 8, 12.

7.6.3 Summary

- To prove a function is integrable we need to find an Archimedean sequence of partitions, i.e. a sequence $\mathcal{P}_n$ satisfying

$$\lim_{n\to\infty}[U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n)] = 0.$$

- To find $\int_a^b f$, we will need to:
  - Find an Archimedean sequence of partitions $\mathcal{P}_n$.
  - Evaluate $L(f, \mathcal{P}_n)$ or $U(f, \mathcal{P}_n)$.
  - Take the limit: $\int_a^b f = \lim_{n\to\infty} L(f, \mathcal{P}_n)$ or $\lim_{n\to\infty} L(f, \mathcal{P}_n)$.
- Monotone and step functions are integrable.

7.7 Properties of Integrals

Theorem 7.5 (Additivity). *Suppose $f : [a, b] \to \mathbb{R}$ is integrable and $c \in (a, b)$. Then, f is integrable on $[a, c]$ and on $[c, b]$, and that*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Theorem 7.6 (Monotonicity). *If $f, g : [a, b] \to \mathbb{R}$ are bounded functions for which*

$$f(x) \le g(x), \text{ for all } x \in [a, b].$$

Then

$$\underline{\int_a^b} f \le \underline{\int_a^b} g, \text{ and } \overline{\int_a^b} f \le \overline{\int_a^b} g.$$

Furthermore, if f and g are integrable, then

$$\int_a^b f \le \int_a^b g.$$

Theorem 7.7 (Linearity). *Suppose α, β are two constants and $f, g : [a, b] \to \mathbb{R}$ are integrable. Then, $\alpha f + \beta g$ is also integrable on $[a, b]$ and that*

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g.$$

Corollary 7.2. *Suppose $f : [a, b] \to \mathbb{R}$ is integrable for which $|f|$ is also integrable on $[a, b]$. Then,*

$$\left| \int_a^b f \right| \le \int_a^b |f|.$$

Theorem 7.8. *Any continuous function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$. Furthermore, the sequence of regular partitions of $[a, b]$ is an Archimedean sequence of partitions for f.*

Theorem 7.9. *Suppose $f, g : [a, b] \to \mathbb{R}$ are bounded functions such that*

$$f(x) = g(x), \text{ for all } x \in (a, b).$$

Suppose f (and hence g) is integrable over $[c, d]$ for every $a < c < d < d$ (e.g. when f is continuous over (a, b)). Then, f and g are integrable on $[a, b]$ and

$$\int_a^b f = \int_a^b g.$$

In other words, the value of $\int_a^b f$ does not depend on the value of f at the endpoints a and b.

Example 7.15. *Consider the function $f : [0,1] \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \sin\left(\dfrac{1}{x(x-1)}\right) & \text{if } 0 < x < 1 \\ 2 & \text{if } x = 1 \end{cases}$$

Prove that f is integrable on $[0,1]$.

Example 7.16. *Prove the following function is integrable on $[0,2]$ and evlauate $\int_0^2 f$.*

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ x-1 & \text{if } 1 \le x < 2 \\ 3 & \text{if } x = 0 \text{ or } x = 2 \end{cases}$$

7.7.1 Warm-ups

Example 7.17. *Show that each of the following functions is integrable over $[0,2]$.*

(a) $f(x) = x$.

(b) $g(x) = \begin{cases} 2 & \text{if } 0 \le x \le 1 \\ -x & \text{if } 1 < x \le 2 \end{cases}$

Solution. (a) This function is continuous, and thus integrable, by a theorem.

(b) This function is decreasing since $-x$ is decreasing and $2 > -x$. Thus by a theorem it is integrable. ☐

7.7.2 More Examples

Example 7.18. *Suppose two bounded functions $f, g : [a,b] \to \mathbb{R}$ satisfy*

$$f(x) \le g(x), \text{ for all } x \in [a,b].$$

Is it true that

$$\overline{\int_a^b} f \le \underline{\int_a^b} g?$$

Solution. This is false. Suppose $f = g$ is any bounded function that is not integrable, e.g. the Dirichlet's function. Then, $f(x) \le g(x)$ for all x. Furthermore $\underline{\int_a^b} f < \overline{\int_a^b} f$. ☐

Example 7.19. *Is it true that for every bounded function $f : [a,b] \to \mathbb{R}$ and every $c \in (a,b)$ we have*

$$\underline{\int_a^b} f = \underline{\int_a^c} f + \underline{\int_c^b} f?$$

Solution. This is true. Let $\mathcal{P}$ be a partition of $[a,b]$ and set $\mathcal{P}^\star = \mathcal{P} \cup \{c\}$. Let $\mathcal{Q}, \mathcal{R}$ be partitions of $[a,c]$ and $[c,b]$, respectively for which

$$\mathcal{P}^\star = \mathcal{Q} \cup \mathcal{R}.$$

By definition we have

$$L(f,\mathcal{P}^\star) = L(f,\mathcal{Q} \cup \mathcal{R}) = L(f,\mathcal{Q}) + L(f,\mathcal{R}).$$

By Refinement Lemma we have

$$L(f,\mathcal{P}) \leq L(f,\mathcal{P}^\star).$$

Combining these and using the fact that

$$L(f,\mathcal{Q}) \leq \underline{\int_a^c} f, \text{ and } L(f,\mathcal{R}) \leq \underline{\int_c^b} f$$

we obtain

$$L(f,\mathcal{P}) \leq \underline{\int_a^c} f + \underline{\int_c^b} f.$$

Since lower integral is the supremum of lower sums we obtain the result. □

Example 7.20. *(a) Is it possible that a function $f : [a,b] \to \mathbb{R}$ is integrable over $[a,b]$, but its square is not?*

(b) Is it possible that a function $f : [a,b] \to \mathbb{R}$ is not integrable over $[a,b]$, but its square is?

(c) Is it possible that $f+g$ and $f-g$ are both integrable over $[a,b]$ but f is not?

(d) Is it possible that two functions are integrable on an interval $[a,b]$ but their product is not integrable?

Solution. (a) This is not possible. Let m, M be infimum and supremum of f. Then, $g : [m,M] \to \mathbb{R}$ defined by $g(x) = x^2$ satisfies

$$|g(x) - g(y)| = |x+y||x-y| \leq 2(|m| + |M|)|x-y|.$$

Therefore, g is Lipschitz. By Example 7.13, f^2 must be integrable.
(b) This is possible. See Example 7.12.
(c) This is not possible. By linearity since $f-g$ and $f+g$ are integrable, so is $f = \frac{1}{2}(f-g) + \frac{1}{2}(f+g)$.
(d) This is not possible. Suppose $f, g : [a,b] \to \mathbb{R}$ are integrable. By the Linearity Theorem, both $f+g$ and $f-g$ are integrable on $[a,b]$. By part (a) and linearity the following function is integrable:

$$(f+g)^2 - (f-g)^2 = 4fg.$$

Therefore, by the Linearity Theorem, fg is integrable. □

Example 7.21. *Suppose $f : [-1,1] \to \mathbb{R}$ is integrable odd function. Prove that*

$$\int_{-1}^1 f = 0.$$

Solution. Since f is integrable on $[-1,1]$, it is integrable on $[-1,0]$ and $[0,1]$ and that

$$\int_{-1}^{1} f = \int_{-1}^{0} f + \int_{0}^{1} f.$$

Let $\mathcal{P} = \{x_0, \ldots, x_n\}$ be a partition of $[0,1]$ and $\mathcal{P}' = \{-x_0, \ldots, -x_0\}$ be its corresponding partition of $[-1,0]$. Note that all functional values of $[-x_i, -x_{i-1}]$ are negative the functional values of $[x_{i-1}, x_i]$ and thus if M_i is the supremum of f over $[x_{i-1}, x_i]$ and m_i' is the infimum of f over $[-x_i, -x_{i-1}]$ then $M_i' = -m_i$. This means for every partition $\mathcal{P}$ of $[0,1]$ its corresponding partition $\mathcal{P}'$ satisfies the following

$$U(f, \mathcal{P}) = -L(f, \mathcal{P}').$$

Therefore, the set of all lower sums of f over $[-1,0]$ is negative of the set of all upper sums of f over $[0,1]$. This means the infimum of the first set is negative the supremum of the second set, i.e.

$$\overline{\int_{-1}^{0}} f = -\underline{\int_{0}^{1}} f \Rightarrow \int_{-1}^{1} f = 0.$$

□

Example 7.22. *Suppose $f : [a,b] \to \mathbb{R}$ is integrable. Prove that $|f| : [a,b] \to \mathbb{R}$ is integrable.*

Solution. Note that by triangle inequality we have:

$$|x| - |y| \leq |x - y|, \text{ and } |y| - |x| \leq |y - x| = |x - y| \Rightarrow -|x - y| \leq |x| - |y| \leq |x - y| \Rightarrow ||x| - |y|| \leq |x - y|.$$

This means $|x|$ is Lipschitz. Therefore, by Example 7.13, the function $|f|$ is integrable on $[a,b]$. □

Example 7.23. *Let $f, g : [a,b] \to \mathbb{R}$ be bounded functions. Which of the following is always true?*

(a) $\underline{\int_a^b}(f+g) = \underline{\int_a^b} f + \underline{\int_a^b} g$.

(b) $\overline{\int_a^b}(f+g) = \overline{\int_a^b} f + \overline{\int_a^b} g$.

Example 7.24. *Let $f, g : [a,b] \to \mathbb{R}$ be bounded functions. Prove the following:*

(a) $\underline{\int_a^b} f + \underline{\int_a^b} g \leq \underline{\int_a^b}(f+g)$.

(b) $\overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g$.

Example 7.25. *Suppose $f : [a,b] \to \mathbb{R}$ is integrable and $g : [a,b] \to \mathbb{R}$ is bounded. Prove:*

(a) $\underline{\int_a^b}(f+g) = \displaystyle\int_a^b f + \underline{\int_a^b} g$.

(b) $\overline{\int_a^b}(f+g) = \displaystyle\int_a^b f + \overline{\int_a^b} g$.

7.8 Exercises

7.8.1 Problems for Grading

The following problems are due Thursday 7/1/2021 before the class starts.

Exercise 7.8 (10 pts). *Page 154, Problem 1.*

Exercise 7.9 (10 pts). *Page 155, Problem 5. (The second term must be $U(g, \mathcal{P}_2)$.)*

Exercise 7.10 (10 pts). *Page 155, Problem 6.*

Exercise 7.11 (10 pts). *Page 159, Problem 3.*

Exercise 7.12 (10 pts). *Page 160, Problem 6.*

7.8.2 Problems for Practice

Exercise 7.13. *Evaluate $\underline{\int}_0^1 f$ and $\overline{\int}_0^1 f$, where f is given by*

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \in \mathbb{Q}^c \end{cases}$$

Hint: Compare the upper sums and lower sums of this function with functions x and $-x$.

Page 155: 2, 3, 4.

Pages 159-160: 1, 2, 4, 5, 7, 8, 9.

7.8.3 Challenege Problems

Exercise 7.14. *Consider the function $f : [a, b] \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}^c \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ is in its simplest form with } m, n \in \mathbb{Z} \text{ and } n > 0 \end{cases}$$

Prove that f is integrable.

Exercise 7.15. *Is it possible that $f : [a, b] \to [a, b]$ is integrable over $[a, b]$, but the composition $f \circ f$ is not?*

Exercise 7.16. *Given an interval $[a, b]$*

(a) Determine all continuous functions for which all lower sums are equal.

(b) Determine all integrable functions for which all lower sums are equal.

Exercise 7.17. *Prove that if $f : [a, b] \to \mathbb{R}$ is integrable, then for every $\epsilon > 0$, there is a continuous function $g : [a, b] \to \mathbb{R}$ for which*

$$g(x) \leq f(x), \text{ for all } x \in [a, b]; \text{ and } \int_a^b (f - g) < \epsilon.$$

Exercise 7.18. *Assume $f : [0, 1] \to \mathbb{R}$ is a continuous function such that $\int_0^1 fg = 0$ for every continuous function $g : [0, 1] \to \mathbb{R}$ which satisfies $g(0) = g(1) = 0$. Prove that $f = 0$.*

7.8.4 Summary

- Additivity of integrals states that $\displaystyle\int_a^b f = \int_a^c f + \int_c^b$.
- Monotonicity of integrals states that $f \leq g$ implies $\displaystyle\int_a^b f \leq \int_a^b g$.
- Linearity of integrals states that if f, g are integrable on $[a, b]$, then for every two constants α, β we have
$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b f.$$
- Continuous functions are integrable.
- If a bounded function f is integrable over any subinterval $[c, d]$ of (a, b) then it is integrable over $[a, b]$ and its integral does not depend on $f(a)$ and $f(b)$.

7.9 Fundamental Theorems of Calculus

Notation: Instead of $\displaystyle\int_a^b f$ we often write $\displaystyle\int_a^b f(x)\ \mathrm{d}x$. Note that the variable x could be changed without changing the value of the integral.

Theorem 7.10 (The First Fundamental Theorem)**.** *Let $F : [a, b] \to \mathbb{R}$ be continuous over $[a, b]$ and differentiable over (a, b). Suppose $F' : (a, b) \to \mathbb{R}$ is bounded and continuous. Then,*
$$\int\limits_a^b F'(x)\ \mathrm{d}x = F(b) - F(a).$$

Remark 7.1. The condition on F' in the First Fundamental Theorem can be made weaker by assuming $F' : (a, b) \to \mathbb{R}$ is bounded and is integrable over each interval $[c, d]$ that lies completely inside (a, b).

Example 7.26. *Evaluate each of the following:*

(a) $\int_a^b x^r\ \mathrm{d}x$, *where* $0 < a < b$ *and* $r \neq -1$ *are real numbers.*

(b) $\int_0^2 f(x)\ \mathrm{d}x$, *where* $f(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1 \\ 2 & \text{if } 1 \leq x \leq 2 \end{cases}$

Theorem 7.11 (The Mean Value Theorem for Integrals)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, there is some $x_0 \in [a, b]$ for which*
$$\frac{1}{b-a} \int_a^b f(x)\ \mathrm{d}x = f(x_0).$$

Theorem 7.12. *Suppose $f : [a, b] \to \mathbb{R}$ is integrable. Then, $F : [a, b] \to \mathbb{R}$ defined by*
$$F(a) = 0, \text{ and } F(x) = \int_a^x f \text{ for all } x \in (a, b]$$
is Lipschitz and thus uniformly continuous over $[a, b]$.

Example 7.27. *Let $f : [0,2] \to \mathbb{R}$ be the function given by*

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ x & \text{if } 1 \le x \le 2 \end{cases}$$

Evaluate $F(x) = \int_0^x f$, and prove that it is continuous.

Theorem 7.13 (The Second Fundamental Theorem)**.** *Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Then,*

$$\frac{d}{dx}\left(\int_a^x f\right) = f(x) \quad \text{for all } x \in (a,b).$$

Example 7.28. *Given a continuous function $f : [a,b] \to \mathbb{R}$ prove that*

$$\frac{d}{dx}\left(\int_x^b f\right) = -f(x) \quad \text{for all } x \in (a,b).$$

Much of what we did before can be extended to integrals of the form $\int_b^a f$, where either $a = b$ or $a < b$.

Definition 7.8. Let $f : [a,b] \to \mathbb{R}$ be integrable. Then, we define

$$\int_b^a f = -\int_a^b f, \text{ and } \int_c^c f = 0 \text{ for all } c \in [a,b].$$

Theorem 7.14. *Suppose $f : [a,b] \to \mathbb{R}$ is integrable and $x_1, x_2, x_3 \in [a,b]$ are fixed. Let $\alpha, \beta \in \mathbb{R}$ be two constants. Then, the following hold:*

- *(Additivity)* $\displaystyle\int_{x_1}^{x_2} f + \int_{x_2}^{x_3} f = \int_{x_1}^{x_3} f.$
- *(Linearity)* $\displaystyle\int_{x_1}^{x_2} (\alpha f + \beta g) = \alpha \int_{x_1}^{x_2} f + \beta \int_{x_1}^{x_2} g.$

Suppose $F : [a,b] \to \mathbb{R}$ is continuous over $[a,b]$ and differentiable over (a,b). Suppose $F' : (a,b) \to \mathbb{R}$ is bounded and continuous. Then,

- *(The First Fundamental Theorem)* $\displaystyle\int_{x_1}^{x_2} F' = F(x_2) - F(x_1).$
- *(The Second Fundamental Theorem)* $\displaystyle\frac{d}{dx}\left(\int_{x_1}^x f\right) = f(x)$ *for all $x \in (a,b)$, if $f : (a,b) \to \mathbb{R}$ is continuous.*

Theorem 7.15. *Let I be an open interval and $f : I \to \mathbb{R}$ be continuous. Let J be an open interval for which $\varphi : J \to \mathbb{R}$ is differentiable with $\varphi(J) \subseteq I$. Given $x_0 \in I$ we have*

$$\frac{d}{dx}\left(\int_{x_0}^{\varphi(x)} f\right) = f(\varphi(x))\varphi'(x).$$

Example 7.29. *Evaluate*

$$\frac{d}{dx}\left(\int_0^{e^{x^2}} \sin(t^2)\ \mathrm{dt}\right), \text{ and } \frac{d}{dx}\left(\int_{x^2}^0 xe^{t^2}\ \mathrm{dt}\right).$$

7.9.1 Warm-ups

Example 7.30. *Evaluate* $\int_0^1 x \, \mathrm{d}x$.

Solution. Consider the function $F : [0,1] \to \mathbb{R}$ given by $F(x) = x^2/2$. We know F is differentiable over $(0,1)$ and integrable over $[0,1]$ since it is a polynomial. Also, $F'(x) = x$. Therefore, by the First Fundamental Theorem

$$\int_0^1 F'(x) \, \mathrm{d}x = F(1) - F(0) \Rightarrow \int_0^1 x \, \mathrm{d}x = 1/2.$$

□

7.9.2 More Examples

Example 7.31. *Evaluate* $\int_0^3 f$*, where* $f : [0,3] \to \mathbb{R}$ *is given by*

$$f(x) = \begin{cases} 1 & \textit{if } 0 \leq x < 1 \\ x - 1 & \textit{if } 1 \leq x < 2 \\ e^x & \textit{if } 2 \leq x \leq 3 \end{cases}$$

Solution. Consider the function $F : [0,1] \to \mathbb{R}$ given by $F(x) = x$. Note that $F'(x) = f(x)$ for all $x \in (0,1)$. Furthermore, F is continuous over $[0,1]$ and differentiable over $(0,1)$. Therefore, by Theorem 7.9 and First Fundamenral Theorem,

$$\int_0^1 f = \int_0^1 1 = F(1) - F(0) = 1.$$

Similarly, using the fact that

$$\frac{d}{dx}\left(\frac{x^2}{2} - x\right) = x - 1, \text{ and } \frac{d}{dx}(e^x) = e^x$$

we obtain the following:

$$\int_1^2 f = \int_1^2 (x-1) \, \mathrm{d}x = \left(\frac{2^2}{2} - 2\right) - \left(\frac{1^2}{2} - 1\right) = \frac{1}{2}, \text{ and}$$

$$\int_2^3 f = \int_2^3 e^x \, \mathrm{d}x = e^3 - e^2.$$

By an exercise, since f is integrable over $[0,1]$, $[1,2]$, and $[2,3]$, it is integrable over $[0,3]$ and thus, by Additivity we have

$$\int_0^3 f = 1 + \frac{1}{2} + e^3 - e^2 = \frac{3}{2} + e^3 - e^2.$$

□

Example 7.32. *Evaluate the derivative of each of the following:*

(a) $\int_0^{x^2} e^{t^2} \, \mathrm{d}t$.

(b) $\int_{e^x}^{10} t^t \, \mathrm{d}t$ *for all* $x > 0$.

Solution. (a) Note that e^{x^2} and x^2 are both continuous on $\mathbb{R}$. Therefore, by Theorem 7.15, we have

$$\frac{d}{dx}\left(\int_0^{x^2} e^{t^2}\ \mathrm{d}t\right) = e^{(x^2)^2}2x = 2xe^{x^4}.$$

(b) Similarly $\varphi(x) = e^x$ is continuous and $f(x) = x^x = \exp(x\ln x)$ is continuous over $(0,\infty)$ and that $f(0,\infty) \subseteq \mathbb{R}$. Therefore, by Theorem 7.15 we have the following:

$$\frac{d}{dx}\left(\int_{e^x}^{10} t^t\ \mathrm{d}t\right) = -\frac{d}{dx}\left(\int_{10}^{e^x} t^t\ \mathrm{d}t\right) = -(e^x)^{e^x}e^x = -e^{xe^x+x}.$$

$\square$

Example 7.33. *Suppose $f : [a,b] \to \mathbb{R}$ is integrable. Prove that there is $x \in [a,b]$ for which*

$$\int_a^x f = \int_x^b f.$$

By an example show that no such x may exist in the open interval (a,b).

Solution. Consider the function $F : [a,b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f \text{ for all } x \in [a,b].$$

By Theorem 7.12, F is Lipschitz and thus continuous. Since $F(a) = \int_a^a f = 0$, the quantity $F(b)/2$ is between $F(a)$ and $F(b)$. By the Intermediate Value Theorem there is $x \in [a,b]$ for which

$$F(x) = \frac{1}{2}\int_a^b f.$$

By additivity we have

$$2\int_a^x f = \int_a^x f + \int_x^b f \Rightarrow \int_a^x f = \int_x^b f.$$

This completes the proof.

Let $f : [-1,1] \to \mathbb{R}$ be given by $f(t) = t$. Since f is continuous over $[-1,1]$ and $F(t) = t^2/2$ is continuous and bounded over $(-1,1)$, and $F'(t) = f(t)$ for all $t \in (-1,1)$, by the First Fundamental Theorem, we have

$$\int_{-1}^x f = F(x) - F(-1) = \frac{x^2-1}{2}, \text{ and } \int_x^1 f = F(1) - F(x) = \frac{1-x^2}{2}.$$

If $\int_{-1}^x f = \int_x^1 f$, then

$$\frac{x^2-1}{2} = \frac{1-x^2}{2} \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

Therefore, no value of x in the open interval $(-1,1)$ satisfies the given condition. $\square$

Example 7.34. *Prove that $e < 4$.*

7.10 Exercises

7.10.1 Problems for Grading

The following problems are due Friday 7/2/2021 before the class starts.

Exercise 7.19 (20 pts). *Page 164, Problem 2.*

Exercise 7.20 (15 pts). *Page 164, Problem 4.*

Exercise 7.21 (10 pts). *Page 164, Problem 5.*

The following problems are due Monday 7/5/2021 at midnight.

Exercise 7.22 (15 pts). *Page 172, Problem 2.*

Exercise 7.23 (10 pts). *Page 173, Problem 3.*

Exercise 7.24 (10 pts). *Page 173, Problem 4.*

Exercise 7.25 (10 pts). *Page 173, Problem 8.*

7.10.2 Problems for Practice

Pages 164-165: 1, 6.

Pages 172-174: 1, 7, 9, 11, 12.

7.10.3 Summary

- The First Fundamental Theorem states that $\displaystyle\int_a^b F' = F(b) - F(a)$, assuming F is continuous over $[a, b]$ and F' is continuous and bounded over (a, b). (This is also true with a weaker assumption on F', but that is rarely used.)
- The Second Fundamental Theorem states that $\displaystyle\frac{d}{dx}\left(\int_a^x f\right) = f(x)$, assuming f is continuous.
- The Mean Value Theorem for Integrals states that $\displaystyle\frac{1}{b-a}\int_a^b f = f(c)$ for some $c \in [a, b]$ assuming $f : [a, b] \to \mathbb{R}$ is continuous.
- We define $\displaystyle\int_b^a f = -\int_a^b f$ and $\displaystyle\int_c^c f = 0$, where $a < b$. With this definition, additivity, linearity, and Fundamental Theorems all hold.

7.11 Substitution and Integration by Parts

Theorem 7.16 (Integration by Substitution). *Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $\varphi : [c, d] \to \mathbb{R}$ be continuous, whose derivative $\varphi' : (c, d) \to \mathbb{R}$ is bounded and continuous. Assume $\varphi((c, d)) \subseteq (a, b)$. Then,*

$$\int_c^d f(\varphi(x))\varphi'(x)\ \mathrm{d}x = \int_{\varphi(c)}^{\varphi(d)} f(x)\ \mathrm{d}x.$$

Theorem 7.17 (Integration by Parts)**.** *Assume $f, g : [a, b] \to \mathbb{R}$ are continuous and their derivatives are bounded and continuous over (a, b). Then,*

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g.$$

Example 7.35. *Evaluate*

$$\int_1^e \ln x \, \mathrm{d}x, \text{ and } \int_0^1 \frac{2x}{x^2+1} \, \mathrm{d}x.$$

7.11.1 More Examples

Example 7.36. *Suppose $f : \mathbb{R} \to \mathbb{R}$ has a continuous second derivative $f'' : \mathbb{R} \to \mathbb{R}$. Prove that*

$$\int_a^b x f''(x) \, dx = bf'(b) + f(a) - af'(a) - f(b).$$

Solution. Since $f' : \mathbb{R} \to \mathbb{R}$ is differentiable it is continuous and bounded over $[a, b]$. Applying Integration by Parts to x and f' we obtain the following:

$$\int_a^b x f'' = bf'(b) - af'(a) - \int_a^b f'.$$

Using the First Fundamental Theorem we obtain

$$\int_a^b f' = f(b) - f(a).$$

Combining the two yields the result. □

7.12 Exercises

7.12.1 Problems for Grading

The following problems are due Thursday 7/8/2021 before the class starts.

Exercise 7.26 (10 pts)**.** *Suppose m, n are two positive integers. Prove that*

$$\int_0^1 x^n (1-x)^m \, \mathrm{d}x = \int_0^1 x^m (1-x)^n \, \mathrm{d}x.$$

Exercise 7.27 (10 pts)**.** *Page 183, Problem 8.*

Exercise 7.28 (10 pts)**.** *Page 183, Problem 9.*

7.12.2 Practice Problems

Pages 182-183: 1, 2, 4, 5.

7.12.3 Summary

- We proved integration by substitution and by parts.
- Make sure you check all assumptions before you use each theorem.

Chapter 8

Approximation by Taylor Polynomials

8.1 Taylor Polynomials

Let I be a neighborhood of a real number x_0. The functions $f, g : I \to \mathbb{R}$ are said to have contact of order at least zero at x_0 if $f(x_0) = g(x_0)$. We say f and g **have contact of order at least** n **at** x_0 if

$$f^{(k)}(x_0) = g^{(k)}(x_0), \text{ for } k = 0, 1, \ldots, n.$$

If in addition $f^{(n+1)}(x_0) \neq g^{(n+1)}(x_0)$, then we say f and g have contact of order n at x_0.

Example 8.1. *The functions*

$$f(x) = e^x, \text{ and } g(x) = 1 + x + \frac{x^2}{2}$$

have contact of order 2 at 0.

Theorem 8.1. *Let I be a neighborhood of x_0, and n be a non-negative integer. Suppose $f : I \to \mathbb{R}$ has n derivatives. Then, there is a unique polynomial $p_n(x)$ of degree at most n that has contact of order at least n with f at x_0. Furthermore, this polynomial is given by*

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Definition 8.1. The polynomial $p_n(x)$ in the theorem above is called the **n-th Taylor polynomial of f at** x_0.

Example 8.2. *Find the n-th Taylor polynomial of each of the following functions at zero.*

(a) $f(x) = e^x$.

(b) $g(x) = \dfrac{1}{1 - x}$.

(c) $h(x) = \ln(1 + x)$.

Ideally we would like to be able to estimate $f(x)$ with $p_n(x)$.

Definition 8.2. Let $f : I \to \mathbb{R}$ be n times differentiable. The **n-th remainder** of f is the function $R_n : I \to \mathbb{R}$ defined by

$$R_n(x) = f(x) - p_n(x).$$

Theorem 8.2 (The Lagrange Remainder Theorem)**.** *Let I be a neighborhood of x_0 and let n be a nonnegative integr. Suppose $f : I \to \mathbb{R}$ has $n+1$ derivatives. Then, for each point $x \neq x_0$ in I, there is a point c strictly between x and x_0 such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Theorem 8.3. *The number e is irrational.*

Theorem 8.4.

$$\lim_{n\to\infty} \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \right) = \gamma \text{ for some } \gamma \in (0,1)$$

Definition 8.3. The number γ defined in Theorem 8.4 is called **Euler's constant**.

Definition 8.4. Given a sequence of real numbers a_n, with $n \geq 0$, the sequence s_n defined by

$$s_n = \sum_{k=0}^{n} a_k$$

is called the sequence of **partial sums** of the series $\sum_{k=0}^{\infty} a_k$. The number a_k is called the k-th term of the series $\sum_{k=0}^{\infty} a_k$. If s_n converges we say the series $\sum_{k=0}^{\infty} a_k$ converges and we define

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} s_n.$$

If s_n diverges we say the series $\sum_{k=0}^{\infty} a_k$ diverges.

Theorem 8.5. *Let I be a neighborhood of a real number x_0 and assume $f : I \to \mathbb{R}$ has derivatives of all orders. Suppose r, M are positive numbers for which $[x_0 - r, x_0 + r] \subseteq I$ and that*

$$|f^{(n)}(x)| \leq M^n \text{ for all } x \in [x_0 - r, x_0 + r], \text{ and for all } n \in \mathbb{N}.$$

Then,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \text{ for all } x \in [x_0 - r, x_0 + r].$$

To prove the above theorem we need the following:

Lemma 8.1. *Given a constant c, the sequence $\frac{c^n}{n!}$ converges to zero.*

Definition 8.5. The series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the **Taylor series** of f at x_0.

Example 8.3. *Prove that for every real number x we have*

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots .$$

Example 8.4. *For what values of x can you prove the Taylor series of $\ln(1+x)$ at zero converge to $\ln(1+x)$?*

8.1.1 Warm-ups

Example 8.5. *Find the largest n for which e^x and e^{x^2} has a contact of order n at 0.*

Solution. Let $f(x) = e^x$ and $g(x) = e^{x^2}$. We have

$$f(0) = g(0) = 1, \text{ and } f'(0) = 1, \text{ and } g'(0) = 0.$$

Therefore, f and g have contact of order at most 0 but not 1. The answer is $n = 0$. □

8.1.2 More Examples

Example 8.6. *Find Taylor polynomials of $\sin x$ and $\cos x$.*

Solution. Let $f(x) = \sin x$. We will see that

$$f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, \text{ and } f^{(4)}(x) = \sin x.$$

Since $f^{(4)}(x) = f(x)$, we conclude that

$$f^{(5)}(x) = f'(x), f^{(6)}(x) = f''(x), f^{(7)}(x) = f'''(x), f^{(8)}(x) = f^{(4)}(x) = f(x), \dots.$$

Therefore, the derivatives of f repeat in cycles of four. This means

$$f^{(n)}(x) = \begin{cases} \sin x & \text{if the remainder of } n \text{ when divided by 4 is 0} \\ \cos x & \text{if the remainder of } n \text{ when divided by 4 is 1} \\ -\sin x & \text{if the remainder of } n \text{ when divided by 4 is 2} \\ -\cos x & \text{if the remainder of } n \text{ when divided by 4 is 3} \end{cases}$$

Using the fact that $\sin 0 = 0$ and $\cos 0 = 1$ we obtain the following:

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

By the formula of the Taylor polynomials we obtain the following for the n-th Taylor polynomial of f at zero:

$$p_n(x) = \sum_{0 \le k \le \frac{n-1}{2}} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

Similarly the n-th Taylor polynomial of $\cos x$ is given as

$$\sum_{0 \le k \le \frac{n}{2}} \frac{(-1)^k x^{2k}}{(2k)!}.$$

Here, by convention, x^0 is set to be 1 (even if $x = 0$.) □

Example 8.7. *Let $x > 0$. Prove that*

$$1 + \frac{x}{3} - \frac{x^2}{9} < \sqrt[3]{1+x} < 1 + \frac{x}{3}$$

Solution. Let $f(x) = \sqrt[3]{1+x}$. By properties of derivatives

$$f'(x) = \frac{1}{3}(1+x)^{-2/3}, f''(x) = \frac{-2}{9}(1+x)^{-5/3}, f'''(x) = \frac{10}{27}(1+x)^{-8/3}.$$

Therefore, the first and second Taylor polynomials of f at zero are

$$p_1(x) = 1 + \frac{1}{3}x, \text{ and } p_2(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2.$$

Let $x > 0$. Applying the Lagrange Remainder Theorem we conclude that there are real numbers c_1, c_2 strictly between 0 and x for which

$$R_1(x) = \frac{f''(c_1)}{2}x^2 = \frac{-1}{9}(1+c_1)^{-5/3}x^2 < 0, \text{ and } R_2(x) = \frac{f'''(c_2)}{3!}x^2 = \frac{5}{81}(1+c_2)^{-8/3}x^2 > 0.$$

Therefore,

$$f(x) < p_1(x), \text{ and } f(x) > p_2(x), \text{ for all } x > 0.$$

This completes the proof. □

Example 8.8. *Prove that*

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \textit{ and } \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \textit{ for all } x \in \mathbb{R},$$

Solution. First, note that by Example 8.6 it is enough to show

$$\forall\, x \in \mathbb{R} \lim_{n\to\infty} R_n(x) = 0,$$

where $R_n(x)$ is the n-th remainder for $\sin x$ (or $\cos x$) at zero. Note that for if $f(x) = \sin x$ or $\cos x$, then

$$|f^{(n)}(x)| = |\sin x| \text{ or } |\cos x| \Rightarrow |f^{(n)}(x)| \leq 1 = 1^n.$$

Therefore, by Theorem 8.5 both functions are equal to their Taylor series over $[-r, r]$ for every $r > 0$. Since this is true for all $r > 0$ the given equalities hold for every $x \in \mathbb{R}$. □

Example 8.9. *Let m, n be two nonnegative integers. Suppose I is a neighborhood of a real number x_0 and $f, g, p, q : I \to \mathbb{R}$ are functions for which f, g have contact of order n at x_0 and p, q have contact of order m at x_0. Prove that fp and gq have contact of order at least* $\min(m, n)$ *at x_0. For every m, n provide an example that shows fp and gq can have contact of order precisely* $\min(m, n)$ *at x_0.*

8.2 Exercises

8.2.1 Problems for Grading

The following problems are due Friday 7/9/2021 before the class starts.

Exercise 8.1 (20 pts). *Page 202, Problem 1.*

Exercise 8.2 (10 pts). *Page 202, Problem 5.*

Exercise 8.3 (10 pts). *Let n be a positive integer. Find the n-th Taylor polynomial of $f(x) = x^n e^x$ at zero.*

Hint: Use the formula for $(fg)^{(k)}$ proved in one of the exercises.

The following problems are due Monday 7/12/2021 before the class starts.

Exercise 8.4 (10 pts). *In class we proved that $\gamma \in [0, 1]$. Prove that $\gamma \in (0, 1)$.*

Exercise 8.5 (10 pts). *Page 208, Problem 5.*

Exercise 8.6 (15 pts). *Page 208, Problem 11.*

Exercise 8.7 (10 pts). *Page 212, Problem 3.*

Exercise 8.8 (10 pts). *Page 212, Problem 4.*

8.2.2 Practice Problems

Exercise 8.9. *Prove that* $\sin 1$ *is irrational.*

Pages 202-203: 2, 3, 4, 6.

Pages 207-209: 1, 3, 6, 7, 10, 11.

Pages 211-212: 1, 2, 5.

8.2.3 Summary

- To find the n-th Taylor polynomial of $f(x)$ at x_0:
  - Find the first few derivatives of f.
  - Guess the general form of the n-th derivative of f.
  - Prove your guess by induction on n.
  - Use the formula $p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$.
- The Lagrange Remainder Theorem states that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}, \text{ for some } c \text{ strictly between } x_0 \text{ and } x.$$

- An infinite sum $\sum_{k=0}^{\infty} a_k$ is defined as the limit of partial sums

$$s_n = a_1 + \cdots + a_n.$$

- To show
$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k, \text{ for all } x \in (x_0 - r, x_0 + r) \qquad (*)$$
Select an arbitrary positive number $s < r$ and show that there is a positive real number M for which
$$|f^{(n)}(x)| \leq M^n, \text{ for all } x \in [x_0 - s, x_0 + s], \text{ and all } n \in \mathbb{N}.$$
Note that M must be independent of n and x, but it could depend on s and x_0.
Allow s to approach r and conclude $(*)$ holds for all $x \in (x_0 - r, x_0 + r)$.

8.3 The Cauchy Integral Remainder Theorem

As we saw in the previous section, the Lagrange Remainder Theorem has its own limitations.

Theorem 8.6 (The Cauchy Integral Remainder Theorem)**.** *Let I be a neighborhood of a real number x_0 and n be a natural number. Suppose the function $f : I \to \mathbb{R}$ has $n + 1$ derivatives and that $f^{(n+1)} : I \to \mathbb{R}$ is continuous. Then for each $x \in I$,*
$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x - t)^n \, dt.$$

Example 8.10. *Prove that for each $x \in (-1, 1]$ we have*
$$\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}.$$

Definition 8.6. Given a real number α and a nonnegative integer n we define $\binom{\alpha}{n}$ as follows:
$$\binom{\alpha}{0} = 1, \text{ and } \binom{\alpha}{n} = \frac{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}{n!} \text{ if } n > 0.$$

Theorem 8.7 (The Binomial Theorem)**.** *For every real number α and every $x \in (-1, 1)$ we have the following:*
$$(1 + x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

In order to prove the above theorem we need the following:

Lemma 8.2 (The Ratio Lemma for Sequences)**.** *Let c_n be a sequence of nonzero real numbers for which $\left|\frac{c_{n+1}}{c_n}\right|$ converges to ℓ, where ℓ is a real number or ∞.*

(a) If $\ell > 1$, then c_n is unbounded and thus it diverges.

(b) If $\ell < 1$, then c_n converges to zero.

Remark 8.1. If $\ell = 1$, then c_n could be divergent or convergent.

Example 8.11. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then f has derivatives of all orders but the only value of x for which

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

is $x = 0$.

Definition 8.7. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **analytic** if

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \text{ for all } x \in \mathbb{R}.$$

Above we showed an example of a function that is not analytic but is infinitely differentiable.

8.3.1 Warm-ups

Example 8.12. *Let $p(x)$ be a polynomial. Prove that for every positive integer n the n-th derivative of $e^{p(x)}$ is of the form $q(x)e^{p(x)}$.*

Solution. This can be proved by induction on n. □

8.3.2 More Examples

Example 8.13. *Let α be a real number. Prove that there is a unique differentiable function $f : (-1, 1) \to \mathbb{R}$ that satisfies*

$$(1 + x)f'(x) = \alpha f(x), \text{ for all } x \in (-1, 1), \text{ and } f(0) = 1.$$

Solution. First, note that if $f(x) = (1 + x)^{\alpha}$, then $f'(x) = \alpha(1 + x)^{\alpha-1}$ and thus

$$(1 + x)f'(x) = (1 + x)\alpha(1 + x)^{\alpha-1} = \alpha(1 + x)^{\alpha} = \alpha f(x).$$

Also, $f(0) = 1$. Thus, this function satisfies the given conditions.

Now assume $f(x)$ satisfies the given conditions. Consider the function $g(x) = f(x)(1 + x)^{-\alpha}$. We have

$$g'(x) = f'(x)(1 + x)^{-\alpha} + f(x)(-\alpha)(1 + x)^{-\alpha-1} = (1 + x)^{-\alpha-1}((1 + x)f'(x) - \alpha f(x)) = 0.$$

By the Identity Criterion, $g(x)$ is a constant function. Since $g(0) = f(0)(1+0)^{-\alpha} = 1$, we have $g(x) = 1$ and thus $f(x) = (1 + x)^{\alpha}$ for all $x \in (-1, 1)$, as desired. □

Example 8.14. *Suppose in the Ratio Lemma for sequences $\ell = 1$. Are there any limitations for what real numbers $\dfrac{c_{n+1}}{c_n}$ could converge? Is it possible that c_n diverges to ∞? Is it possible that $\lim\limits_{n\to\infty} c_n$ does not exist at all?*

Solution. Given a nonzero real number r consider the constant sequence $c_n = r$. Then, $c_{n+1}/c_n = 1$ and $c_n \to r$.

Let $c_n = \dfrac{1}{n}$ and note that

$$\frac{c_{n+1}}{c_n} = \frac{n}{n+1} = 1 - \frac{1}{n+1} \to 1, \text{ and } c_n \to 0.$$

If $c_n = (-1)^n$, then $\left|\dfrac{c_{n+1}}{c_n}\right| = 1$ and c_n does not have a limit.

Similarly if $c_n = n$ or $c_n = -n$, then we could have $\ell = 1$ and c_n diverging to ∞ or $-\infty$.

Therefore it is possible that $\ell = 1$ and c_n converges to any arbitrary real number or diverges to $\pm\infty$ or its limit does not exist. □

Example 8.15. *Prove that for every positive integer n we have*

$$\int_0^1 (1-x^2)^n \, \mathrm{d}x \geq \frac{2}{3\sqrt{n}}.$$

Solution. By the Bernoulli's Inequality, the monotonicity of integrals, and the First Fundamental Theorem we have the following:

$$\forall\, x \in [0,1]\ (1-x^2)^n \geq 1 - nx^2 \Rightarrow \int_0^1 (1-x^2)^n \, \mathrm{d}x \geq \int_0^1 (1-nx^2) \, \mathrm{d}x = 1 - \frac{n}{3}.$$

Here we are using the fact that the derivative of $x - nx^3/3$ is $1 - nx^2$ which is continuous over $\mathbb{R}$.

This, however is not very helpful when n is large as the right hand side is negative when $n > 4$. So, we should make sure to only consider the integral over the region that the integrand is positive. In other words, we will make sure $1 - nx^2 \geq 0$ or $x \leq 1/\sqrt{n}$. Here is the solution:

$$\int_0^1 (1-x^2)^n \, \mathrm{d}x \geq \int_0^{1/\sqrt{n}} (1-x^2)^n \, \mathrm{d}x \geq \int_0^{1/\sqrt{n}} (1-nx^2) \, \mathrm{d}x = \frac{1}{\sqrt{n}} - \frac{n}{3n\sqrt{n}} = \frac{2}{3\sqrt{n}}.$$

This completes the proof. □

Example 8.16. *Give an example of an infinitely differentiable strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ for which $f^{(n)}(0) = 0$ for all $n \geq 0$.*

8.4 Exercises

8.4.1 Problems for Grading

The following problems are due Tuesday 7/13/2021 before the class starts.

Exercise 8.10 (10 pts)**.** *Page 215, Problem 2.*

Exercise 8.11 (10 pts)**.** *Page 220, Problem 4.*

Exercise 8.12 (10 pts)**.** *Page 220, Problem 5.*

Exercise 8.13 (10 pts)**.** *Page 221, Problem 9.*

The following problems are due Thursday 7/15/2021 before the class starts.

Exercise 8.14 (10 pts)**.** *Page 223, Problem 2.*

Exercise 8.15. *Page 223, Problem 3.*

Exercise 8.16. *Page 223, Problem 4.*

Exercise 8.17. *We gave an example of a function whose derivatives are all zero at point* 0 *but is not equal to its Taylor series. Is there a function* $f : \mathbb{R} \to \mathbb{R}$ *for which* $f^{(n)}(0) = 1$ *for all* n *but is not equal to its Taylor series? Given a sequence of real numbers* a_n*, is there a function* $f : \mathbb{R} \to \mathbb{R}$ *for which* $f^{(n)}(0) = a_n$ *for all* n *and* f *is not equal to its Taylor series?*

8.4.2 Problems for Practice

Page 215: 1, 3, 4, 5.

Page 220-221: 2, 6, 7, 8.

8.4.3 Summary

- The Cauchy Integral Remainder Theorem states that

$$f(x) = p_n(x) + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \, \mathrm{d}t,$$

  assuming $f^{(n+1)}$ is continuous on a neighborhood of x_0.

- The Cauchy Integral Remainder Theorem is generally more difficult to work with but is a stronger theorem compared to the Lagrange Remainder Theorem.

- There is a function that is differentiable infinitely many times but is never equal to its Taylor series except at the center x_0.

- The Binomial Theorem states that $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ for all $\alpha \in \mathbb{R}$ and all $x \in (-1, 1)$.

Chapter 9

Sequences and Series of Functions

By linearity of limits we obtain the following:

Theorem 9.1 (Linearity for Series). *Suppose* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *converge to real numbers* a *and* b*, respectively, and let* α, β *be two real numbers. Then, the series* $\sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)$ *converges to* $\alpha a + \beta b$*.*

Sometimes it is impossible to find the limit of a sequence even though the sequence is convergent. The Monotone Convergence Theorem allows us to show a monotone sequence converges without finding its limit. How about other sequences? The following addresses this issue:

Definition 9.1. A sequence a_n is said to be **Cauchy** if the following holds:

$$\forall\ \epsilon > 0\ \exists\ N \in \mathbb{N} \text{ such that } n, m \geq N \Rightarrow |a_n - a_m| < \epsilon.$$

Theorem 9.2 (The Cauchy Convergence Criterion for Sequences). *A sequence is convergent if and only if it is Cauchy.*

Remark 9.1. The above theorem is specific to real numbers. For example a sequence of rationals that converges to $\sqrt{2}$ is Cauchy but it does not converge to a rational number.

Applying the above test to partial sums of a series we obtain the following:

Theorem 9.3 (The Cauchy Convergence Criterion for Series). *A series* $\sum_{n=0}^{\infty} a_n$ *converges if and only if*

$$\forall\ \epsilon > 0\ \exists\ N \in \mathbb{N} \textit{ such that } \forall\ m, n \in \mathbb{N}\ m \geq n \geq N \Rightarrow |a_n + a_{n+1} + \cdots + a_m| < \epsilon.$$

Remark 9.2. In order to determine if a series converges we may ignore the first few terms of the series. In other words, for every $N \in \mathbb{N}$, the series $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=N}^{\infty} a_n$ converges.

Theorem 9.4 (Test for Divergence). *If* $\sum_{n=0}^{\infty} a_n$ *converges, then* $a_n \to 0$*. Consequently, if* a_n *does not converge to zero, then the series* $\sum_{n=0}^{\infty} a_n$ *diverges.*

Example 9.1. *Given a real number* r *prove that* $\sum_{n=0}^{\infty} r^n$ *converges if and only if* $|r| < 1$*.*

Example 9.2. *By an example show that the converse of Test for Divergence is not true.*

Theorem 9.5 (The Comparison Test)**.** *Suppose $a_n \geq b_n \geq 0$ for all n.*

(a) If $\sum_{n=0}^{\infty} a_n$ converges, then so does $\sum_{n=0}^{\infty} b_n$.

(b) If $\sum_{n=0}^{\infty} b_n$ diverges, then so does $\sum_{n=0}^{\infty} a_n$.

Example 9.3. *Determine if each of the following is convergent.*

(a) $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}$.

(b) $\sum_{n=2}^{\infty} \frac{1}{n - \sqrt{n}}$.

Theorem 9.6 (The Integral Test)**.** *Suppose a_n is a sequence of nonnegative real numbers. Suppose there is a natural number N for which $f(n) = a_n$ for all $n \geq N$, where $f : [N, \infty) \to \mathbb{R}$ is a continuous and decreasing function for which $f(x) \geq 0$ for all $x \geq N$. Then, $\sum_{n=0}^{\infty} a_n$ converges if and only if the sequence*

$$\int_N^n f \text{ with } n = N, N+1, \ldots$$

is bounded.

Theorem 9.7 (The p-Test)**.** *Given a real number p, the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

Example 9.4. *For what values of p does the series $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ converge?*

Theorem 9.8 (The Alternating Series Test)**.** *Suppose a_n is a sequence that is decreasing and converges to zero. Then the series*

$$a_0 - a_1 + a_2 - \cdots = \sum_{n=0}^{\infty} (-1)^n a_n$$

converges.

Example 9.5. *Prove that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.*

Theorem 9.9 (The Absolute Convergence Test)**.** *Suppose a_n is a sequence for which $\sum_{n=0}^{\infty} |a_n|$ converges, then so does $\sum_{n=0}^{\infty} a_n$.*

Example 9.6. *Prove that $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent.*

Definition 9.2. A series $\sum_{n=0}^{\infty} a_n$ is called **absolutely convergent** if $\sum_{n=0}^{\infty} |a_n|$ converges. If $\sum_{n=0}^{\infty} a_n$ converges but $\sum_{n=0}^{\infty} |a_n|$ diverges, we say $\sum_{n=0}^{\infty} a_n$ **converges conditionally**.

Theorem 9.10 (The Ratio Test)**.** *Suppose a_n is a sequence of nonzero numbers for which $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \ell$ exists.*

(a) If $\ell < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely.

(a) If $\ell > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges.

Theorem 9.11 (The Root Test)**.** *Suppose a_n is a sequence of numbers for which $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \ell$ exists.*

(a) If $\ell < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely.

(a) If $\ell > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges.

Theorem 9.12 (The Limit Comparison Test)**.** *Suppose a_n, b_n are two sequence of positive numbers for which*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \ell \textit{ is a positive real number.}$$

Then, $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} b_n$ converges.

9.0.1 Warm-ups

Example 9.7. *In the Comparison Test we assume $a_n \geq b_n \geq 0$. Is the assumption that b_n is nonnegative necessary?*

Solution. The answer is yes. For example let a_n be any sequence of positive numbers for which $\sum_{n=0}^{\infty} a_n$ converges and let $b_n = -1$. Then, $\sum_{n=0}^{\infty} b_n = \sum -1$ diverges even though $\sum_{n=0}^{\infty} a_n$ converges. □

Example 9.8. *Determine if the series $\sum_{n=1}^{\infty} \sqrt[n]{n}$ converges.*

Solution. Note that $\sqrt[n]{n} \geq 1$ and thus the sequence $\sqrt[n]{n}$ does not converge to zero. Therefore, by the Test for Divergence the series $\sum_{n=1}^{\infty} \sqrt[n]{n}$ does not converge. □

9.0.2 More Examples

Example 9.9. *Determine if each series converges.*

(a) $\sum_{n=1}^{\infty} \frac{n^2}{n!}$.

(b) $\sum_{n=}^{\infty} \frac{1}{(\ln n)^n}$.

(c) $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$.

(d) $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$.

Example 9.10. *Suppose a_n is a decreasing sequence of nonnegative numbers. Prove that $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.*

Example 9.11. *Suppose the two series $\sum_{n=0}^{\infty} a_n$, and $\sum_{n=0}^{\infty} b_n$ converge. Is it true that $\sum_{n=0}^{\infty} a_n b_n$ must converge?*

Example 9.12. *Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are analytic and α, β are real numbers. Prove that $\alpha f + \beta g$ is analytic.*

Example 9.13. *Determine if the series converges:*

$$\sum_{n=1}^{\infty} \frac{\sin n}{n\sqrt{n}}.$$

9.1 Exercises

9.1.1 Problems for Grading

The following problems are due Friday 7/16/2021 before the class starts.

Exercise 9.1 (35 pts)**.** *Page 239, Problem 1.*

Exercise 9.2 (10 pts)**.** *Page 240, Problem 2.*

Exercise 9.3 (10 pts)**.** *Prove the Limit Comparison Test: Suppose a_n, b_n are two sequence of positive numbers for which*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \ell \text{ is a positive real number.}$$

Then, $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} b_n$ converges.

9.1.2 Problems for Practice

Exercise 9.4. *Suppose a_n is a sequence that converges to a number $a \in (-1, 1)$. Prove that $\sum_{n=1}^{\infty} a_n^n$ converges.*

Exercise 9.5. *Suppose a_n is a sequence of nonnegative real numbers for which $\sum_{n=1}^{\infty} a_n$ converges. Is it true that there must exist a natural number N for which $a_n \leq \frac{1}{n}$ for all $N \geq n$?*

Page 240: 3, 6, 7.

9.1.3 Challenge Problems

Exercise 9.6. *Define the sequence a_n by $a_n = \int_1^n (2^{1/x} - 1)\, dx$. Is this sequence convergent?*

Exercise 9.7. *Determine if $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ diverges, converges conditionally, or converges absolutely.*

Exercise 9.8. *Prove that if $\sum_{n=0}^{\infty} a_n^2$ and $\sum_{n=0}^{\infty} b_n^2$ both converge, then $\sum_{n=0}^{\infty} a_n b_n$ also converges.*

Exercise 9.9. *Prove that if* $\sum_{n=0}^{\infty} a_n^2$ *converges, then* $\sum_{n=0}^{\infty} \frac{a_n}{n^p}$ *converges for all* $p > 1/2$.

Exercise 9.10. *Suppose* a_n *is a decreasing sequence that converges to zero. Prove that if* $\sum_{n=0}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} na_n = 0$.

Exercise 9.11. *Prove that if* a_n *is a sequence of nonnegative numbers for which* $\sum_{n=0}^{\infty} a_n$ *diverges, then the series* $\sum_{n=0}^{\infty} \frac{a_n}{1+a_n}$ *also diverges.*

Exercise 9.12. *Suppose* a_n *is a sequence of nonnegative numbers for which* $\sum_{n=1}^{\infty} a_n$ *converges. Prove that the sequence* $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ *also converges.*

9.1.4 Summary

- To determine if a series converges start with approximating the general term of the series.
- Before applying a convergence test make sure you check all the conditions.
- If the terms of a series involve factorials or powers of constants, you should consider using the Ratio Test.
- If the terms involve rational functions consider using the Limit Comparison Test or the Comparison Test.
- $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. If $\sum a_n$ converges but $\sum |a_n|$ diverges we say $\sum a_n$ converges conditionally.

9.2 Pointwise and Uniform Convergence

Definition 9.3. Given a sequence of functions $f_n : D \to \mathbb{R}$ and a function $f : D \to \mathbb{R}$ we say f_n **converges pointwise** to f on D if

$$f(x) = \lim_{n\to\infty} f_n(x), \text{ for all } x \in D.$$

We say f is the **pointwise limit** of f_n. This is sometimes denoted by

$$f_n \xrightarrow{\text{p.w.}} f$$

Example 9.14. *Find the pointwise limit of each sequence of functions:*

(a) $f_n : [0,1] \to \mathbb{R}$ *given by* $f_n(x) = x^n$.

(b) $g_n : \mathbb{R} \to \mathbb{R}$ *given by* $f_n(x) = e^{-nx^2}$.

(c) $h_n : [0,1] \to \mathbb{R}$ *given by*

$$h_n(x) = \begin{cases} 0 & \textit{if } x = \dfrac{m}{2^n} \textit{ for some integer } m \\ 1 & \textit{otherwise} \end{cases}$$

Remark 9.3. The above examples show that the pointwise limit of a sequence of continuous, differentiable or integrable functions may not be continuous, differentiable or integrable, respectively.

Definition 9.4. We say a sequence of functions $f_n : D \to \mathbb{R}$ **converges uniformly** to the function $f : D \to \mathbb{R}$ if the following holds:

$$\forall\, \epsilon > 0\ \exists\, N \in \mathbb{N}\ \forall\, n \geq N\ \forall\, x \in D\ |f_n(x) - f(x)| < \epsilon.$$

In that case we write $f_n \xrightarrow{u} f$.

Remark 9.4. Note that by definition of uniform convergence, if f_n uniformly converges to f, then f is the pointwise limit of f_n.

Example 9.15. *Check if each of the convergences in Example 9.14 is uniform.*

Example 9.16. *Find the pointwise limit of the sequence of functions $f_n : [-0.5, 0.5] \to \mathbb{R}$ given by $f_n(x) = x^n$. Prove that this convergence is uniform.*

Definition 9.5. A sequence of functions $f_n : D \to \mathbb{R}$ is called **uniformly Cauchy** if

$$\forall\, \epsilon > 0\ \exists\, N \in \mathbb{N} \text{ such that } \forall\, n, m \geq N\ \forall\, x \in D\ |f_n(x) - f_m(x)| < \epsilon.$$

Theorem 9.13 (The Weierstrass Uniform Convergence Criterion)**.** *A sequence of functions $f_n : D \to \mathbb{R}$ is uniformly convergent if and only if it is uniformly Cauchy.*

Example 9.17. *Prove that $f_n(x) = \sum_{k=1}^{n} \dfrac{x^k}{k^2 \cdot 3^k}$ uniformly converges to $f(x) = \sum_{k=1}^{\infty} \dfrac{x^k}{k^2 \cdot 3^k}$ over $[-2, 2]$. Use that to prove the function $\sum_{k=1}^{\infty} \dfrac{x^k}{k^2 \cdot 3^k}$ is continuous over $[-2, 2]$.*

Theorem 9.14. *Suppose a sequence of functions $f_n : D \to \mathbb{R}$ converges uniformly to $f : D \to \mathbb{R}$, then:*

(a) If f_n is continuous for all n, then f is continuous.

(b) If $D = [a, b]$ is a closed and bounded interval and f_n is integrable over $[a, b]$ for all n, then f is integrable over $[a, b]$. Furthermore, $\displaystyle\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

Remark 9.5. With the assumptions of the above example,

$$\lim_{n\to\infty} \int_b^a f_n = \int_b^a f.$$

Theorem 9.15. *Let I be an open interval. Suppose $f_n : I \to \mathbb{R}$ is a sequence of differentiable functions with continuous derivatives that satisfy both of the following:*

- *$f_n \xrightarrow{p.w.} f$ over I, and*
- *$f_n' \xrightarrow{u} g$ over I.*

Then, f is differentiable and $f' = g$.

Theorem 9.16. *Let I be an open interval, $f_n : I \to \mathbb{R}$ be a sequence of continuously differentiable functions for which both of the following holds:*

- $f_n \xrightarrow{p.w.} f$ *over* I, *and*
- f_n' *is uniformly Cauchy on* I.

Then, f *is continuously differentiable and* $f'(x) = \lim_{n\to\infty} f_n'(x)$.

9.2.1 Warm-ups

Example 9.18. *Suppose* $f_n, f : D \to \mathbb{R}$ *are functions for which* $f_n \xrightarrow{u} f$. *Prove that* $f_n \xrightarrow{p.w.} f$, *but the converse is not true.*

Solution. By definition, for every $\epsilon > 0$, there is $N \in \mathbb{N}$ for which for all $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$, for all $x \in D$. This means $f_n(x) \to f(x)$ and thus $f_n \xrightarrow{p.w.} f$.

We saw in an example that $f_n(x) = x^n$ converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$

This convergence is not uniform, though since all f_n's are continuous but f is not. □

9.2.2 More Examples

Example 9.19. *Consider the sequence of functions* $f_n : \mathbb{R} \to \mathbb{R}$ *given by* $f_n(x) = \dfrac{\sin(nx+1)}{\sqrt{n+1}}$.

(a) Find the pointwise limit of $f_n(x)$.

(b) Is the convergence in part (a) uniform?

Solution. (a) We see that

$$|f(x)| \leq \frac{1}{\sqrt{n+1}}.$$

Applying the definition of limit and the Comparison Lemma we conclude that $f_n \xrightarrow{p.w.} 0$. (Show this!)
(b) Since we can make $1/\sqrt{n+1}$ less than ϵ for all x, the convergence must be uniform. We will now prove this. Given $\epsilon > 0$, by the Archimedean Property, there is $N \in \mathbb{N}$ for which $1/\epsilon < N$. If $n \geq N^2$, then

$$|f(x)| = \frac{|\sin(nx+1)|}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} \leq \frac{1}{N} < \epsilon.$$

Therefore, $f_n \xrightarrow{u} 0$. □

Example 9.20. *Find the pointwise limit of the sequence of functions* $f_n : (0,\infty) \to \mathbb{R}$ *given by* $f_n(x) = \dfrac{1}{nx}$. *Is the convergence uniform?*

Solution. By properties of limit

$$\lim_{n\to\infty} \frac{1}{nx} = \frac{1}{x} \lim_{n\to\infty} \frac{1}{n} = 0.$$

This convergence is not uniform. Suppose the convergence were uniform. Thus, for $\epsilon = 1$, there must be $N \in \mathbb{N}$ for which for all $n \geq N$ we have

$$|\frac{1}{nx} - 0| < 1, \text{ for all } x > 0.$$

Setting $x = 1/n$ we obtain a contradiction $1 < 1$. Therefore, the convergence is not uniform. □

Example 9.21. *Let $f_n : D \to \mathbb{R}$ be a sequence of functions and $f : D \to \mathbb{R}$ be a function. Suppose a_n is a sequence of real numbers that converges to zero. Suppose in addition that*

$$|f_n(x) - f(x)| \leq a_n, \textit{ for all } x \in D \textit{ and all } n \in N.$$

Prove that $f_n \xrightarrow{u} f$.

Example 9.22. *Give an example of a sequence of continuous functions $f_n : (0,1) \to \mathbb{R}$ that converges to a continuous function $f : (0,1) \to \mathbb{R}$ but the convergence is not uniform.*

Example 9.23. *Prove that the series*

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2 + x^{2n}}$$

converges uniformly over $\mathbb{R}$.

9.3 Exercises

9.3.1 Problems for Grading

The following problems are due Monday 7/19/2021 before the class starts.

Exercise 9.13 (10 pts)**.** *Page 248, Problem 1.*

Exercise 9.14 (10 pts)**.** *Page 248, Problem 2.*

Exercise 9.15 (10 pts)**.** *Page 249, Problem 6.*

Exercise 9.16 (10 pts)**.** *Page 249, Problem 8.*

The following problems are due Tuesday 7/20/2021 before the class starts.

Exercise 9.17 (10 pts)**.** *Page 254, Problem 1.*

Exercise 9.18 (10 pts)**.** *Page 254, Problem 3.*

Exercise 9.19 (10 pts)**.** *Page 254, Problem 4.*

9.3.2 Problems for Practice

Page 244: 2, 4.

Page 249: 3, 4, 5, 7.

Page 254: 2, 5.

9.3.3 Summary

- To prove $f_n \xrightarrow{p.w.} f$ you need to show given $x \in D$, the sequence $f_n(x)$ approaches $f(x)$.
- To find the uniform limit of a sequence of functions f_n:
  - Find the pointwise limit f of f_n.
  - Start with an arbitrary $\epsilon > 0$ and show there is $N \in \mathbb{N}$ for which $|f_n(x) - f(x)| < \epsilon$ for all $x \in D$.
  - N may depend on ϵ but cannot depend on x.
- To prove a sequence $f_n : D \to \mathbb{R}$ is uniformly Cauchy:
  - Let $\epsilon > 0$ be fix.
  - Show there is $N \in \mathbb{N}$ for which
  $$\text{If } m, n \geq N \Rightarrow |f_n(x) - f_m(x)| < \epsilon \text{ for all } x \in D.$$
  - N may only depend on ϵ and cannot depend on x. (That is why it is called "uniform".)
- To prove $f_n \to f$ is not uniform, use proof by contradiction.
  - Write down what it means using the ϵ-N definition.
  - Choose an appropriate ϵ and x to achieve a contradiction.
  - To find appropriate values of x look for places that could cause a problem. You may want to use limits if finding an explicit value of x is hard.
- If a sequence is continuous (resp. integrable) its uniform limit is continuous (resp. integrable).
- The uniform limit of a sequence of differentiable functions may not be differentiable. We need to know:
  - $f_n \xrightarrow{p.w.} f$, and
  - f_n' is uniformly Cauchy.
  - Note that the domain must be an open interval, since we are discussing differentiability.

9.4 Power Series

Definition 9.6. Given a sequence of real numbers $a_n, n \geq 0$ and a real number x_0, the **power series at** x_0 **associated with the sequence** a_n is defined as

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

for all x that the series converges. The set of all values of x for which the above series converges is said to be the **domain of convergence of** $f(x)$.

Remark 9.6. In the above sum, $(x - x_0)^0$ is considered as 1, even when $x = x_0$.

Remark 9.7. Note that the above sum is always convergent for $x = x_0$, and thus x_0 is in the domain of convergence of the power series $f(x)$ above.

Example 9.24. *Find the domain of convergence of* $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

Definition 9.7. Assume A is a subset of the domain of convergence of

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k.$$

Set

$$s_n(x) = \sum_{k=0}^{n} a_k (x - x_0)^k, \text{ for all } n \geq 0, \text{ for all } x \in A.$$

We say $\sum_{k=0}^{\infty} a_k (x - x_0)^k$ converges uniformly on A if

$$s_n \xrightarrow{u} f, \text{ on } A.$$

Note: From now on we will only focus on power series centered at zero, but all results hold for power series centered at other points as well.

Theorem 9.17. *Suppose* $\sum_{k=0}^{\infty} a_k x^k$ *converges for* $x = x_0$*, where* $x_0 \neq 0$*, and* r *is a positive real number for which* $r < |x_0|$*. Then, both series*

$$\sum_{k=0}^{\infty} a_k x^k, \text{ and } \sum_{k=1}^{\infty} k a_k x^{k-1}$$

converge uniformly on $[-r, r]$.

Theorem 9.18. *Suppose* $(-r, r)$ *lies in the domain of convergence of* $\sum_{k=0}^{\infty} a_k x^k$*. Define*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \text{ for all } x \in (-r, r).$$

Then, f *is infinitely differentiable over* $(-r, r)$ *and*

$$f^{(n)}(x) = \sum_{k=n}^{\infty} a_k \frac{d^n(x^k)}{dx^n}, \text{ for all } n \in \mathbb{N}.$$

Furthermore,

$$a_n = \frac{f^{(n)}(0)}{n!}, \text{ for all } n \geq 0.$$

Example 9.25. *Find a formula for* $\sum_{k=1}^{\infty} kx^k$ *for all* $x \in (-1, 1)$.

Theorem 9.19. *The domain of convergence of every power series* $\sum_{k=0} a_k x^k$ *is one of the following:*

- $\{0\}$, *i. e. the power series converges only for* $x = 0$.
- $\mathbb{R}$, *i. e. the power series converges for all* $x \in \mathbb{R}$.
- $(-r, r), [-r, r), (-r, r]$, *or* $[-r, r]$, *for some positive real number* r.

Remark 9.8. The above theorem shows that the domain of convergence is always an interval. For that reason we sometimes use "interval of convergence" instead of "domain of convergence". Furthermore, we say the **radius of convergence** is zero, ∞, and r in each case, respectively.

Example 9.26. *Find the domain of convergence of each power series:*

(a) $\sum_{n=2}^{\infty} \frac{x^n}{n \ln n}$.

(b) $\sum_{n=0}^{\infty} n! x^n$.

(c) $\sum_{n=1}^{\infty} n x^n$.

9.4.1 Warm-ups

Example 9.27. *Give an example of a power series whose domain of convergense is* $\mathbb{R}$ *and one whose domain of convergence is* $\{1\}$.

9.4.2 More Examples

Example 9.28. *In this example we provide a different proof for the Binomial formula.*
Let α *be a real number. Consider the power series*

$$f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

(a) Prove that $(-1, 1)$ *lies in the domain of convergence of* $f(x)$.

(b) Prove that $(1 + x) f'(x) = \alpha f(x)$, *and* $f(0) = 1$.

(c) Use part (b) to prove the Binomial Theorem.

Example 9.29. *Suppose* r *is a positive real number and*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \text{ for all } x \in (-r, r).$$

Prove that if f *is an even function, then* $a_n = 0$ *for all odd* n, *and if* f *is an odd function, then* $a_n = 0$ *for all even* n.

9.5 Exercises

9.5.1 Problems for Grading

The following problems are due Thursday 7/22/2021 before the class starts.

Exercise 9.20 (15 pts)**.** *Page 262, Problem 1.*

Exercise 9.21 (10 pts)**.** *Page 262, Problem 7.*

Exercise 9.22 (10 pts)**.** *Page 263, Problem 13.*

9.5.2 Problems for Practice

Exercise 9.23. *Is there an analytic function $f : \mathbb{R} \to \mathbb{R}$ for which $f^{(n)}(0) = n$ for all $n \geq 0$? If there is, find all such functions.*

Pages 262-263: 4, 5, 8, 10, 12, 14.

9.5.3 Challenge Problems

Exercise 9.24. *Find the domain of convergence of $\sum_{n=2}^{\infty} \frac{x^n}{n \ln n}$.*

Exercise 9.25. *Show that the following series converges uniformly on $\mathbb{R}$.*

$$\sum_{n=1}^{\infty} \frac{x}{n + n^2 x^2}.$$

9.5.4 summary

- Domain of convergence of a power series is the set of all points where the power series converges.
- If a power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ converges over an open interval, then $f(x)$ is infinitely differentiable, and f can be differentiated term-by-term, and $a_n = \frac{f^{(n)}(0)}{n!}$.
- To find the domain of convergence of a power series we use the Ratio (or Root) test. We would like the limit $\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right|$ to be less than 1. There are three cases:
  - The limit is 0, independent of x. In that case the domain of convergence is $\mathbb{R}$.
  - The limit is ∞ for all $x \neq 0$. In which case, the domain of convergence is $\{0\}$.
  - The limit depends on x. In which case, we set the limit to be less than 1. That yields an open interval for possible values of x. Then, we would have to check the endpoints of this interval in order to find the domain of convergence.

Appendix A

Textbook Erratum and Comments

In addition to the errata that can be found here, I have also noticed the following:

Page 16. x could be zero, which makes zx rational. To avoid that, one could use $a+z$ and $b+z$ instead of a/z and b/z.

Page 28. C does not need to be non-negative. The proof for $C \leq 0$ works the same way as when $C = 0$.

Page 20, Problem 11. The Bernoulli Inequality holds for all $b \geq -1$. It can be proved by induction on n.

It appears that the existence of the n-th root of a natural number has not been proved prior to page 33, but in page 33, Problem 10 its existence is needed.

Page 42: In the last sentence of Problem 8, I think it would be better to replace the word "series" by "sequence" since as far as I can tell a series is not defined yet.

Page 81: Example 3.32 can be solved faster using Theorem 2.20, i.e. the Sequential Density of $\mathbb{Q}$, (and exercise 3, page 37 for the second part.)

Page 106: In Theorem 4.22, one may only assume that f is differentiable and f is twice differentiable at x_0. (In other words, we do not need f to be twice differentiable over I.)

Page 155, Problem 5: $U(f, P_2)$ must be $U(g, P_2)$.

Page 183: For problem 9, perhaps the assumption that $f : (0, \infty) \to \mathbb{R}$ differentiable is needed in order for us to be able to invoke problem 8? Both problems are true without the differentiability of f but much harder to prove, as far as I can tell.

Page 212: Problems 3 and 4 are probably meant to be worded similar to problem 5(b). The current wording says the Taylor expansion converges, but it does not say what it converges to.

Page 244: Problem 5 is the same as problem 27, page 22.

Page 263: Problem 11 is valid for $\alpha \in (-1, 1)$. It is also essentially the same as Problem 9 in the same page. This problem is also the same as Problem 7, page 42.