Math 445 Summary and Homework

February 6, 2021

Notations

- $\wedge$, conjunction.
- $\vee$, disjunction.
- $\rightarrow$, implication.
- $\neg$, negation.
- $\mathbb{N} = \{0, 1, \ldots\}$, the set of non-negative integers.
- $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, the set of integers.
- $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, \text{ and } n \neq 0\}$, the set of all rational numbers.
- $\mathbb{R}$, the set of all real numbers.
- $a \mid b$, a divides b.
- $\text{rem}(x, y)$ the remainder when x is divided by y.
- $\beta(x, y)$ Gödel's β-function.

Contents

This note may contain some typos. Feel free to message me if you see any typos.

1 Week 1

1.1 Connectives and Sentences of Sentential Logic

Definition 1.1. The **symbols** of a sentential logic $\mathcal{S}$ are

- A (finite or countable) set of sentences usually denoted by $\mathcal{A} = \{S_0, S_1, S_2, \ldots\}$. Each of the S_i's is called an **atomic sentence**.
- The **sentential connectives** $\vee, \wedge, \rightarrow$, and $\neg$.
- Parenthesis (and).

Definition 1.2. The **sentences** of $\mathcal{S}$ are defined as follows:

(i) All atomic sentences are sentences.

(ii) If φ is a sentence, then so is $\neg\varphi$.

(iii) If φ and ψ are sentences, then so are $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, and $(\varphi \rightarrow \psi)$.

(iv) Nothing else is a sentence.

If $\mathcal{B} \subseteq \mathcal{A}$ is a set of atomic sentences of $\mathcal{S}$, then $\overline{\mathcal{B}}$ is the set of all sentences that only use the atomic sentences of $\mathcal{B}$.

Example 1.1. Check if each of the following is a sentence. If they are write down at least two history for the sentence.

a. $((S_1 \vee S_2) \wedge \neg S_3)$

b. $S_1 \to \neg S_2$

c. $(S_2 \vee (S_3 \to \wedge S_2))$

d. $(S_2 \wedge \neg S_1)$

Notation: The outer most parenthesis for a sentence is typically omitted. For example, instead of $(S_1 \wedge S_2)$ we often write $S_1 \wedge S_2$.

Definition 1.3. The **length** of a sentence is the number of non-parenthetical symbols that appear in the sentence. For example the length of $S_1 \vee (S_2 \wedge \neg S_1)$ is 6.

Definition 1.4. A **history** of a sentence is a sequence of sentences for which each element of this sequence is either an atomic sentence or is obtained by applying (ii) or (iii) in Definition 1.2 to two terms of the sequence prior to that term.

Example 1.2. Write two histories for the sentence $S_1 \vee (S_2 \wedge \neg S_1)$.

Example 1.3. By inserting parentheses, in how many ways can we turn $S_1 \vee S_2 \to S_3$ into a sentence?

1.2 Truth Assignments

Definition 1.5. A **truth assignment** for $\mathcal{A}$ is any function $h : \mathcal{A} \to \{T, F\}$.

Theorem 1.1. *Suppose $\mathcal{B}$ is a set of atomic sentences, and $h : \mathcal{B} \to \{T, F\}$ is a truth assignment. Then, there is precisely one function $\overline{h} : \overline{\mathcal{B}} \to \{T, F\}$ satisfying all of the following. For every atomic sentence S and every two sentences φ and ψ:*

(i) $\overline{h}(S) = h(S)$.

(ii) $\overline{h}(\neg\varphi) = T$ *if and only if* $\overline{h}(\varphi) = F$.

(iii) $\overline{h}(\varphi \wedge \psi) = T$ *if and only if* $\overline{h}(\varphi) = \overline{h}(\psi) = T$.

(iv) $\overline{h}(\varphi \vee \psi) = F$ *if and only if* $\overline{h}(\varphi) = \overline{h}(\psi) = F$.

(v) $\overline{h}(\varphi \to \psi) = F$ *if and only if* $\overline{h}(\varphi) = T$*, and* $\overline{h}(\psi) = F$.

We will skip the proof of this theorem for now.

Example 1.4. Suppose $\mathcal{A} = \{A, B, C\}$. Define a truth assignment $h : \mathcal{A} \to \{T, F\}$ by $h(A) = T, h(B) = h(C) = F$. Find $\overline{h}((A \vee \neg B) \to C)$.

Definition 1.6. We say a truth assignment h **satisfies** a sentence θ, or h **models** θ, if $\overline{h}(\theta) = T$, in which case we write $h \vDash \theta$.

Definition 1.7. Let Σ be a set of sentences, and h be a truth assignment. We say h **models** Σ, if h models θ for all $\theta \in \Sigma$. In that case we write $h \vDash \Sigma$.

Example 1.5. Let A, B, C be atomic sentences. Find a truth assignment that models $\{A \vee B, B \to C, C \wedge \neg A\}$ or show no such truth assignment exists.

1.3 Tautologies, Satisfiability, and Truth Tables

Definition 1.8. A sentence θ is a **tautology** or **valid** if every truth assignment models θ, in which case we write $\vDash \theta$. A sentence θ is called a **contradiction** if $h(\theta) = F$ for every truth assignment h.

Example 1.6. Prove that $\varphi \vee \neg\varphi$ is a tautology for every sentence φ.

Definition 1.9. A sentence θ is said to be **satisfiable** if $h \vDash \theta$ for some truth assignment h.

Example 1.7. Let $\mathcal{A} = \{A, B, C\}$. Prove that $(A \wedge \neg B) \to C$ is satisfiable.

Theorem 1.2. *Let θ be a sentence. Then*

a. θ is satisfiable if and only if $\neg\theta$ is not a tautology.

b. θ is a tautology if and only if $\neg\theta$ is a contradiction.

Definition 1.10. Let θ be a sentence that has n atomic sentences. A **truth table** for θ is a table whose first row consists of a history of θ that starts with all n atomic sentences that appear in θ. The first n columns of this table list all 2^n possible truth assignments of these n atomic sentences. Each row determines the truth value of the corresponding sentence with respect to the given truth assignment.

Example 1.8. Given $\mathcal{A} = \{A, B, C\}$. By drawing a truth table, find out the propertion of truth assignments that model $(\neg A \wedge B) \to C$.

1.4 More Examples

Example 1.9. Prove that:

a. In every sentence at least one atomic sentence appears.

b. It is impossible for a sentence to end with a connective. (Recall that the outer parentheses can be removed.)

Solution. Let θ be a sentence. We will prove both claims by induction on the length of θ.

a. **Basis step:** Note that sentences created from (ii) and (iii) in Definition 1.2 have more than two non-parenthetical symbols and thus their length is more than 1. Therefore, θ must be an atomic sentence, which completes the proof of the basis step.

Inductive step: If θ is an atomic sentence, then we are done. Otherwise, θ is one of $\neg\varphi, \varphi \wedge \psi, \varphi \vee \psi$, or $\varphi \to \psi$. In all cases, φ has less non-parenthetical symbols than θ and thus, by inductive hypothesis, an atomic sentence appears in φ. Therefore, an atomic sentence appears in θ. This completed the proof.

b. **Basis step:** If length of θ is 1, since by (a) it must have an atomic sentence, it cannot have any connectives. **Inductive step:** By assumption θ is either an atomic sentence (which does not contain any connectives) or one of $\neg\varphi, \varphi \wedge \psi, \varphi \vee \psi$, or $\varphi \to \psi$. If θ were to end with a connective, then φ or ψ must also end with a connective. However, the lengths of both of these sentences φ and ψ are less than the length of θ. This violates the inductive hypothesis. Therefore, θ cannot end with a connective. □

Example 1.10. Find all sentences of length 1 and 2.

Solution. We will show that only atomic sentences are those sentences of length 1. Let θ be a sentence of length 1. If θ is not atomic, it must be obtained by at least one application of (ii) or (iii). This means θ ie one of $\neg\varphi, \varphi \wedge \psi, \varphi \vee \psi$, or $\varphi \to \psi$. In all cases it means the length of θ is more than the length of φ. However we know (by the previous example) the length of each sentence is at least 1. Thus, the length of θ is at least 2, a contradiction.

We will show sentences of form $\neg S_i$ are the only sentences of length 2. First note that $\neg S_i$ has length 2, since it contains two non-parenthetical symbols $\neg$ and S_i. Suppose θ is a sentence of length 2. It cannot be atomic since atomic sentences have only one symbol. Thus, θ must be one of $\neg\varphi, \varphi \wedge \psi, \varphi \vee \psi$, or $\varphi \to \psi$. Since length of each of these $\varphi \wedge \psi, \varphi \vee \psi$, or $\varphi \to \psi$ is at least 3, $\theta = \neg\varphi$, where φ is a sentence of length 1. By what we proved above φ must be an atomic sentence. This completes the proof of the claim. □

Example 1.11. Let ψ be a sentence, and θ be a satisfiable sentence. Prove that $\psi \to \theta$ is satisfiable.

Solution. Note that since θ is satisfiable, there is a truth assignment h for which $\overline{h}(\theta) = T$. By definition of $\overline{h}$, we know $\overline{h}(\psi \to \theta) = T$. Therefore, $h \vDash \psi \to \theta$ and thus $\psi \to \theta$ is satisfiable. □

Example 1.12. Let $\mathcal{A} = \{S_1, S_2, \ldots, S_n\}$. How many truth assignments $h : \mathcal{A} \to \{T, F\}$ are there?

Solution. Note that each $h(S_i)$ could be either T or F. Thus, the number of possible truth assignments is 2^n. □

1.5 Exercises

All students are expected to do all of the exercises listed in the following two sections.

1.5.1 Problems for Grading

The following problems must be submitted on Friday 9/11/2020 before the beginning of class. The submission will be on Gradescope via Elms. **Late submission will not be accepted.**

Instructions for submission: To submit your solutions please note the following:

- Each problem must go on a separate page.
- It is highly recommended (but not required) that you LaTeX your homework.
- If you are not typing your work (which is fine) please make sure your work is legible.
- To submit your homework go to Elms. Hit "GradeScope" on the left panel. That should allow you to upload a PDF file of your homework.
- You could use the (free) DocScan app to scan and upload your homework.
- Sometime in the next few days run a test and make sure this all works out so you do not face any issues right before the deadline.
- Homework must be submitted before the class starts on the due date. GradeScope will not allow late submissions.
- You can read more about submitting homework on Gradescope here.

All answers and proofs must be complete and fully justified.

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

Exercise 1.1 (10 pts)**.** *For each part of this problem, replace each atomic sentence S_i by a sentence from precalculus (involving integers, real numbers, etc.) that makes the statement true. Then replace each atomic sentence S_i by a sentence that make the statement false. Explain your answers.*

a. $((\neg S_1) \vee S_2) \to S_3$

b. $(\neg S_2 \wedge S_3) \vee (S_3 \to S_1)$.

Exercise 1.2 (10 pts)**.** *Suppose $\mathcal{A} = \{A, B, C\}$. How many possible histories of the sentence $(A \wedge B) \to \neg C$ are there that start with three atomic sentences? Write down two of them that start with A, B, C.*

Exercise 1.3 (10 pts)**.** *Suppose $\mathcal{A} = \{A, B, C\}$. Each of the following expressions can either be turned into a sentence by adding parentheses or it cannot. If it cannot, explain why it cannot. If it can, determine all possible ways that this can be done. Make sure your justification is complete.*

a. $\neg\neg A \wedge B \to C$

b. $B\neg \to A \vee C$.

Exercise 1.4 (10 pts)**.** *Let $\mathcal{A} = \{A, B, C\}$ and that the truth assignment function h is defined by $h(A) = h(C) = F$, and $h(B) = T$. Find $\overline{h}((A \vee \neg C) \to (B \wedge \neg A))$. As usual show all of your steps.*

Exercise 1.5 (10 pts)**.** *Suppose $\mathcal{A} = \{A, B, C\}$. Prove that $A \to ((A \vee B) \to C)$ is satisfiable.*

Exercise 1.6 (10 pts)**.** *Suppose $\mathcal{A} = \{A, B, C\}$. Prove that the statement $\theta = (A \wedge B) \to (A \vee C)$ is a tautology in two ways:*

a. Using the truth table.

b. By assuming there is a truth assignment h for which $h \not\models \theta$ and arriving at a contradiction.

Exercise 1.7 (10 pt)**.** *Prove that in every sentence, the number of open parenthesis symbols "(" is the same as the number of close parenthesis symbols ")".*

Hint: Use induction on the length of the sentence. See Examples 1.9 and 1.10.

Exercise 1.8 (10 pts)**.** *Suppose $\mathcal{A} = \{S_1, S_2, \ldots, S_n\}$.*

a. How many truth assignments $h : \mathcal{A} \to \{T, F\}$ are there that model S_1?

b. How many truth assignments $h : \mathcal{A} \to \{T, F\}$ are there that model $\neg(S_1 \vee S_2 \vee \cdots \vee S_n)$?

1.5.2 Problems for Practice

Exercise 1.9. *Let $\mathcal{A} = \{A, B, C\}$. Determine if each sentence is tautology, contradiction, or satisfiable.*

a. $A \to (A \wedge B)$.

b. $(\neg A \wedge \neg B) \wedge (A \vee C)$

Exercise 1.10. *Find all sentences of length 3.*

2 Week 2

2.1 Logical Consequences

Definition 2.1. We say a sentence θ is a **logical consequence** of a set of sentences Σ if every truth assignment that models Σ also models θ. In that case we write $\Sigma \models \theta$. When $\Sigma = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ is a finite set, instead of $\Sigma \models \theta$ we write $\varphi_1, \varphi_2, \ldots, \varphi_n \models \theta$.

Example 2.1. Prove each of the following:

a. $\{(\varphi \vee \psi) \wedge \neg\varphi\} \models \psi$.

b. $\{\varphi \to \neg\psi, \psi \to \varphi\} \not\models \varphi \to \psi$.

Theorem 2.1. *Let Σ and Γ be sets of sentences and θ be a sentence. Then,*

a. If $\theta \in \Sigma$, then $\Sigma \models \theta$.

b. If $\Sigma \models \varphi$ for all $\varphi \in \Gamma$, and $\Gamma \models \theta$, then $\Sigma \models \theta$.

Example 2.2. Suppose φ is a satisfiable sentence. Prove that there are atomic sentences $A_1, \ldots, A_n, B_1, \ldots, B_m$ for which

$$A_1 \wedge \cdots \wedge A_n \wedge \neg B_1 \wedge \cdots \wedge \neg B_m \vDash \varphi.$$

Solution. Since φ is satisfiable, there is a truth assignment h that models φ. Suppose $A_1, \ldots, A_n$ are all atomic sentences of φ that are modeled by h, and $B_1, \ldots, B_m$ are all atomic sentences of φ that are not modeled by h. We claim

$$A_1 \wedge \cdots \wedge A_n \wedge \neg B_1 \wedge \cdots \wedge \neg B_m \vDash \varphi.$$

Note that the truth value of φ depends only on the truth values of the atomic sentences $A_1, \ldots, A_n, B_1, \ldots, B_m$, since these are the only atomic sentences that appear in φ. Now, assume v is a truth assignment that models $A_1 \wedge \cdots \wedge A_n \wedge \neg B_1 \wedge \cdots \wedge \neg B_m$. Since $\overline{v}(A_i) = \overline{h}(A_i)$, and $\overline{h}(B_i) = \overline{v}(B_i)$, and $\overline{v}(\varphi)$ only depends on $\overline{v}(A_i)$ and $\overline{v}(B_i)$ we conclude that $\overline{v}(\varphi) = \overline{h}(\varphi) = T$, as desired. $\square$

The following theorem shows a connection between logical consequence $\vDash$ and satisfiability.

Theorem 2.2. *Let Σ be a set of sentences and θ be a sentence. Then,*

a. $\Sigma \vDash \theta$ if and only if $\Sigma \cup \{\neg\theta\}$ is not satisfiable.

b. $\Sigma \nvDash \neg\theta$ if and only if $\Sigma \cup \{\theta\}$ is satisfiable.

The following shows an important connection between logical consequence $\vDash$ and implication $\rightarrow$.

Theorem 2.3. *Let Σ be a set of sentences and θ, φ be two sentences. Then, $\Sigma \cup \{\varphi\} \vDash \theta$ if and only if $\Sigma \vDash \varphi \rightarrow \theta$.*

Example 2.3 (Important)**.** Prove each of the following logical consequences:

a. $\varphi \rightarrow \psi, \psi \rightarrow \theta \vDash \varphi \rightarrow \theta$.

b. $\varphi \vDash \psi \rightarrow \varphi$.

c. $\neg\psi \vDash \psi \rightarrow \phi$.

d. $\neg\varphi \rightarrow \varphi \vDash \varphi$.

e. $\varphi \rightarrow \psi, \neg\varphi \rightarrow \psi \vDash \psi$.

f. $\varphi \rightarrow \psi \vDash \neg\psi \rightarrow \neg\varphi$.

g. $\neg\psi \rightarrow \neg\varphi \vDash \varphi \rightarrow \psi$.

2.2 Logical Equivalence

Definition 2.2. We say two sentences θ and φ are **logically equivalent** if $\overline{h}(\theta) = \overline{h}(\varphi)$ for every truth assignment h. In that case we write $\theta \equiv \varphi$.

Theorem 2.4. *Let φ, ψ, and θ be three sentences, τ be a tautology and c be a contradiction. Then,*

a. $(\varphi \wedge \psi) \equiv (\psi \wedge \varphi)$ *and* $(\varphi \vee \psi) \equiv (\psi \vee \varphi)$. *(Commutative Laws.)*

b. $(\varphi \wedge \psi) \wedge \theta \equiv \varphi \wedge (\psi \wedge \theta)$ *and* $(\varphi \vee \psi) \vee \theta \equiv \varphi \vee (\psi \vee \theta)$. *(Associative Laws.)*

c. $\neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi)$ *and* $\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi)$. *(De Morgan's Laws.)*

d. $\varphi \wedge (\psi \vee \theta) \equiv (\varphi \wedge \psi) \vee (\varphi \wedge \theta)$ *and* $\varphi \vee (\psi \wedge \theta) \equiv (\varphi \vee \psi) \wedge (\varphi \vee \theta)$. *(Distributive Laws.)*

e. $\varphi \rightarrow \psi \equiv \neg\varphi \vee \psi$. *(Implication-Disjunction Law.)*

f. $\neg\neg\varphi \equiv \varphi$ *(Double Negation Law.)*

g. $\varphi \vee \neg\varphi \equiv \tau$, *and* $\varphi \wedge \neg\varphi \equiv c$, *and* . *(Inverse Laws.)*

h. $\varphi \wedge \tau \equiv \varphi \vee c \equiv \varphi$, *and* $\varphi \wedge c \equiv c$, *and* $\varphi \vee \tau \equiv \tau$. *(Identity Laws.)*

i. $\varphi \vee \varphi \equiv \varphi \wedge \varphi \equiv \varphi$. *(Idempotent Laws.)*

Example 2.4. Write a sentence that is equivalent to $(A \vee B) \rightarrow B$ and does not use $\rightarrow$ or $\vee$.

Theorem 2.5. *Given any sentence θ, there is a sentence $\theta^\star$ for which $\theta \equiv \theta^\star$, and that $\theta^\star$ does not use any symbols other than $\neg, \rightarrow, (,)$, and the atomic sentences that appear in θ.*

Remark: By associativity all different placements of parentheses in the sentence $\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n$ give logically equivalent sentences. So, we will often omit the parentheses in such instances. We will also denote $\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n$ by $\bigvee\limits_{i=1}^{n} \varphi_i$. Similarly we will denote $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$ by $\bigwedge_{i=1}^{n} \varphi_i$.

Definition 2.3. (i) We say a sentence $\theta_1 \wedge \theta_2 \wedge \cdot \wedge \theta_n$ is in **conjunctive normal form** (or CNF for shorts) if each θ_i is an atomic sentence, negation of an atomic sentence, or disjunction of atomic sentences and negations of atomic sentences.
(ii) We say a sentence $\theta_1 \vee \theta_2 \vee \cdot \vee \theta_n$ is in **disjunctive normal form** (or DNF for shorts) if each θ_i is an atomic sentence, negation of an atomic sentence, or conjunction of atomic sentences and negations of atomic sentences.

Example 2.5. Let, A, B, C be atomic sentences. Determine if each sentence is in DNF, CNF or neither.

1. $A \vee \neg B$
2. $(A \rightarrow B) \vee \neg C$
3. $(A \wedge \neg B) \vee (\neg C \wedge B)$

Example 2.6. Create two sentences, one in DNF and one in CNF whose truth tables are as follows.

A	B	C	θ
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	T

Definition 2.4. We say two set of sentences are equivalent if they are satisfied by precisely the same truth assignments.

Theorem 2.6. *Two set of sentences Σ and Γ are equivalent if and only if $\Gamma \vDash \theta$ for every $\theta \in \Sigma$, and $\Sigma \vDash \varphi$ for all $\varphi \in \Gamma$.*

2.3 Proof by Induction

Theorem 2.7. *Suppose Σ is a set of sentences for which*

- *Every atomic sentence is in Σ,*
- *If $\varphi \in \Sigma$, then $\neg\varphi \in \Sigma$, and*
- *If $\varphi, \theta \in \Sigma$, then $\varphi \vee \theta, \varphi \to \theta, \varphi \wedge \theta$ are all in Σ.*

Then Σ is the set of all sentences.

Theorem 2.8. *Fix a natural number n and let φ_n be a sentence. For any sentence θ we define a sentence $\theta^\star$ by substituting all occurances of the atomic sentence S_n in θ by φ_n.*

a. Let h be a truth assignment and define the truth assignment $h^\star$ by $h^\star(S_n) = \overline{h}(\varphi)$, and $h^\star(S_i) = h(S_i)$ for all $i \neq n$. Then $\overline{h^\star}(\theta) = \overline{h}(\theta^\star)$.

b. If $\vDash \theta$, then $\vDash \theta^\star$.

2.4 More Examples

Example 2.7. Let φ be a sentence. Prove that the number of instances of connectives $\vee, \wedge, \to$ that appear in φ is one less than the number of instances of atomic sentences that appear in φ.

Example 2.8. Let θ be a sentence for which no atomic sentence appears in θ more than once. Prove that θ is satisfiable but it is not a tautology.

2.5 Exercises

2.5.1 Problem for Grading

The following problems must be submitted on Friday 9/18/2020 before the beginning of class. The submission will be on Gradescope via Elms. **Late submission will not be accepted.**

For all of the problems below, A, B, C, D are atomic sentences; θ, ϕ, ψ are arbitrary sentences; and Σ is a set of sentences.

Exercise 2.1 (10 pts)**.** *Prove that* $\{A \vee B, A \vee C\} \not\models (A \to C)$ *by finding a truth assignment that model the left side but not the right side.*

Exercise 2.2 (5 pts)**.** *In class we proved one direction of Theorem 5.1 (from the online textbook). Carefully prove the other direction stated below:*

$$\text{If } \Sigma \models (\theta \to \phi), \text{ then } (\Sigma \cup \{\theta\}) \models \phi.$$

Exercise 2.3 (10 pts)**.** *Prove* $(\phi \to \theta) \models (\neg\theta \to \neg\phi)$ *is two different ways:*

a. *Using a truth table.*

b. *Using Lemma 5.1, Theorem 5.1 and Corollary 5.1 as needed.*

Exercise 2.4 (15 pts)**.** *Determine if each statement is true or false. If it is true prove it. If it is false find a counterexample. (For giving counterexamples you may want to use a specific truth assignment.)*

a. $(A \vee B) \wedge A \equiv A$.

b. *If* $\Sigma \models \phi$ *or* $\Sigma \models \theta$, *then* $\Sigma \models (\phi \vee \theta)$.

c. $\neg A \wedge \neg B \equiv \neg(A \wedge B)$.

Exercise 2.5 (10 pts)**.** *Consider the sentence* $\theta = (\neg A \to B) \to (C \to (D \wedge \neg B))$.

a. *Use logical equivalences in the last page to find a sentence in CNF that is equivalent to* θ.

b. *Find a sentence in DNF that is equivalent to* θ.

Exercise 2.6 (10 pts)**.** *Find two sentences, one in DNF and one in CNF that are equivalent to a sentence* θ *whose truth table is given below.*

A	B	C	θ
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	T

Exercise 2.7 (10 pts)**.** *Using logical equivalences in the last page, prove that the following statement is a tautology:*

$$(((\phi \vee \neg\theta) \wedge \theta) \to (\phi \wedge \theta)) \wedge ((\phi \vee \theta) \to ((\phi \wedge \neg\theta) \vee \theta))$$

Note: *In each step, you must specify which rule you are using.*

Exercise 2.8 (10 pts)**.** *Using induction (Theorem 7.1) prove that for every sentence θ, there is a sentence $\theta^\star$, for which $\theta \equiv \theta^\star$ and $\theta^\star$ contains the same atomic sentences as θ and uses only the connectives $\neg$ and $\to$.*

2.5.2 Challenge Problems

Challenge problems are for those who want to get more out of this class.

Exercise 2.9. *Is it true that every sentence is equivalent to a sentence whose only connectives are $\vee, \wedge, \to$?*

3 Week 3

3.1 A Formal Proof System

In order to prove the Completeness Theorem we need to provide a set of axioms that we are able to use to deduce all tautologies from those axioms using certain predetermined rules.

Since we know every sentence is equivalent to a sentence that uses only $\neg$ and $\to$ we only focus on the sentences that do not have the connectives $\wedge$ and $\vee$.

Definition 3.1. The set Λ_0 of **logical axioms** of $\mathcal{S}$ consists of all sentences of form

1. $\varphi \to (\psi \to \varphi)$
2. $(\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta))$
3. $(\neg\varphi \to \psi) \to ((\neg\varphi \to \neg\psi) \to \varphi)$

Theorem 3.1. *Every sentence in Λ_0 defined in the above definition is a tautology.*

Definition 3.2. **Modus ponens** is the rule that allows us to deduce ψ from φ and $\varphi \to \psi$.

Definition 3.3. A **logical deduction** (or simply a deduction) in $\mathcal{S}$ is a finite sequence $\varphi_1, \varphi_2, \ldots, \varphi_n$ of sentences such that for each i with $1 \leq i \leq n$ one of the following holds:

- $\varphi_i \in \Lambda_0$, or
- φ_i is obtained by an application of modus ponens to two sentences that appear earlier in the sequence, i.e. there are $j, k < i$ for which $\varphi_k = (\varphi_j \to \varphi_i)$.

Definition 3.4. We say a sentence φ is **logically deducible** (written as $\vdash \varphi$) if there is a deduction whose last sentence is φ.

Example 3.1. For every two sentences φ and ψ, prove that:

a. $\vdash (\varphi \to \psi) \to (\varphi \to \varphi)$.

b. $\vdash (\varphi \to \varphi)$

Scratch: For part (a) we look at the axioms and see which one could give us this sentence on the right side of the implication. We notice that substituting $\theta = \varphi$ in Axiom 2 gives us just that. But doing so changes the left side of the implication to $\varphi \to (\psi \to \varphi)$ which is precisely Axiom 1.

For (b), we see that in part (a) we have $\varphi \to \varphi$ in the right side of an implication. So, can we find *some* ψ that makes $\varphi \to \psi$ deducible? Axiom 1 again helps.

Solution. a. $\varphi_1 = \varphi \to (\psi \to \varphi)$ is an instance of Axiom 1. $\varphi_2 = (\varphi \to (\psi \to \varphi)) \to ((\varphi \to \psi) \to (\varphi \to \varphi))$ is an instance of Axiom 2. Applying modus ponens to φ_1, and φ_2 we obtain $\varphi_3 = (\varphi \to \psi) \to (\varphi \to \varphi)$. Therefore, $\varphi_1, \varphi_2, \varphi_3$ is a deduction, and thus $\vdash (\varphi \to \psi) \to (\varphi \to \varphi)$.

b. Let φ_1, φ_2, and φ_3 be as in part (a) when ψ is substituted by $\varphi \to \varphi$. Note that $\varphi_4 = \varphi \to \psi$ is an instance of Axiom 1. Applying modus ponens to φ_4 and φ_3 we obtain $\varphi \to \varphi$. Thus, $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi \to \varphi$ is a deduction, and thus $\vdash \varphi \to \varphi$. □

Theorem 3.2 (Modus Ponens for Deductions)**.** *Let φ, ψ be sentences. If $\vdash \varphi$, and $\vdash \varphi \to \psi$, then $\vdash \psi$.*

Theorem 3.3 (Soundness)**.** *If a sentence is deducible, then it is a tautology. (In other words, $\vdash \theta$ implies $\models \theta$.)*

Definition 3.5. Let Σ be a set of sentences. A **deduction from** Σ is a sequence $\varphi_1, \varphi_2, \ldots, \varphi_n$ of sentences such that for each i

- $\varphi_i \in \Lambda_0 \cup \Sigma$, or
- there are $j, k < i$ for which φ_i follows from φ_j and φ_k by an application of modus ponens. In other words, $\varphi_k = \varphi_j \to \varphi_i$.

Remark. Note that $\vdash \varphi$ if and only if $\emptyset \vdash \varphi$.

Lemma 3.1. *If $\Sigma \subseteq \Gamma$ are two sets of sentences and θ is a sentence for which $\Sigma \vdash \theta$, then $\Gamma \vdash \theta$.*

Definition 3.6. Let Σ be a set of sentences. We say a sentence θ is **deducible from** Σ (written $\Sigma \vdash \theta$) if there is a deduction from Σ whose last sentence is θ.

Theorem 3.4 (Modus Ponens for Deductions from Hypotheses)**.** *Suppose Σ is a set of sentences, and φ, ψ are two sentences. If $\Sigma \vdash \varphi$, and $\Sigma \vdash \varphi \to \psi$, then $\Sigma \vdash \psi$.*

Theorem 3.5 (Soundness)**.** *Suppose θ is a sentence and Σ is a set of sentences. If $\Sigma \vdash \theta$, then $\Sigma \vDash \theta$.*

Theorem 3.6 (Deduction Theorem)**.** *Suppose Σ is a set of sentences and φ, ψ are two sentences. Then, $\Sigma \vdash \varphi \to \psi$ if and only if $\Sigma \cup \{\varphi\} \vdash \psi$.*

Theorem 3.7. *For every three sentences φ, ψ, and θ, all of the following sentences are deducible.*

a. $(\neg\varphi \to \varphi) \to \varphi$.

b. $\varphi \to (\neg\varphi \to \psi)$.

c. $(\varphi \to (\psi \to \theta)) \to (\psi \to (\varphi \to \theta))$.

d. $\neg\neg\varphi \to \varphi$.

e. $\varphi \to \neg\neg\varphi$.

f. $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$.

g. $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$.

Proof. a. By Deduction Theorem, it is enough to show $\neg\varphi \to \varphi \vdash \varphi$. Axiom 3 shows $\vdash (\neg\varphi \to \varphi) \to ((\neg\varphi \to \neg\varphi) \to \varphi)$. Applying Lemma 3.1 and modus ponens we obtain $\neg\varphi \to \varphi \vdash (\neg\varphi \to \neg\varphi) \to \varphi$. By an example we know $\vdash \neg\varphi \to \neg\varphi$. Another application of Lemma 3.1 and modus ponens implies $\neg\varphi \to \varphi \vdash \varphi$, as desired.

b. By the Deduction Theorem it is enough to show $\varphi \vdash \neg\varphi \to \psi$. Applying the Deduction Theorem again we obtain that it is enough to prove $\varphi, \neg\varphi \vdash \psi$.

[Scratch: We see that Axiom 3 can be used. In order to get ψ, we need to substitute θ by ψ. The first two sentences have φ and $\neg\varphi$ to right of the implication, which is good, because two applications of Axiom 1 could give us those sentences. So, here is the rest of the solution:]

By Axiom 3 we have $\vdash (\neg\psi \to \varphi) \to ((\neg\psi \to \neg\varphi) \to \psi)$ $(*)$. By Axiom 1 $\vdash \varphi \to (\neg\psi \to \varphi)$. By Deduction Theorem, $\varphi \vdash \neg\psi \to \varphi$. Applying modus ponens to this and $(*)$ we obtain $\varphi \vdash (\neg\psi \to \neg\varphi) \to \psi$ $(**)$. By Axiom 1 we know $\vdash \neg\varphi \to (\psi \to \neg\varphi)$. The Deduction Theorem implies $\neg\varphi \vdash \neg\psi \to \neg\varphi$. Combining this and $(**)$ and Lemma 3.1 we obtain $\varphi, \neg\varphi \vdash \psi$, as desired.

c. Two applications of Deduction Theorem imply that it is enough to prove $\varphi \to (\psi \to \theta), \psi, \varphi \vdash \theta$. By modus ponens $\varphi \to (\psi \to \theta), \varphi \vdash \psi \to \theta$. Since $\varphi \to (\psi \to \theta), \psi, \varphi \vdash \psi$, another application of modus ponens gives us $\varphi \to (\psi \to \theta), \psi, \varphi \vdash \theta$, as desired.

d. [Scratch: The third axiom seems useful as it is the only one with negations. We keep φ as the last sentence appearing in this axiom. Changing ψ to $\neg\varphi$ makes the first sentence $\neg\varphi \to \neg\varphi$ and the second sentence to $\neg\varphi \to \neg\neg\varphi$, both of which can be deducted from $\neg\neg\varphi$.]
By Deduction Theorem, it is enough to prove $\neg\neg\varphi \vdash \varphi$. By Axiom 3, we have $\vdash (\neg\varphi \to \neg\varphi) \to ((\neg\varphi \to \neg\neg\varphi) \to \varphi)$. By an example $\vdash \neg\varphi \to \neg\varphi$. Combining these two and modus ponens we obtain $\vdash (\neg\varphi \to \neg\neg\varphi) \to \varphi$. Axiom 1 and Deduction Theorem imply $\neg\neg\varphi \vdash \neg\varphi \to \neg\neg\varphi$. Modus ponens along with Lemma 3.1 implies $\neg\neg\varphi \vdash \varphi$, as desired.

e. By Deduction Theorem, it is enough to prove $\varphi \vdash \neg\neg\varphi$.

Scratch: Similar to the previous part, it seems like we need to use Axiom 3 in a way that it ends with $\neg\neg\varphi$. Replacing φ by $\neg\neg\varphi$. We need to now choose ψ so that both $\neg\neg\neg\varphi \to \psi$ and $\neg\neg\neg\varphi \to \neg\psi$ are deductible from φ. Choosing $\psi = \varphi$ works.

By Axiom 3 for sentences $\neg\neg\varphi$ and φ, we obtain $\vdash (\neg\neg\neg\varphi \to \varphi) \to ((\neg\neg\neg\varphi \to \neg\varphi) \to \neg\neg\varphi)$. Axiom 1, and the Deduction Theorem imply $\varphi \vdash \neg\neg\neg\varphi \to \varphi$ and thus using modes ponens and Lemma 3.1 we conclude $\varphi \vdash (\neg\neg\neg\varphi \to \neg\varphi) \to \neg\neg\varphi$. Note that by part (d), we know $\vdash \neg\neg\neg\varphi \to \neg\varphi$. Applying modes ponens and Lemma 3.1 we obtain $\varphi \vdash \neg\neg\varphi$, as desired.

f. Using the Deduction Theorem twice we conclude it is enough to prove $\varphi \to \psi, \neg\psi \vdash \neg\varphi$. For simplicity let $\Sigma = \{\varphi \to \psi, \neg\psi\}$.

[Scratch: Similar to the previous part, we need to use Axiom 3 with $\neg\varphi$ instead of φ. So, we need to see if we can show $\Sigma \vdash \neg\neg\varphi \to \psi$ and $\Sigma \vdash \neg\varphi \to \neg\psi$. The second one follows from axiom 1. The firs one needs a "replacement" of $\neg\neg\varphi$ by φ, but this is not allowed. So, we should find a way around it. We know $\varphi \vdash \neg\neg\varphi$ and vice-versa. So, we could use use Deduction Theorem twice and get the result. This yields the following solution:]

By Axiom 3 we have $\vdash (\neg\neg\varphi \to \psi) \to ((\neg\neg\varphi \to \neg\psi) \to \neg\varphi)$ $(*)$. Note that by part (e) $\neg\neg\varphi \vdash \varphi$, and thus $\Sigma \cup \{\neg\neg\varphi\} \vdash \varphi$. Combining this with the fact that $\varphi \to \psi \in \Sigma$, we obtain $\Sigma \cup \{\neg\neg\varphi\} \vdash \psi$. Therefore, by Deduction Theorem, $\Sigma \vdash \neg\neg\varphi \to \psi$. Applying modus ponens to the last deduction and $(*)$ we obtain $\Sigma \vdash (\neg\neg\varphi \to \neg\psi) \to \neg\varphi$ $(**)$. By Axiom 1 and Deduction Theorem, $\neg\psi \vdash \neg\neg\varphi \to \neg\psi$. Since $\neg\psi \in \Sigma$, by

Lemma 3.1 we obtain $\Sigma \vdash \neg\neg\varphi \to \neg\psi$. Combining this with (∗∗) we conclude $\Sigma \vdash \neg\varphi$, as desired.

g. By Deduction Theorem it is enough to prove $\neg\psi \to \neg\varphi \vdash \varphi \to \psi$. By part (f) $\neg\psi \to \neg\varphi \vdash \neg\neg\varphi \to \neg\neg\psi$ (∗).

[We would like to somehow replace $\neg\neg\varphi$ and $\neg\neg\psi$ by φ and ψ, respectively. Note that this cannot be done by saying "since $\neg\neg \equiv \varphi$ then we can replace it by φ". However you could do that using the two facts that $\neg\neg\varphi \vdash \varphi$ and $\varphi \vdash \neg\neg\varphi$. Here is how we turn this into a complete solution:]

By Deduction Theorem, it is enough to show $\neg\psi \to \neg\varphi, \varphi \vdash \psi$. For simplicity let $\Sigma = \{\neg\psi \to \neg\varphi, \varphi\}$. Since $\varphi \in \Sigma$, we have $\Sigma \vdash \varphi$. By part (e) we know $\vdash \varphi \to \neg\neg\varphi$. By modus ponens, and Lemma 3.1 we have $\Sigma \vdash \neg\neg\varphi$. Using this, (∗), Lemma 3.1, and modus ponens we obtain $\Sigma \vdash \neg\neg\psi$. By part (d) we know $\vdash \neg\neg\psi \to \psi$. Applying modus ponens, and Lemma 3.1 we obtain $\Sigma \vdash \psi$, as desired. □

Theorem 3.8. *For every set of sentences Σ and every two sentences φ, and ψ, we have $\Sigma \vdash \neg(\varphi \to \psi)$ if and only if $\Sigma \vdash \varphi$ and $\Sigma \vdash \neg\psi$.*

Proof. Exercise. □

3.2 Consistent Sets

The objective is to prove the Completeness Theorem stated below:

Theorem 3.9 (The Completeness Theorem)**.** *Let Σ be a set of sentences and φ be a sentence. Then, $\Sigma \vdash \varphi$ if and only if $\Sigma \models \varphi$.*

One direction of the above theorem is already proved as the Soundness Theorem. The idea is to relate deducibility with what is called "consistency" and show this concept is the same as satisfiability. We already have a relation between satisfiability and logical consequences (Theorem 2.2 (a)).

Definition 3.7. A set of sentences Σ is said to be **inconsistent** if $\Sigma \vdash \varphi$ and $\Sigma \vdash \neg\varphi$ for some sentence φ. A set that is not inconsistent is called **consistent**.

Theorem 3.10. *A set of sentences Σ is inconsistent if and only if $\Sigma \vdash \psi$ for every sentence ψ.*

Theorem 3.11 (Finiteness)**.** *Let Σ be a set of sentences and φ be a sentence.*

a. If $\Sigma \vdash \varphi$, then there is a finite subset Σ_0 of Σ for which $\Sigma_0 \vdash \varphi$.

b. Σ is consistent if and only if every finite subset of Σ is consistent.

Theorem 3.12. *Suppose Σ is a consistent set of sentences and φ is a sentence. Then, $\Sigma \cup \{\varphi\}$ or $\Sigma \cup \{\neg\varphi\}$ is consistent.*

Proof. Suppose on the contrary that $\Sigma \cup \{\varphi\}$ and $\Sigma \cup \{\neg\varphi\}$ are both inconsistent. Thus, $\Sigma \cup \{\varphi\} \vdash \neg\varphi$ and $\Sigma \cup \{\neg\varphi\} \vdash \varphi$, by Theorem 3.10. By Deduction Theorem, $\Sigma \vdash \varphi \to \neg\varphi$ (∗) and $\Sigma \vdash \neg\varphi \to \varphi$ (∗∗). By

Theorem 3.7 $\vdash (\neg\varphi \to \varphi) \to \varphi$. By Lemma 3.1, (∗∗), and modus ponens, $\Sigma \vdash \varphi$. Combining this with (∗) we obtain $\Sigma \vdash \neg\varphi$. This means Σ is inconsistent, a contradiction. □

Using the above theorem we will extend any consistent set to a maximal consistent set.

Definition 3.8. A set of sentences Γ is said to be **maximal consistent** if Γ is consistent and for every sentence θ, either $\theta \in \Gamma$ or $\neg\theta \in \Gamma$.

Theorem 3.13. *Every consistent set of sentences is contained in a maximal consistent set of sentences.*

Proof. By Exercise 3.5 the set of sentences can be enumerated as

$$\varphi_1, \varphi_2, \ldots \quad (*)$$

For every natural number n we create a consistent set Γ_n by $\Gamma_0 = \Sigma$, and

$$\Gamma_n = \begin{cases} \Gamma_{n-1} \cup \{\varphi_n\} & \text{if } \Gamma_{n-1} \cup \{\varphi_n\} \text{ is consistent} \\ \Gamma_{n-1} \cup \{\neg\varphi_n\} & \text{otherwise} \end{cases}$$

Note that if Γ_{n-1} is consistent by Theorem 3.12 at least one of $\Gamma_{n-1} \cup \{\varphi_n\}$ or $\Gamma_{n-1} \cup \{\neg\varphi_n\}$ is consistent. Thus, the above definition is valid and each Γ_n is consistent. Let $\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n$. Note that since $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$, every finite set of sentences is in Γ_k for some k, and thus it is consistent. By Finiteness Theorem, Γ is consistent. Since the list (∗) contains all sentences, for each sentence θ either $\theta \in \Gamma$ or $\neg\theta \in \Gamma$, as desired. □

Theorem 3.14. *Assume Γ is a maximal consistent set of sentences. Then Γ is satisfiable.*

Proof. Let h be a truth assignment for which $h(A) = T$ if and only if $A \in \Gamma$, for every atomic sentence A. We will prove by induction on the length of sentence θ that $\overline{h}(\theta) = T$ if and only if $\theta \in \Gamma$.

Basis step. Suppose θ has length 1. Thus, θ is atomic. By the way h is defined $\overline{h}(\theta) = T$ if and only if $\theta \in \Gamma$, as desired.

Inductive step. The case where θ is atomic was dealt with in the basis step. Since we are only using two connectives, there are two cases.

Case I. $\theta = \neg\varphi$. If $\theta \in \Gamma$, then $\varphi \notin \Gamma$ since Γ is consistent. By inductive hypothesis, $\overline{h}(\varphi) = F$ and hence $\overline{h}(\theta) = T$. If $\theta \notin \Gamma$, then since Γ is maximal, $\varphi \in \Gamma$. By inductive hypothesis $\overline{h}(\varphi) = T$ and thus $\overline{h}(\theta) = F$.

Case II. $\theta = (\varphi \to \psi)$. Note that lengths of $\psi, \varphi, \neg\psi$ and $\neg\varphi$ are all less than length of θ.

Suppose $\theta \in \Gamma$. If $\psi \in \Gamma$ or $\neg\varphi \in \Gamma$, then by inductive hypothesis $\overline{h}(\psi) = T$ or $\overline{h}(\neg\varphi) = T$. In both cases $\overline{h}(\theta) = T$. Otherwise, by maximality of Γ we have $\neg\psi \in \Gamma$ and $\varphi \in \Gamma$. Thus, By Theorem 3.8, $\Gamma \vdash \neg(\varphi \to \psi)$, which contradicts the fact that Γ is consistent.

Suppose $\theta \notin \Gamma$. Since Γ is maximal, $\neg(\varphi \to \psi) \in \Gamma$. Therefore, by Theorem 3.8, $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\psi$. Since Γ is maximal consistent, $\varphi, \neg\psi \in \Gamma$. By inductive hypothesis, $\overline{h}(\varphi) = T$, and $\overline{h}(\psi) = F$. This means $\overline{h}(\varphi \to \psi) = F$, as desired. □

3.3 More Examples

Example 3.2. Prove each of the following deductions.

a. $\vdash \underbrace{\neg \cdots \neg}_{n \text{ times}} \varphi \to \varphi$, if n is even.

b. $\vdash \underbrace{\neg \cdots \neg}_{n \text{ times}} \varphi \to \neg\varphi$, if n is odd.

c. $\vdash ((\varphi \to \psi) \to \varphi) \to \varphi$

Solution. We will prove (a) and (b) by induction on n. If $n = 0$, then by an example $\vdash \varphi \to \varphi$. If $n = 1$, then $\vdash \neg\varphi \to \neg\varphi$ by the same example. This completes the proof of the basis step.

Suppose $n \geq 2$ is an integer. By Theorem 3.7(d) and Deduction Theorem

$$\underbrace{\neg \cdots \neg}_{n \text{ times}} \varphi \vdash \underbrace{\neg \cdots \neg}_{n-2 \text{ times}} \varphi \quad (*)$$

Suppose n is even. Therefore, $n - 2$ is even and thus, by inductive hypotheses $\vdash \underbrace{\neg \cdots \neg}_{n-2 \text{ times}} \varphi \to \varphi$. Using Lemma 3.1, $(*)$ and modus ponens we obtain that $\underbrace{\neg \cdots \neg}_{n \text{ times}} \varphi \vdash \varphi$. The result for when n is even follows using the Deduction Theorem.

Similarly when n is odd, $\vdash \underbrace{\neg \cdots \neg}_{n \text{ times}} \varphi \to \neg\varphi$, as desired.

(c) By Deduction Theorem it is enough to show $(\varphi \to \psi) \to \varphi \vdash \varphi$.

By Theorem 3.7 and Deduction Theorem, $(\varphi \to \psi) \to \varphi \vdash \neg\varphi \to \neg(\varphi \to \psi)$ $(*)$. We also know that $\neg\varphi, \varphi \vdash \psi$, by Theorem 3.10, hence by Deduction Theorem $\neg\varphi \vdash \varphi \to \psi$, and thus $\vdash \neg\varphi \to (\varphi \to \psi)$ $(**)$. Using Axiom 3 we obtain $\vdash (\neg\varphi \to (\varphi \to \psi)) \to ((\neg\varphi \to \neg(\varphi \to \psi)) \to \varphi)$. Combining this with $(**)$ we obtain $\vdash (\neg\varphi \to \neg(\varphi \to \psi)) \to \varphi$. This along with $(*)$ and modus ponens implies $(\varphi \to \psi) \to \varphi \vdash \varphi$, as desired. □

3.4 Exercises

All students are expected to do all of the exercises listed in the following two sections.

3.4.1 Problems for grading

The following problems must be submitted on Friday 9/25/2020 before the beginning of class. The submission will be on Gradescope via Elms. **Late submission will not be accepted.**

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

For practice on deducibility check the proof of Theorem 3.7.

Do not use the Completeness Theorem in your solutions.

Exercise 3.1 (10 pts)**.** *Prove that every axiom of* Λ_0 *is a tautology.*

Exercise 3.2 (10 pts)**.** *Prove the Theorem: For every set of sentences* Σ *and every two sentences* φ*, and* ψ*, we have* $\Sigma \vdash \neg(\varphi \to \psi)$ *if and only if* $\Sigma \vdash \varphi$ *and* $\Sigma \vdash \neg\psi$*.*

Exercise 3.3 (15 pts)**.** *Prove that for every two sentences* φ *and* ψ*, we have*

a. $\vdash (\varphi \to \psi) \to ((\psi \to \theta) \to (\varphi \to \theta))$.

b. $\vdash (\neg\neg\varphi \to \psi) \to (\varphi \to \psi)$.

c. $\vdash \varphi \to (\neg\varphi \to \psi)$

Exercise 3.4 (10 pts)**.** *Show that for a set of sentences* Σ *and two sentences* φ *and* θ*, if* $\Sigma \cup \{\varphi\} \vdash \theta$ *and* $\Sigma \cup \{\neg\varphi\} \vdash \theta$*, then* $\Sigma \vdash \theta$*.*

Definition 3.9. An infinite set A is called **countable** if its elements can be enumerated. In other words, if $A = \{a_1, a_2, a_3, \ldots\}$.

Theorem 3.15. *If* $A_1, A_2, A_3, \ldots$ *is a sequence of countable sets. Then* $\bigcup_{n=1}^{\infty} A_n$ *is countable.*

Proof. List the elements of each set as follows:

$$\begin{aligned} A_1 &= \{a_{11}, a_{12}, a_{13}, \ldots\} \\ A_2 &= \{a_{21}, a_{22}, a_{23}, \ldots\} \\ A_3 &= \{a_{31}, a_{32}, a_{33}, \ldots\} \\ &\vdots \end{aligned}$$

The elements of the union can be listed as

$$\underbrace{a_{11}}_{\text{sum}=2}, \underbrace{a_{12}, a_{21}}_{\text{sum}=3}, \underbrace{a_{13}, a_{22}, a_{31}}_{\text{sum}=4}, \underbrace{a_{14}, a_{23}, a_{32}, a_{41}}_{\text{sum}=5}, \underbrace{a_{15}, a_{24}, a_{33}, a_{42}, a_{51}}_{\text{sum}=6} \ldots,$$

where in each step the elements whose index sums are n are listed. □

Exercise 3.5 (10 pts)**.** *Let* $A_1, A_2, \ldots, A_n$ *be countable sets. Prove that* $A_1 \times A_2 \times \cdots \times A_n$ *is countable.*

Note: $A_1 \times A_2 \times \cdots \times A_n$ is the set of all n-tuples whose i-th component is in A_i for all i.

Hint: Induct on n. For $n = 2$, write down $A_1 \times A_2$ as a union of a countable number of countable sets.

Exercise 3.6 (10 pts)**.** *Let A be a countable set. Prove that the set consisting of all finite sequences whose terms are from A is countable. Deduce that $\overline{\mathcal{S}}$, the set of all sentences, is countable.*

Exercise 3.7 (10 pts)**.** *Let A and B be two atomic sentences. Define a sequence φ_n of sentences by $\varphi_0 = A \to B$, and $\varphi_n = (\varphi_{n-1} \to A)$. Determine (with proof) for which natural numbers n we have $\vdash \varphi_n$.*

Hint: First try $n = 0, 1, 2, 3$.

Exercise 3.8 (15 pts)**.** *Let n be a positive integer. Prove the following:*

a. If $n \geq 2$, then there is a sentence in Λ_0 that is of the form Axiom (1) and has precisely n implication symbols.

b. If $n \geq 6$, and $n \neq 7$, then there is a sentence in Λ_0 that is of the form Axiom (2) and has precisely n implication symbols.

c. If $n \geq 4$, and $n \neq 5$, then there is a sentence in Λ_0 that is of the form Axiom (3) and has precisely n implication symbols.

3.4.2 Problems for Practice

Exercise 3.9. *Determine if each sentence is deducible for all sentences φ, ψ, θ.*

a. $\varphi \to (\neg\psi \to \neg\neg\varphi)$

b. $\neg(\theta \to \neg\varphi) \to \theta$

c. $(\neg\neg\theta \to \psi) \to (\varphi \to \theta)$

4 Week 4

4.1 Completeness and Compactness Theorems

The following theorem relates consistency with deducibility.

Theorem 4.1. *A sentence φ is deducible from a set of sentences Σ if and only if $\Sigma \cup \{\neg\varphi\}$ is inconsistent.*

Proof. Suppose $\Sigma \vdash \varphi$. By Lemma 3.1, $\Sigma \cup \{\neg\varphi\} \vdash \varphi$. Since $\Sigma \cup \{\neg\varphi\} \vdash \neg\varphi$, we conclude that $\Sigma \cup \{\neg\varphi\}$ is inconsistent, as desire.

Now, suppose $\Sigma \cup \{\neg\varphi\}$ is inconsistent. By Theorem 3.10, $\Sigma \cup \{\neg\varphi\} \vdash \varphi$. By Deduction Theorem, $\Sigma \vdash \neg\varphi \to \varphi$. By Theorem 3.7 (a), $\vdash (\neg\varphi \to \varphi) \to \varphi$. Using the Deduction Theorem we obtain $\Sigma \vdash \varphi$, as desired. ☐

Theorem 4.2. *A set of sentences Σ is consistent if and only if it is satisfiable.*

Proof. Suppose Σ is consistent. By Theorem 3.13, Σ is contained in a maximal consistent set Γ. By Theorem 3.14, Γ is satisfiable and hence there is a truth assignment h that models Γ. Since $\Sigma \subseteq \Gamma$, h also models Σ.

Suppose Σ is satisfiable. Let h be a truth assignment that models Σ. If Σ were not consistent, then $\Sigma \vdash \varphi$ and $\Sigma \vdash \neg\varphi$ for some sentence φ. By Soundness Theorem, $\Sigma \vDash \varphi$ and $\Sigma \vDash \neg\varphi$. Since $h \vDash \Sigma$, we have $h \vDash \varphi$ and $h \vDash \neg\varphi$, which is a contradiction. □

Proof of the Completeness Theorem. By Soundness Theorem, $\Sigma \vdash \varphi$ implies $\Sigma \vDash \varphi$.

Suppose $\Sigma \vDash \varphi$. By Theorem 2.2, $\Sigma \cup \{\neg\varphi\}$ is not satisfiable. Therefore, by Theorem 4.2, $\Sigma \cup \{\neg\varphi\}$ is inconsistent. By Theorem 4.1, $\Sigma \vdash \varphi$, as desired. □

One of the most important consequences of the Completeness Theorem and the Finiteness Theorem is the Compactness Theorem:

Theorem 4.3 (Compactness)**.** *Let Σ be a set of sentences and θ be a sentence.*

a. Σ is satisfiable if and only if all finite subsets of Σ are satisfiable.

b. $\Sigma \vDash \theta$ if and only if $\Sigma_0 \vDash \theta$ for some finite subset Σ_0 of Σ.

Example 4.1. Suppose Σ is a set of sentences for which every truth assignment models at least one element of Σ. Then, there are sentences $\varphi_1, \varphi_2, \dots, \varphi_n \in \Sigma$ for which $\vDash \varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n$.

Solution. Let $\Gamma = \{\neg\theta \mid \theta \in \Sigma\}$. By assumption, Γ is not satisfiable. By Theorem 4.2, Γ is inconsistent. By Finiteness Theorem, there is a finite subset Γ_0 of Γ that is inconsistent. Therefore, Γ_0 is not satisfiable. Let $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ be the set of all sentences whose negations are in Γ_0. Then, since Γ_0 is not satisfiable, for every truth assignment h we have $\overline{h}(\neg\varphi_1 \wedge \cdots \wedge \neg\varphi_n) = F$. This means, $\overline{h}(\varphi_1 \vee \cdots \vee \varphi_n) = T$, and hence $\varphi_1 \vee \cdots \varphi_n$ is a tautology. □

4.2 First Order Logic

4.2.1 Basics of a language

Definition 4.1. The **symbols of a first order language** $\mathcal{L}$ are as follows:

- A collection of symbols for **functions**, each of specified arity.
- A collection of symbols for **relations**, each of specified arity. We require all languages to have the binary relation $=$.
- A collection of symbols for **constants**.

- A countable set of **variables** $v_1, v_2, \ldots$.
- The **quantifiers** $\forall$ and $\exists$.
- **Sentential connectives** $\neg, \wedge, \vee, \rightarrow$.
- **Parentheses** and **comma**: (,), and ,.

We allow a language to not have any function symbols, constants, or relation symbols other than $=$.

Definition 4.2. The set of all constants, function symbols, relation symbols other than $=$ of a language $\mathcal{L}$ is called the **non-logical symbols of** $\mathcal{L}$ and is denoted by $\mathcal{L}^{nl}$.

Definition 4.3. An **$\mathcal{L}$-structure** $\mathcal{A}$ is a non-empty set A, called the **domain** or **universe** along with an n-ary relation $R^{\mathcal{A}}$ for every n-ary relation symbol R of $\mathcal{L}$, an n-ary function $F^{\mathcal{A}}$ for every n-ary function symbol F of $\mathcal{L}$, a distinguished element $c^{\mathcal{A}} \in A$ for every constant c of $\mathcal{L}$. No other functions, relations or named elements are in this $\mathcal{L}$-structure.

Example 4.2. The following are all examples of structures.

a. $\mathcal{Z} = (\mathbb{Z}, s, 0)$, where s is the unary successor function defined by $s(n) = n + 1$, and 0 is the integer 0. $\mathcal{Z}$ is an $\mathcal{L}$-structure, where $\mathcal{L}^{nl} = \{S, c\}$, S is a unary function symbol, c is a constant symbol, $S^{\mathcal{Z}} = s$, and $c^{\mathcal{Z}} = 0$

b. $\mathcal{N} = (\mathbb{N}, +, \cdot, <, 0, 1)$, where $+$ and $\cdot$ are the binary addition and multiplication functions. $<$ is the binary relation "less than" and 0 and 1 are zero and one in natural numbers. $\mathcal{N}$ is an $\mathcal{L}$-structure, where $\mathcal{L}^{nl} = \{F, G, R, c_1, c_2\}$, with F and G binary function symbols, R a binary relation symbol, and c_1, c_2 two constant symbols. $F^{\mathcal{N}} = +, G^{\mathcal{N}} = \cdot, R^{\mathcal{N}} =<, c_1^{\mathcal{N}} = 0$, and $c_2^{\mathcal{N}} = 1$.

c. $\mathcal{A} = (A, P^{\mathcal{A}})$, where A is a nonempty set and $P^{\mathcal{A}}$ is a unary function. $\mathcal{A}$ is an $\mathcal{L}$-structure, where $\mathcal{L}^{nl} = \{P\}$, where P is a unary relation symbol.

Example 4.3. Let A be a set of size n for some positive integer n. How many $\mathcal{L}$-structures of form $(A, P^{\mathcal{A}})$ are there for which P is a unary relation symbol?

Definition 4.4. The **terms** of a language $\mathcal{L}$ (or $\mathcal{L}$-terms) are defined as follows:

- Every constant and variable of $\mathcal{L}$ is a term.
- If F is an n-ary function of $\mathcal{L}$, and $t_1, t_2, \ldots, t_n$ are terms, then $F(t_1, t_2, \ldots, t_n)$ is a term.
- Nothing else is a term.

Definition 4.5. A sequence of terms showing how a term is built from constants, variables, (b), and (c) in the Definition 4.4 is called a **history** of that term.

Example 4.4. If F is a binary function symbol, c_1, c_2 are constants and v_1, v_2, v_3 are variables, then $F(c_1, F(v_1, v_2))$, and $F(F(c_2, v_1), F(c_1, F(v_2, v_3)))$ are terms.

The set of all terms of $\mathcal{L}$ is denoted by $Tm_{\mathcal{L}}$.

Example 4.5. $\mathcal{N} = (\mathbb{N}, +, \cdot, 0)$ is an $\mathcal{L}$-structure where $\mathcal{L}^{nl} = \{F, G, c\}$. The term $F(x, G(y, c))$ in this structure is the same as $x + (y \cdot 0)$.

Definition 4.6. The **atomic formulas** of $\mathcal{L}$ are all expressions of the form $R(t_1, t_2, \ldots, t_n)$, where R is an n-ary relation and $t_1, t_2, \ldots, t_n$ are terms. For simplicity we write $= (x, y)$ as $(x = y)$.

Definition 4.7. The **formulas** of $\mathcal{L}$ (or $\mathcal{L}$-formulas) are defined as follows:

a. Any atomic formula of $\mathcal{L}$ is a formula.

b. If φ and ψ are formulas, then so are $\neg\varphi, \varphi \wedge \psi, \varphi \vee \psi$, and $\varphi \rightarrow \psi$.

c. If φ is a formula then $\forall\ v_n\ \varphi$ and $\exists\ v_n\ \varphi$ are both formulas for every $n \in \mathbb{N}$.

Definition 4.8. The set of all formulas of a language $\mathcal{L}$ is denoted by $Fm_{\mathcal{L}}$.

Definition 4.9. A sequence of formulas showing how a formula φ is built from atomic formulas, (b), and (c) in Definition 4.7 is called a **history** of φ. A formula in a history of φ is called a **subformula** of φ.

Definition 4.10. An occurrence of a variable x in a formula φ is called **bound** if this occurrence is in a subformula ψ of φ that begins with a quantifier on x (i. e. $\exists x$ or $\forall x$). An occurrence is **free** if it is not bound. Given a formula φ if $x_1, x_2, \ldots, x_n$ are all variables that appear free in φ, then we often write $\varphi(x_1, x_2, \ldots, x_n)$ instead of φ.

Example 4.6. In formula $(\forall x \exists y R(x, y, z)) \rightarrow (\exists z F(x, z) = y)$, the first and second occurrences of x and y are both bound. The first occurrence of z is free. The last occurrences of x and y are both free, and the second and third occurrences of z are both bound.

Definition 4.11. A **sentence** is a formula in which no variable occurs free.

4.2.2 Interpretations

Definition 4.12. Let t be an $\mathcal{L}$-term and $x_1, \ldots, x_n$ be all variables that appear in t. Then we sometimes write t as $t(x_1, x_2, \ldots, x_n)$ and treat that as an n-ary function.

Definition 4.13. Given a term $t(x_1, \ldots, x_n)$ and a structure $\mathcal{A}$ with universe A, and $a_1, \ldots, a_n \in A$, the value $t^{\mathcal{A}}(a_1, \ldots, a_n)$ is obtained by replacing every function symbol F by $F^{\mathcal{A}}$, every constant symbol c by $c^{\mathcal{A}}$, and each x_i by a_i.

Example 4.7. Let $\mathcal{L}^{nl} = \{F, G, c\}$, and let $\mathcal{A} = (\mathbb{N}, +, ., 1)$ be an $\mathcal{L}$-structure. Suppose $t(x, y, z) = F(G(c, x), G(y, z))$ is an $\mathcal{L}$-term. Then $t^{\mathcal{A}}(n, k, m) = n + km$.

Definition 4.14. Let $\mathcal{A}$ be an $\mathcal{L}$-structure and A be its universe. Let $\varphi(x_1, x_2, \ldots, x_n)$ (or φ for short) be an $\mathcal{L}$-formula, and $a_1, a_2, \ldots, a_n \in A$. We say $a_1, a_2, \ldots, a_n$ **satisfies** φ as follows:

a. If $\varphi = R(t_1, t_2, \ldots, t_k)$, then $a_1, \ldots, a_n$ satisfies φ if and only if after substituting each x_i by a_i and each t_i by $t_i^{\mathcal{A}}$ and R by $R^{\mathcal{A}}$ the relation $R^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_k^{\mathcal{A}})$ holds in $\mathcal{A}$.

b. If $\varphi = \neg\psi$, then $a_1, \ldots, a_n$ satisfies φ if and only if $a_1, \ldots, a_n$ does not satisfy ψ.

c. If $\varphi = (\theta \wedge \psi)$, then $a_1, \ldots, a_n$ satisfies φ if and only if $a_1, \ldots, a_n$ satisfies both θ and ψ. Similar for when $\varphi = (\theta \rightarrow \psi)$ and $\varphi = (\theta \vee \psi)$.

d. If $\varphi = \forall x\psi$, then $a_1, \ldots, a_n$ satisfies φ if and only if for every $b \in A$, $a_1, \ldots, a_n, b$ satisfies $\psi(x_1, \ldots, x_n, x)$.

e. If $\varphi = \exists x\psi$, then $a_1, \ldots, a_n$ satisfies φ if and only if there is $b \in A$ for which $a_1, \ldots, a_n, b$ satisfies $\psi(x_1, \ldots, x_n, x)$.

Note that in (d) and (e), the element b only replaces those occurrences of x in ψ that are free.

In shorts, the above definition means, to see if $a_1, a_2, \ldots, a_n$ satisfy φ, we substitute free variables of φ by $a_1, a_2, \ldots, a_n$ and interpret all the quantifiers and see if the obtained sentence is true in the given structure.

Definition 4.15. With the notations of the above definition, when $a_1, \ldots, a_n$ satisfies φ, we write $\mathcal{A} \models \varphi(a_1, \ldots, a_n)$ or we say $\varphi^{\mathcal{A}}(a_1, \ldots, a_n)$ holds. If φ is a sentence, and the empty sequence satisfies φ then we write $\mathcal{A} \models \varphi$ and we say $\mathcal{A}$ **models** φ.

Example 4.8. Let $\mathcal{L}^{nl} = \{R\}$, where R is a binary relation. Determine all structures that model each of the following sentences.

a. $\forall x \forall y (R(x, y) \rightarrow R(y, x))$.

b. $\forall x R(x, x)$.

c. $\forall x \forall y \forall z ((R(x, y) \wedge R(y, z)) \rightarrow R(x, z))$.

Definition 4.16. Given a language $\mathcal{L}$, we say an $\mathcal{L}$-formula $\varphi(x)$ defines a subset B of the universe, provided b satisfies $\varphi(x)$ if and only if $b \in B$.

Example 4.9. Write down an $\mathcal{L}$-formula that defines the universe.

Scratch: We need to find a formula that is satisfied by every element of the universe. $x = x$ is a good one.
Solution. Consider the formula $\varphi(x)$ given by $x = x$. If a is an element of the universe, then $a = a$ and thus a satisfies $\varphi(x)$, as desired. □

Example 4.10. Let $\mathcal{L}^{nl} = \{F\}$, where F is a binary function, and $\mathcal{N} = (\mathbb{N}, +)$. Write down an $\mathcal{L}$-formula that defines $\{0\}$.

Scratch: We should find a property of zero that no other number has, but we are only allowed to use addition. $0 + 0 = 0$ seems to be an appropriate one.
Solution. Consider $\varphi(x)$ to be $F(x, x) = x$.
$\mathcal{N} \models \varphi(a)$, for some $a \in \mathbb{N}$ if and only if $F^{\mathcal{N}}(a, a) = a$, which is the same as $a + a = a$, or $a = 0$. □

4.3 More Examples

Example 4.11. Let $\mathcal{L}^{nl} = \{R\}$, write down an $\mathcal{L}$-formula that models $\{1\}$ in $(\mathbb{N}, >)$.

Scratch: We must find a way to say there is precisely one element less than 1. So, we will say there is one element less than 1 and everything else is more than 1.

Solution. Consider the sentence $\varphi(x)$ defined by $(\exists y R(x,y)) \wedge (\forall y \forall z((R(x,y) \wedge R(x,z)) \to y = z)$.
$n \in \mathbb{N}$ satisfies $\varphi(x)$ if and only if n satisfies both $(\exists y R(x,y))$ and $(\forall y \forall z((R(x,y) \wedge R(x,z)) \to y = z)$. The first one means there is $m \in \mathbb{N}$ for which $n > m$, which is equivalent to saying $n > 0$. The second one is saying for every $m, k \in \mathbb{N}$ if $n > k$ and $n > m$, then $m = k$. This means there is at most one element less than n. This means $n < 2$. Combining the two we obtain that $n = 1$ if and only if n satisfies $\varphi(x)$, as desired. □

Example 4.12. Let n be a positive integer. Suppose $\mathcal{L}^{nl} = \{R\}$, where R is an n-ary relation. Find all terms and atomic formulas of $\mathcal{L}$.

Solution. By definition terms are either variables, constants or functions evaluated at terms. Since there are no constants or functions the only terms of $\mathcal{L}$ are variables $v_1, v_2, \ldots$.

Atomic formulas are all formulas of form $R(t_1, \ldots, t_n)$ or $t_1 = t_2$, where t_i's are terms. Since the only terms are variables, the only atomic formulas are $R(x_1, \ldots, x_n)$ and $x_1 = x_2$, where $x_1, \ldots, x_n$ are (not necessarily distinct) arbitrary variables. □

4.4 Exercises

4.4.1 Problems for grading

The following problems must be submitted on Friday 10/2/2020 before the beginning of class. The submission will be on Gradescope via Elms. **Late submission will not be accepted.**

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

Exercise 4.1 (10 pts)**.** *Suppose φ is a sentence that is not a tautology.*

a. Prove that there is a maximal consistent set Γ that does not contain φ.

b. What is the intersection of all maximal consistent sets of sentences?

Exercise 4.2 (10 pts)**.** *Suppose Σ_1, Σ_2 are sets of sentences for which Σ_1 is satisfiable but $\Sigma_1 \cup \{\neg\varphi \mid \varphi \in \Sigma_2\}$ is not satisfiable. Prove that there are $\varphi_1, \varphi_2, \ldots, \varphi_n \in \Sigma_2$ for which $\Sigma_1 \models \varphi_1 \vee \cdots \vee \varphi_n$.*

Hint: Use The Compactness Theorem.

The following problems are in First Order Logic.

Exercise 4.3 (10 pts). *Suppose $\mathcal{L}^{nl} = \{R, c\}$, where R is a binary relation symbol, and c is a constant. Let n be a positive integer. How many $\mathcal{L}$-structures with $A = \{1, 2, \ldots, n\}$ as the universe are there?*

Exercise 4.4 (20 pts). *Let $\mathcal{L} = \{F,\ G,\ R,\ S,\ c,\ d\}$, where F is a binary function symbol, G is a unary function symbol, R is a binary relation symbol, S is a unary relation symbol, and c and d are constants.*
For each of the following, identify whether it is a term, a formula, or neither. If it is a formula, determine whether it is a sentence. If it is a formula which is not a sentence, identify which variables are free and which are bound.

a. $\forall x(S(x) \wedge R(c, F(G(y), y)))$

b. S

c. $R(c, F(G(d, y)))$

d. $\forall x \forall y \neg R(x, y)$

Exercise 4.5 (15 pts). *Suppose $\mathcal{L}^{nl} = \{R\}$, where R is a binary relation symbol. For each of the following three sentences state its meaning in English and give an example of a model in which that sentence holds, but the other two do not.*

a. $\forall x \exists y(R(y, x) \wedge \forall z(R(z, x) \rightarrow z = y))$

b. $\exists x \forall y(\neg R(x, y))$

c. $\forall x \forall y(R(x, y) \rightarrow \exists z(R(x, z) \wedge R(z, y)))$

Exercise 4.6 (10 pts). *Suppose $\mathcal{L}$ contains the binary relation symbol S and the constant c. Let $\mathcal{N}$ be an $\mathcal{L}$-structure with universe $\mathbb{N}$, $S^{\mathcal{N}} = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x + 1 = y\}$ and $c^{\mathcal{N}} = 0$. Let $\mathcal{M}$ be an $\mathcal{L}$-structure with universe $\mathbb{N}$, $S^{\mathcal{M}} = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x + 1 = y\}$ and $c^{\mathcal{M}} = 5$. Find an $\mathcal{L}$-formula $\psi(x)$ such that $\mathcal{M} \models \psi(c)$ but $\mathcal{N} \not\models \psi(c)$.*

Exercise 4.7 (15 pts). *Let $\mathcal{L}^{nl} = \{R\}$. Consider the $\mathcal{L}$-structure $\mathcal{M} = (\mathbb{N},\ <)$.*

a. Find an $\mathcal{L}$-formula $\varphi(x)$ such that for all $a \in \mathbb{N}$, $\mathcal{M} \models \varphi(a)$ if and only if $a = 0$.

b. Find an $\mathcal{L}$-formula $\varphi(x)$ such that for all $a \in \mathbb{N}$, $\mathcal{M} \models \varphi(a)$ if and only if $a = 0$ or $a = 1$.

c. Write an $\mathcal{L}$-formula $\varphi(x, y)$ such that for all $a, b \in \mathbb{N}$, $\mathcal{M} \models \varphi(a, b)$ if and only if $a = b + 1$.

5 Week 5

5.1 Translating from English

Example 5.1. Write sentences in first order logic that their translations are each of the following:

a. Every prime number is odd.

b. There is precisely one element in a set.

c. A function is one-to-one.

d. Some even numbers are prime.

Definition 5.1. Let $\mathcal{L}$ be a first order language, and θ be an $\mathcal{L}$-sentence.

- We say θ is **satisfiable** if $\mathcal{A} \vDash \theta$ for some $\mathcal{L}$-structure $\mathcal{A}$.
- We say θ is **valid** if $\mathcal{A} \vDash \theta$ for every $\mathcal{L}$-structure $\mathcal{A}$.

Theorem 5.1. *A sentence θ is satisfiable if and only if $\neg\theta$ is not valid. Similarly $\neg\theta$ is satisfiable if and only if θ is not valid.*

The above theorem is often used to check validity of a sentence. We often assume a sentence is not valid and see what the consequences are. If we get a contradiction that means the sentence is valid. Otherwise, we may be able to see the sentence is not valid and come up with an example of a structure that does not model the sentence.

Example 5.2. Let P, Q be unary relation symbols and R be a binary relation symbol. Determine if each of the following sentences are valid, satisfiable or neither.

a. $\forall x(P(x) \to Q(x)) \to (\forall x P(x) \to \forall x Q(x))$

b. $\exists x(P(x) \to Q(x)) \to (\exists x P(x) \to \exists x Q(x))$

c. $\forall x \exists y R(x, y) \to \exists x R(x, x)$

d. $\forall x \forall y R(x, y) \to \forall y \forall x R(x, y)$

Definition 5.2. Let $\mathcal{L}$ be a first order language. We say that an $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ (φ for short) **is modeled** by a structure $\mathcal{A}$, written as $\mathcal{A} \vDash \varphi$, if $\mathcal{A} \vDash \forall x_1 \cdots \forall x_n \varphi$. The formula φ is called **valid**, written as $\vDash \varphi$, if every $\mathcal{L}$-structure models φ. A set of formulas Σ is said to be **satisfiable** if there is a structure that models all formulas of Σ.

Remark. Given a sentence θ and a structure $\mathcal{A}$, the empty sequence either satisfies θ or does not. This means $\mathcal{A} \vDash \theta$ or $\mathcal{A} \vDash \neg\theta$. However if θ is a formula, then it is not the case that $\mathcal{A} \vDash \theta$ or $\mathcal{A} \vDash \neg\theta$. For example consider the formula $P(x)$, where P is a unary relation symbol. If the universe is $\{1, 2\}$, and $P = \{1\}$, then $\forall x P(x)$ and $\forall x \neg P(x)$ both fail. This implies that $\mathcal{A} \nvDash P(x)$ and $\mathcal{A} \nvDash \neg P(x)$.

Definition 5.3. Let $\mathcal{L}$ be a first order language. An $\mathcal{L}$-formula φ is said to be a **logical consequence** of Σ, if $\mathcal{A} \vDash \varphi$ for every structure $\mathcal{A}$ that models Σ.

Theorem 5.2. *Let Σ be a set of sentences and θ be a sentence. Then $\Sigma \vDash \theta$ if and only if $\Sigma \cup \{\neg\theta\}$ is not satisfiable.*

Definition 5.4. We say two $\mathcal{L}$-formulas are equivalent, written as $\varphi \equiv \psi$, whenever $\vDash \varphi \to \psi$ and $\vDash \psi \to \varphi$.

Example 5.3. Prove that for every two formulas φ and ψ

a. $\varphi \vee \psi \equiv \neg\varphi \to \psi$

b. $\varphi \wedge \psi \equiv \neg(\varphi \to \neg\psi)$

c. $\exists x\varphi \equiv \neg\forall x\neg\varphi$

Theorem 5.3. *For any formula φ, there is a formula φ^* such that $\varphi \equiv \varphi^*$, and φ^* does not use $\exists, \wedge$, and $\vee$.*

5.2 More Examples

Example 5.4. Let P be a unary relation symbol and R be a binary relation symbol. Determine if each of the following is true or false.

a. $\models R(x,y) \to \forall xR(x,y)$

b. $\models \forall x\forall y(R(x,y) \wedge P(x)) \to \forall xR(x,x)$

c. $\models (\forall xP(x) \to \forall xQ(x)) \to \forall x(P(x) \to Q(x))$

a. **Scratch:** Suppose $\mathcal{A} \not\models R(x,y) \to \forall xR(x,y)$. This means there are a_1, a_2 in the universe A such that the sentence $R(a_1,a_2) \to \forall xR(x,a_2)$ is false. Which means if $R(a_1,a_2)$ holds, but not for all $b \in A$, $R(b,a_2)$ holds. In other words, $R(a_1,a_2)$ holds but $R(b,a_2)$ does not hold for some b. This is clearly possible.

b. **Scratch:** If for all a and b in the universe, $R(a,b)$ and $P(a)$, then setting $a = b$ gives us $R(a,a)$, which means this must be true. We will turn this into a formal proof.

c. **Scratch:** Let's see what happens if $\mathcal{A}$ does not model this sentence. This means $\mathcal{A}$ models $\forall xP(x) \to \forall xQ(x)$ but not $\forall x(P(x) \to Q(x))$. Therefore, there is an element a for which $P(a)$ holds but $Q(a)$ does not. We can create an example that $\forall xP(x)$ and $\forall xQ(x)$ are both false.

Solution. a. This is false. Consider an structure $\mathcal{A}$ with $A = 1,2$, and $R^{\mathcal{A}} = \{(1,1)\}$. Clearly $R(1,1)$ holds but $R(2,1)$ does not. This means $\forall xR(x,1)$ does not hold. Thus $\mathcal{A}$ does not model $R(x,y) \to \forall xR(x,y)$.

b. We will prove this is true. Suppose $\mathcal{A}$ is a structure for which the empty sequence models $\forall x\forall y(R(x,y) \wedge P(x))$. This means for every a, b in the universe $R(a,b)$ and $P(a)$ both hold. Thus setting $b = a$ we conclude that $R(a,a)$ holds. Therefore, $\forall xR(x,x)$ is modeled by $\mathcal{A}$.

c. This is false. Let $\mathcal{A}$ be a structure whose universe is $A = \{1,2\}$, $P^{\mathcal{A}} = \{1\}$, and $Q^{\mathcal{A}} = \{2\}$. We see that $\forall xP(x)$ and $\forall xQ(x)$ are both false in this structure. Also $P(1) \to Q(1)$ is false. This means $\forall xP(x) \to \forall xQ(x)$ is true, but $\forall x(P(x) \to Q(x))$ is false. Therefore, in this structure the given sentence is not modeled. □

Example 5.5. Suppose P and Q are unary relation symbols, and $\mathcal{A}$ is a structure that does not model $\exists x(P(x) \to Q(x)) \to (\exists x P(x) \to \exists x Q(x))$. Prove that $Q^{\mathcal{A}}$ is empty.

Solution. By assumption $\mathcal{A}$ models $\exists x(P(x) \to Q(x))$, but it does not model $\exists x P(x) \to \exists x Q(x)$. Therefore, $\exists x P(x)$ is true in this structure and $\exists x Q(x)$ is false. The latter means $Q^{\mathcal{A}}$ is empty. □

Example 5.6. Write down a formula $\varphi(x)$ that defines the empty set.

Solution. $\neg(x = x)$ is such a formula. (Why?) □

Example 5.7. Given a formula $\varphi(x)$, write down a sentence that interprets

"There exists a unique x for which $\varphi(x)$".

Solution. Consider the formula $(\exists x \varphi(x)) \wedge \forall x \forall y((\varphi(x) \wedge \varphi(y)) \to x = y)$.
If a structure $\mathcal{A}$ models this sentence, then it must model $\exists x \varphi(x)$, which means there is an element a for which $\varphi(a)$ is true. $\mathcal{A}$ must also model $\forall x \forall y((\varphi(x) \wedge \varphi(y)) \to x = y)$, which means if for two elements a, b we have $\varphi(a)$ and $\varphi(b)$ are true, then $a = b$. This implies there is not more than one element of the universe that satisfies $\varphi(x)$.

Combining these two we obtain the result. □

Example 5.8. Let $\mathcal{L}^{nl} = \{R, c\}$, where R is a binary relation symbol and c is a constant symbol. Consider the $\mathcal{L}$-structure $\mathcal{A} = (\mathbb{N}, |, 1)$, where $|$ is the dividing relation.

a. Write down an $\mathcal{L}$-formula that defines the constant 0.

b. Write down an $\mathcal{L}$-formula that defines the set of all prime numbers in the structure $\mathcal{A}$.

c. Write down an $\mathcal{L}$-formula that defines all integers with at least two distinct prime factors.

Scratch: a. Zero is the only integer that is divisible by everything.

b. For a natural number p to be prime we need to say p has precisely two divisors.

c. We will use part (b).

Solution. a. Consider the formula $\varphi(x)$ given by $\forall y R(y, x)$. A natural number a satisfies $\varphi(x)$ if and only if $R^{\mathcal{A}}(n, a)$ hold for all $n \in \mathbb{N}$. Taking $n = 0$, gives us $a = 0 \times k$ for some natural number k which means $a = 0$. Furthermore, if $a = 0$, then $0 = 0 \times n$ and thus n divides 0, which means this formula defines $\{0\}$.

b. Let $\psi(x)$ be the formula $\neg(x = c) \wedge \forall y(R(y, x) \rightarrow ((y = x) \vee (y = c)))$.

A natural number a satisfies $\psi(x)$ if and only if $a \neq 1$, and if b divides a, then either $b = 1$ or $b = a$. This is precisely the definition of a prime number. Therefore, $\psi(x)$ defines the set of all primes.

c. Let $\theta(x)$ be the formula $\exists x_1 \exists x_2 (x_1 \neq x_2) \wedge R(x_1, x) \wedge R(x_2 \wedge x) \wedge \psi(x_1) \wedge \psi(x_2)$ □

5.3 Exercises

5.3.1 Problems for grading

The following problems must be submitted on Wednesday 10/7/2020 before the beginning of class. The submission will be on Gradescope via Elms. **Late submission will not be accepted.**

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

Exercise 5.1 (15 pts)**.** *Let $\mathcal{L}^{nl} = \{F, R\}$, where F is a unary function symbol and R is a binary relation symbol. Let*

$$\theta = \forall x \exists y F(y) = x, \quad \textit{and} \quad \varphi = \exists x \forall y (R(x, y) \vee x = y).$$

Determine if θ and φ are true in each of the following $\mathcal{L}$-structures.

a. $A = \mathbb{Q}$, $F(x) = x^2$, and $R(a, b)$ holds iff $a < b$.

b. $A = \mathbb{N}$, $F(x) = x + 1$, and $R(a, b)$ holds iff a divides b.

c. $A = \mathbb{R}$, $F(x) = 3x$, and $R(a, b)$ holds iff $a^2 + b = 0$.

Exercise 5.2 (10 pts)**.** *Let $\varphi(x)$ and $\psi(x)$ be two formulas. Using the definition, prove that $\varphi(x) \vee \psi(x) \equiv \neg\varphi(x) \rightarrow \psi(x)$.*

Exercise 5.3 (20 pts)**.** *Let $\mathcal{L}^{nl} = \{F, G\}$, where F and G are binary function symbols. Suppose in an $\mathcal{L}$-structure $\mathcal{A}$ the universe is $A = \mathbb{R}$, $F^{\mathcal{A}}(x, y) = xy$, and $G^{\mathcal{A}}(x, y) = x + y$.*

a. Write a formula $\theta(x)$ that is satisfied only by 0.

b. Write a formula $\alpha(x)$ that is satisfied only by 1.

c. Write a formula $\psi(x)$ that is satisfied only by non-negative real numbers.

d. Write a formula $\varphi(x, y)$ that is satisfied by (a, b) iff $a \leq b$.

Exercise 5.4 (10 pts)**.** *Prove or disprove each of the following*

a. $\models \exists x R(x, x) \rightarrow R(y, y)$.

b. $\models \forall x \forall y R(x, y) \rightarrow \exists y R(y, y)$.

6 Week 6

6.1 Properties of Validity and Logical Consequences

In this section we will see some examples of valid formulas and some properties of logical consequences.

Tautologies: Suppose θ is a tautology in sentential logic that only uses atomic sentences $A_1, \ldots, A_n$. Let $\varphi_1, \ldots, \varphi_n$ be formulas. If we replace each atomic sentence A_i of θ by the formula φ_i we obtain a formula θ^* that is also valid. The reason is that every sequence of elements $a_1, \ldots, a_n$ in the universe of $\mathcal{A}$ either satisfy $\varphi_i^{\mathcal{A}}$ or it does not. However in either case, since θ is a tautology, $\theta^\star$ will be satisfied for every sequence of elements in the universe.

Every such formula is called a tautology.

Example 6.1. $(\exists x P(x) \to Q(y)) \to (\neg Q(y) \to \neg \exists x P(x))$ is a tautology.

Modes Ponens: Suppose Σ is a set of sentences and φ, ψ are two formulas. If $\Sigma \vDash \varphi \to \psi$, and $\Sigma \vDash \varphi$, then $\Sigma \vDash \psi$.

Suppose $\mathcal{A}$ is a structure that models Σ. Since $\Sigma \vDash \varphi \to \psi$ and $\Sigma \vDash \varphi$, by definition we must have $\mathcal{A} \vDash \varphi \to \psi$, and $\mathcal{A} \vDash \varphi$. We know that every sequence $a_1, \ldots, a_n$ of elements of the universe of $\mathcal{A}$ satisfies $\varphi \to \psi$ and φ, which means the sequence must also satisfy ψ.

Example 6.2. Let Σ be a set of sentences, and φ, ψ be two formulas such that $\Sigma \vDash \varphi$. Prove that $\Sigma \vDash \psi \to \varphi$.

Universal Quantification: (a) If φ is a formula for which x does not occur free, then $\varphi \equiv \forall x \varphi$.
(b) Suppose Σ is a set of sentences and φ a formula for which $\Sigma \vDash \varphi$. Then $\Sigma \vDash \forall x \varphi$.

Note that in (a) it is important that x does not occur free in φ.

Example 6.3. Give an example of a formula φ for which $\varphi \not\equiv \forall x \varphi$.

Solution. Consider the formula $P(x)$, where P is a unary relation. Consider the structure $\mathcal{A} = (\{1, 2\}, P^{\mathcal{A}})$, where $P^{\mathcal{A}} = \{1\}$. We know $P(1)$ holds but $P(2)$ does not. Thus 1 does not satisfy $P(x) \to \forall x P(x)$. Therefore, $\nvDash P(x) \to \forall x P(x)$. Hence $P(x) \not\equiv \forall x P(x)$. $\square$

Substitution: Suppose $\varphi(x)$ is a formula and $t(z_1, \ldots, z_n)$ (or t for short) is a term. Then,

$$\vDash \forall x \varphi(x) \to \varphi(t(z_1, \ldots, z_n)),$$

provided no occurrence of $z_1, \ldots, z_n$ in t is bound in $\varphi(t)$.

The same holds if φ has multiple free variables. In other words $\models \forall x \varphi(x, x_1, \ldots, x_n) \rightarrow \varphi(t, x_1, \ldots, x_n)$ provided no new occurrence in $\varphi(t, x_1, \ldots, x_n)$ of a variable in t is bound.

Example 6.4. $\not\models \forall x \exists y R(x, y) \rightarrow \exists y R(y, y)$.

6.2 A Formal Proof System

Definition 6.1. We say a formula φ is a **generalization** of a formula ψ if for some $n \geq 0$, and some variables $x_1, \ldots, x_n$, $\varphi = \forall x_1 \forall x_2 \cdots \forall x_n \psi$.

Note that φ is a generalization of itself.

Definition 6.2. The set $\Lambda_{\mathcal{L}}$ of **logical axioms** of a first order language $\mathcal{L}$ consists of all generalizations of the following formulas, where φ and ψ are formulas, $x, y, x_1, \ldots, x_m, y_1, \ldots, y_m$ are variables, and t is a term.

(i) All tautologies.

(ii) (Substitution Axiom) $\forall x \varphi(x, \ldots) \rightarrow \varphi(t, \ldots)$, where no new occurance in $\varphi(t, \ldots)$ of a variable in t is bound.

(iii) (Distribution of Universal Quantifier Axiom) $\forall x(\varphi \rightarrow \psi) \rightarrow (\forall x \varphi \rightarrow \forall x \psi)$.

(iv) (Generalization Axiom) $\varphi \rightarrow \forall x \varphi$, where x does not occur free in φ.

(v) (Equality Axioms) $x = x$; $(x = y \rightarrow y = x)$; $(x = y \rightarrow (y = z \rightarrow x = z))$; $x_1 = y_1 \rightarrow (x_2 = y_2 \rightarrow \cdots (x_m = y_m \rightarrow (R(x_1, \ldots, x_m) \rightarrow R(y_1, y_2, \ldots, y_m)) \cdots)$, for every m-ary relation R.

Note: In the book they have not included generalizations of axioms for equality (but they should have!)

Note: For any formula ψ, the formula $\exists x \psi$ is short hand for $\neg \forall x \neg \psi$.

Remark. Note that generalization of an axiom is an axiom itself.

Theorem 6.1. *Suppose $\varphi \in \Lambda_{\mathcal{L}}$ for a first order language $\mathcal{L}$. Then $\models \varphi$.*

Definition 6.3. A **(logical) deduction** is a finite sequence $\varphi_1, \ldots, \varphi_n$ of $\mathcal{L}$-formulas such that for every $i \leq n$ one of the following holds.

- $\varphi_i \in \Lambda_{\mathcal{L}}$.
- φ_i is obtained from φ_j and φ_k for two $j, k < i$, by an application of modus ponens. In other words, $\varphi_k = \varphi_j \rightarrow \varphi_i$.

Definition 6.4. A formula φ is said to be **deducible**, written $\vdash \varphi$, if there is a deduction whose last formula is φ.

Example 6.5. Prove $\vdash P(x) \to \exists x P(x)$.

Solution. We need to show $\vdash P(x) \to \neg\forall x \neg P(x)$.

Using the tautology $(A \to \neg B) \to (B \to \neg A)$ we obtain

$$\vdash (\forall x \neg P(x) \to \neg P(x)) \to (P(x) \to \neg\forall x \neg P(x))$$

By the Substitution Axiom we know $\vdash \forall x \neg P(x) \to \neg P(x)$. Applying Modus Ponens we conclude that $\vdash P(x) \to \neg\forall x \neg P(x)$, as desired. $\square$

Theorem 6.2 (Soundness)**.** *If a formula φ is deducible then it is valid. In other words, $\vdash \varphi$ implies $\vDash \varphi$.*

Similar to before we could define deduction from hypotheses.

Definition 6.5. Let Γ be a set of formulas. A **(logical) deduction** from Γ is a finite sequence $\varphi_1, \ldots, \varphi_n$ of formulas such that for every $i \leq n$ one of the following holds.

- $\varphi_i \in \Lambda_{\mathcal{L}} \cup \Gamma$.
- φ_i is obtained from φ_j and φ_k for two $j, k < i$, by an application of modus ponens. In other words, $\varphi_k = \varphi_j \to \varphi_i$.

If there is a deduction from Γ whose last formula is φ, we say φ is deducible from Γ and we write $\Gamma \vdash \varphi$.

Lemma 6.1. *If $\Sigma \subseteq \Gamma$ are two sets of formulas and θ is a formula for which $\Sigma \vdash \theta$, then $\Gamma \vdash \theta$.*

Theorem 6.3 (Modus Ponens)**.** *Suppose Γ is a set of formulas, and φ, ψ are two formulas. If $\Gamma \vdash \varphi \to \psi$, and $\Gamma \vdash \varphi$, then $\Gamma \vdash \psi$.*

Theorem 6.4 (Soundness)**.** *If for a formula φ and a set of sentences Σ we have $\Sigma \vdash \varphi$, then $\Sigma \vDash \varphi$.*

Theorem 6.5 (The Deduction Theorem)**.** *Assume Γ is a set of formulas, and φ, ψ are two formulas. Then, $\Gamma \vdash (\varphi \to \psi)$ if and only if $\Gamma \cup \{\varphi\} \vdash \psi$.*

A proof similar to the Finiteness Theorem in sentential logic works for the following theorem.

Theorem 6.6. *If a formula φ can be deduced from a set of formulas Γ, then a finite subset of Γ deduces φ.*

Theorem 6.7 (Generalization Theorem)**.** *Let Γ be a set of formulas, φ be a formula, and x be a variable that does not occur free in any of the formulas of Γ. Then, $\Gamma \vdash \varphi$ if and only if $\Gamma \vdash \forall x \varphi$.*

6.3 More Examples

Example 6.6. Prove or disprove: $\vdash \forall x(P(x) \to Q(x)) \to (\exists x P(x) \to \exists y Q(y))$.

Scratch: If for every x, $P(x) \to Q(x)$ and there is a x for which $P(x)$, then $Q(x)$ must hold. Thus, this must be deducible.

To prove that note that by contraposition $\exists x P(x) \to \exists y Q(y)$ must be equivalent to $\forall y \neg Q(y) \to \forall x \neg P(x)$. This allows us to use Deduction Theorem and thus we need to prove

$$\forall x(P(x) \to Q(x)), \forall y \neg Q(y) \vdash \forall x \neg P(x).$$

Substitution Axiom gives us $P(x) \to Q(x)$ and $\neg Q(x)$. Then combine this with contraposition and Modus Ponens to obtain $\neg P(x)$. Then apply the Generalization Theorem.

Solution. By Deduction Theorem and definition of $\exists$, it is enough to prove

$$\forall x(P(x) \to Q(x)) \vdash \neg \forall x \neg P(x) \to \neg \forall y \neg Q(y).$$

Using the tautology $(A \to B) \to (\neg B \to \neg A)$ we obtain

$$\vdash (\forall y \neg Q(y) \to \forall x \neg P(x)) \to (\neg \forall x \neg P(x) \to \neg \forall y \neg Q(y))$$

By Modus Ponens it is enough to prove

$$\forall x(P(x) \to Q(x)) \vdash \forall y \neg Q(y) \to \forall x \neg P(x).$$

By Deduction Theorem, it is enough to prove

$$\forall x(P(x) \to Q(x)), \forall y \neg Q(y) \vdash \forall x \neg P(x).$$

Let $\Sigma = \{\forall x(P(x) \to Q(x)), \forall y \neg Q(y)\}$. By the Gerenalization Theorem, since x does not occur free in any of the formulas of Σ, it is enough to prove $\Sigma \vdash \neg P(x)$. By Substitution Axiom we obtain

$$\Sigma \vdash P(x) \to Q(x), \text{ and } \Sigma \vdash \neg Q(x).$$

Using tautology $(A \to B) \to (\neg B \to \neg A)$, we have

$$\vdash (P(x) \to Q(x)) \to (\neg Q(x) \to \neg P(x)).$$

By Modus Ponens we have $\Sigma \vdash \neg Q(x) \to \neg P(x)$. Applying Modus Ponens again we obtain $\Sigma \vdash \neg P(x)$, as desired. $\square$

Example 6.7. Prove or disprove:

a. $\models (\exists x P(x) \to \forall x Q(x)) \to \forall x(P(x) \to Q(x))$.

b. $\models (P(x) \to \forall y Q(y)) \to (\exists x P(x) \to \exists y Q(y))$.

Solution. a. This is true. Suppose $\mathcal{A} \not\models (\exists x P(x) \to \forall x Q(x)) \to \forall x(P(x) \to Q(x))$. This means $\mathcal{A} \models \exists x P(x) \to \forall x Q(x)$, and $\mathcal{A} \not\models \forall x(P(x) \to Q(x))$. Therefore, there is c for which $P(c) \to Q(c)$ is false. This means $P(c)$ holds, but $Q(c)$ does not. This implies $\exists x P(x)$ is true, and $\forall x Q(x)$ is not true. This is implies $\mathcal{A} \nvDash \exists x P(x) \to \forall x Q(x)$, which is a contradiction.

b. This is false. Let $A = \{1, 2\}$, $P^{\mathcal{A}} = \{1\}$, and $Q^{\mathcal{A}} = \emptyset$. $P(2)$ does not hold, and $\forall y Q(y)$ is not satisfied. Thus, $\mathcal{A} \vDash P(2) \to \forall y Q(y)$. Also, note that $P(1)$ holds but $Q(1)$ and $Q(2)$ both fail, which means $\exists x P(x) \to \exists y Q(y)$ is not satisfied. Thus, $\mathcal{A}$ does not model the given formula. □

Example 6.8. Determine if each of the following is true.

a. $\vdash P(x) \to P(y)$.

b. $\vdash \varphi \to \forall x \varphi$.

c. $\vdash \forall x \varphi(x) \to \forall y \varphi(y)$.

d. $\vdash \forall x \forall y R(x, y) \to \forall x \forall y R(y, x)$

Scratch: a. This means we need to see whether the sentence $\forall x \forall y (P(x) \to P(y))$ is deducible or not. This means if $P(x)$ holds for some x, then $P(y)$ also holds, but that is not true, because P may hold for one value, but not hold for other values.

b. We know the Generalization Axiom requires x to not occur free in φ for this to be an Axiom. So, this is probably false.

c. This seems true, because x is just a place-holder! However it could be the case that y becomes bound by a different quantifier other than the outer most $\forall y$.

d. This seems to be true.

Solution. a. This is false. Assume it were true. By the Soundness Theorem $\vDash \forall x \forall y (P(x) \to P(y))$. Consider the structure $\mathcal{A}$ with $A = \{1, 2\}$, $P^{\mathcal{A}} = \{1\}$. We know $P^{\mathcal{A}}(1)$ holds, but $P^{\mathcal{A}}(2)$ does not hold. Therefore, $\mathcal{A}$ does not model $P^{\mathcal{A}}(1) \to P^{\mathcal{A}}(2)$.

b. Let the structure be the same as the one in part (a), and let φ be the same as $P(x)$. Then, $P(1)$ holds, but $\forall x P(x)$ does not.

c. This is false. Take $\varphi(x) = \exists y R(x, y)$. Consider a structure $\mathcal{A}$ with universe $A = \mathbb{Z}$, and $R^{\mathcal{A}} =<$. We know that for every integer x, there is an integer $y = x + 1$ for which $x < y$. Thus, $\forall x \exists y R(x, y)$ holds in this structure. However there does not exist any y for which $y < y$. Therefore, $\forall y \exists y R(y, y)$ is not satisfied.

d. This is true. By the Generalization Theorem it is enough to prove $\forall x\forall y R(x,y) \vdash R(y,x)$, since the formula on the left is a sentence and has no free variables. Take two new variables z, t. Applying the Substitution Axiom twice we obtain that $\forall x\forall y R(x,y) \vdash R(z,t)$. By the Generalization Theorem, we see that $\forall x\forall y R(x,y) \vdash \forall z\forall t R(z,t)$. By the Substitution Axiom we have $\vdash \forall z\forall t R(z,t) \to \forall t R(y,t)$. By Modus Ponens, we obtain $\forall x\forall y R(x,y) \vdash \forall t R(y,t)$. Substitution Axiom again implies $\vdash \forall t R(y,t) \to R(y,x)$. Applying Modus Ponens again we obtain $\forall x\forall y R(x,y) \vdash R(y,x)$, as desired. □

6.4 Exercises

6.4.1 Problems for grading

Exercise 6.1 (25 pts). *Let $\mathcal{L}^{nl} = \{P, Q, F, c\}$ where P is a unary relation symbol, Q is a binary relation symbol, F is a binary function symbol, and c is a constant. Each of the following formulas is an instance of an axiom from $\Lambda_{\mathcal{L}}$. Identify which axiom, and explain why the formula really is an instance of it.*

a. $\forall x((\exists y Q(x,y)) \to (P(x) \to \exists y Q(x,y)))$

b. $\forall y((\forall x \exists z(P(z) \to \neg Q(x,y))) \to (\exists z(P(z) \to \neg Q(F(c,y),y))))$

c. $\forall y(\forall x(P(x) \to Q(y,x)) \to (\forall x P(x) \to \forall x Q(y,x)))$

d. $\forall y((P(y) \to \exists z Q(z,c)) \to \forall x(P(y) \to \exists z Q(z,c)))$

e. $(x = y) \to ((z = w) \to (F(x,z) = c \to F(y,w) = c))$

Exercise 6.2 (10 pts). *Suppose φ and ψ are two formulas in an $\mathcal{L}$-structure and x is a variable. Let Σ be a set of sentences such that $\Sigma \vdash \forall x\varphi$ and that $\Sigma \vdash \forall x(\varphi \to \psi)$. Prove that $\Sigma \vdash \forall x\psi$.*

Exercise 6.3 (20 pts). *Let $\mathcal{L}^{nl} = \{<\}$, where $<$ is a binary relation satisfying all of the following. (For simplicity $<(x,y)$ is denotes by $x < y$.)*

(i) $\theta_1 = \forall x \neg(x < x)$

(ii) $\theta_2 = \forall x\forall y((x < y) \vee (y < x) \vee (x = y))$

(iii) $\theta_3 = \forall x\forall y\forall z(((x < y) \wedge (y < z)) \to (x < z))$

For simplicity denote $\neg(x = y)$ by $x \neq y$.
Let $\Sigma = \{\theta_1, \theta_2, \theta_3, \exists x(x = x), \forall x\exists y(x < y)\}$.

a. Let ψ_2 be the sentence $\exists x_1\exists x_2(x_1 \neq x_2)$. Show $\Sigma \vDash \psi_2$.

b. Let ψ_3 be the sentence $\exists x_1\exists x_2\exists x_3((x_1 \neq x_2) \wedge (x_1 \neq x_3) \wedge (x_2 \neq x_3))$. Show $\Sigma \vDash \psi_3$.

c. For each $n \geq 2$, let ψ_n be the sentence

$$\exists x_1\exists x_2 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} (x_i \neq x_j).$$

Show that for all $n \geq 2$, $\Sigma \vDash \psi_n$ (Hint: Use induction on n.)

d. Conclude that if an $\mathcal{L}$-structure models Σ then the universe must be infinite. (Hint: Argue by contradiction.)

7 Week 7

7.1 Theorems on Deducibility

Theorem 7.1 (Generalization on Constants)**.** *Suppose Γ is a set of formulas, $\varphi(x, x_1, \ldots, x_n)$ is a formula, and c is a constant that does not appear anywhere in $\varphi(x, x_1, \ldots, x_n)$ or any of the formulas of Γ. If $\Gamma \vdash \varphi(c, x_1, \ldots, x_n)$, then $\Gamma \vdash \forall x \varphi(x, x_1, \ldots, x_n)$.*

Proof. Exercise. □

Definition 7.1. A set of formulas Γ is said to be **inconsistent** if $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ for some formula φ. It is called **consistent** if it is not inconsistent.

Theorem 7.2. *A set of formulas Γ is inconsistent if and only if $\Gamma \vdash \theta$ for every formula θ.*

Proof. The proof is identical to the case of Sentential Logic. □

Theorem 7.3. *Let Γ be a set of formulas and φ, ψ be two formuals. Then,*

a. $\Gamma \vdash \varphi$, and $\Gamma \vdash \neg\psi$ if and only if $\Gamma \vdash \neg(\varphi \to \psi)$.

b. (Double negation) $\Gamma \vdash \varphi$ if and only if $\Gamma \vdash \neg\neg\varphi$.

c. (Contraposition) $\Gamma \cup \{\varphi\} \vdash \psi$ if and only if $\Gamma \cup \{\neg\psi\} \vdash \neg\varphi$.

d. (Proof by Contradiction) $\Gamma \vdash \varphi$ if and only if $\Gamma \cup \{\neg\varphi\}$ is inconsistent. $\Gamma \vdash \neg\varphi$ if and only if $\Gamma \cup \{\varphi\}$ is inconsistent.

Definition 7.2. A set of sentences Γ is called a **maximal consistent set of sentences** if Γ is consistent and for every sentence φ, we have $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.
Similarly a set of formulas Γ is called a **maximal consistent set of formulas** if Γ is consistent and for every formula φ, we have $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Example 7.1. Prove or disprove

a. $\vdash \forall x \exists y R(x, y) \to \exists y \forall x R(x, y)$.

b. $\vdash \exists x \forall y R(x, y) \to \forall y \exists x R(x, y)$.

7.2 Proof of the Completeness Theorem

The objective of this section is to prove the Completeness Theorem.

Theorem 7.4 (The Completeness Theorem)**.** *Let Σ be a set of sentences and θ be a sentence. Then, $\Sigma \vdash \theta$ if and only if $\Sigma \vDash \theta$.*

The Soundness Theorem proves one direction of the Completeness Theorem. For the other direction, assume $\Sigma \models \theta$. By a theorem $\Sigma \cup \{\neg\theta\}$ is not satisfiable. We know by a theorem $\Sigma \vdash \theta$ if and only if $\Sigma \cup \{\neg\theta\}$ is inconsistent. Therefore, we need to show every set of sentences that is not satisfiable is inconsistent. In other words, we need to show every consistent set of sentences is satisfiable. This means in order to prove the Compltleness Theorem we need to prove the following:

Theorem 7.5 (Model Existence)**.** *Suppose Σ is a consistent set of sentences. Then, there is a structure that models Σ. In other words, Σ is satisfiable.*

Before we can prove the Model Existence Theorem we need the following theorems, all of which can be proved in a similar manner to the ones in sentential logic.

Theorem 7.6. *If Σ is a consistent set of formulas, and θ is a formula, then either $\Sigma \cup \{\theta\}$ or $\Sigma \cup \{\neg\theta\}$ is consistent.*

Theorem 7.7.

- *Every consistent set of sentences is contained in a maximal consistent set of sentences.*
- *Every consistent set of formulas is contained in a maximal consistent set of formulas.*

Theorem 7.8 (Finiteness Theorem)**.** *Let Σ be a set of sentences and θ be a sentence.*

a. If $\Sigma \vdash \theta$, then θ is deducible from a finite subset of Σ.

b. Σ is consistent if and only if every finite subset of Σ is consistent.

Theorem 7.9. *Suppose Γ is a maximal consistent set of sentences, φ, θ are two sentences. Then,*

a. $\Gamma \vdash \varphi$ if and only if $\varphi \in \Gamma$.

b. $\varphi \to \theta \in \Gamma$ if and only if $\theta \in \Gamma$ or $\neg\varphi \in \Gamma$.

We present the proof of the Completeness Theorem when the only non-logical symbol of the language $\mathcal{L}$ is a binary relation symbol R.

Suppose Σ is a consistent set of sentences. We first add some constants to the language. Let

$$\mathcal{L}' = \mathcal{L} \cup \{c_1, c_2, \ldots\}.$$

Claim. Σ is a consistent set of sentences in $\mathcal{L}'$.

Let $\psi_0(x_0), \psi_1(x_1), \ldots$ be a sequence listing all $\mathcal{L}'$-formulas with one free variable. We define a sequence of consistent sets of formulas recursively as follows: $\Sigma_0 = \Sigma$
$\Sigma_{n+1} = \Sigma_n \cup \{\exists x_n \psi_n(x_n) \to \psi_n(c_{i_n})\}$, where i_n is the smallest natural number for which c_{i_n} does not appear in any of the formuals in Σ_n nor in $\psi_n(x_n)$.

Claim. Σ_n is consistent for all natural numbers n, and thus $\Sigma' = \bigcup_{n=1}^{\infty} \Sigma_n$ is consistent.

Let Γ be a maximal consistent set of sentences containing Σ'.

Claim. Γ satisfies the following:

For every formula $\varphi(x)$, we have $\forall x\varphi(x) \in \Gamma$ if and only if $\varphi(c_n) \in \Gamma$ for every $n \in \mathbb{N}$.

Suppose $\forall x\varphi(x) \in \Gamma$. Since $\forall x\varphi(x) \to \varphi(c_n)$ is an instant of the Substitution Axiom, by Modus Ponens we conclude that $\Gamma \vdash \varphi(c_n)$. (Note that c_n is a term with no variables, so the Substitution Axiom can be applied.)

Suppose $\varphi(c_n) \in \Gamma$ for all n. Since Γ is consistent, $\neg\varphi(c_n) \notin \Gamma$. By assumption $\neg\varphi(x)$ is $\psi_m(x_m)$ for some m. Since x and x_m are the only free variables present in $\neg\varphi(x)$ and $\psi_m(x_m)$, respectively x and x_m must be the same variables. So, we will use x instead of x_m, from now on.

We know $\psi_m(c_{i_m}) \notin \Gamma$. Since $\exists x\psi_m(x) \to \psi_m(c_{i_m}) \in \Gamma$, by Theorem 7.9, $\neg\exists x\psi_m(x) \in \Gamma$. Substituting $\exists$ with $\neg\forall\neg$ we conclude that $\neg\neg\forall x\neg\psi_m(x) \in \Gamma$. By Double Negation Theorem and Theorem 7.9 we conclude that $\forall x\neg\psi_m(x) \in \Gamma$, and thus

$$\forall x\neg\neg\varphi(x) \in \Gamma \qquad (*)$$

On the other hand, by the Substitution Axiom and the Deduction Theorem we have $\forall x\neg\neg\varphi(x) \vdash \neg\neg\varphi(x)$. By the Double Negation Theorem we have $\forall x\neg\neg\varphi(x) \vdash \varphi(x)$. By the Generalization Theorem we obtain that $\forall x\neg\neg\varphi(x) \vdash \forall x\varphi(x)$. Therefore, by the Deduction Theorem we obtain $\vdash \forall x\neg\neg\varphi(x) \to \forall x\varphi(x)$. Combing this with $(*)$ we conclude that $\Gamma \vdash \forall x\varphi(x)$ and hence $\forall x\varphi(x) \in \Gamma$, as desired.

We will now define a structure $\mathcal{A}$ that models Γ as follows:

Let the universe will be a subset A of $\mathbb{N}$ defined below:

$A = \{n \in \mathbb{N} \mid \text{ if } (c_n = c_k) \in \Gamma, \text{ then } n \leq k\}$. Let $c_n^{\mathcal{A}} = m$, where m is the smallest integer with $(c_n = c_m) \in \Gamma$. Note that $(c_n = c_n)$ is an axiom and thus $\Gamma \vdash (c_n = c_n)$, which implies $(c_n = c_n) \in \Gamma$, by Theorem 7.9. Therefore, $c_n^{\mathcal{A}}$ is well-defined. Note also that $(c_n = c_m) \in \Gamma$ if and only if $(c_m = c_n) \in \Gamma$ by Theorem 7.9 and Equality Axioms.

The equality relation on A is defined as usual.

We need to define the relation R. For every $m, n \in A$ the relation $R^{\mathcal{A}}(m, n)$ holds if and only if $R(c_m, c_n) \in \Gamma$.

We will now prove by induction that $\mathcal{A} \vDash \theta$ for every $\theta \in \Gamma$.

Basis step: If θ is an atomic formula, then θ is either $t_1 = t_2$ or $R(t_1, t_2)$ for two terms t_1, t_2. Since θ is a sentence, t_1 and t_2 cannot have any free variables. Thus, they must be constants.

We see that $R(c_m, c_n) \in \Gamma$ if and only if $R^{\mathcal{A}}(c_m^{\mathcal{A}}, c_n^{\mathcal{A}})$ holds if and only if $\mathcal{A} \vDash R(c_m, c_n)$.

We also see that $(c_m = c_n) \in \Gamma$ if and only if $c_m^{\mathcal{A}} = c_n^{\mathcal{A}}$ (why?) if and only if $\mathcal{A} \vDash (c_m = c_n)$.

The inductive step is done using the fact that Γ is maximal consistent, Theorem 7.9, and the fact that $\forall x \varphi(x) \in \Gamma$ if and only if $\varphi(c_n) \in \Gamma$ for all n.

7.3 More Examples

Example 7.2. Show that every maximal consistent set of sentences is contained in a maximal consistent set of formulas, and all sentences in a maximal consistent set of formulas forms a maximal consistent set of sentences.

Solution. Suppose Σ is a maximal consistent set of sentences. Since every sentence is a formula, Σ is also a consistent set of formulas. By Theorem 7.7, Σ is contained in a maximal consistent set of formulas.

Now, suppose Γ is a maximal consistent set of formulas and let Σ be the set consisting of all sentences of Γ. We will prove that Σ is a maximal consistent set of sentences. Since Γ is consistent and $\Sigma \subseteq \Gamma$, the set Σ is also consistent. Suppose, θ is a sentence. Since Γ is maximal, either $\theta \in \Gamma$ or $\neg\theta \in \Gamma$. Since θ is a sentence, and Σ consists of all sentences in Γ, we conclude that $\theta \in \Sigma$ or $\neg\theta \in \Sigma$. Thus, Σ is a maximal consistent set of sentences. $\square$

Example 7.3. Prove or disprove: $\vdash \exists x \forall y R(x, y) \to \exists x R(x, x)$.

Solution. Using the tautology $\vdash (A \to B) \to (\neg B \to \neg A)$, we see that

$$\vdash (\forall x \neg R(x, x) \to \forall x \neg \forall y R(x, y)) \to (\neg \forall x \neg \forall y R(x, y) \to \neg \forall x \neg R(x, x))$$

Note that since $\exists x = \neg \forall x \neg$, by Modus Ponens it is enough to prove $\vdash \forall x \neg R(x, x) \to \forall x \neg \forall y R(x, y)$. By Deduction Theorem it is enough to prove $\forall x \neg R(x, x) \vdash \forall x \neg \forall y R(x, y)$. Since x does not occur free in $\forall x \neg R(x, x)$ by the Generalization Theorem it is enough to prove $\forall x \neg R(x, x) \vdash \neg \forall y R(x, y)$. By the Proof by Contradiction Theorem (Theorem 7.3 part (d)) it is enough to show $\Sigma = \{\forall x \neg R(x, x), \forall y R(x, y)\}$ is inconsistent. By Substitution Axiom we have $\vdash \forall x \neg R(x, x) \to \neg R(x, x)$, and $\vdash \forall y R(x, y) \to R(x, x)$. Therefore, by Deduction Theorem, $\Sigma \vdash \neg R(x, x)$, and $\Sigma \vdash R(x, x)$. Therefore, Σ is inconsitent, as desired. $\square$

Example 7.4. Let φ be a formula. Without using the Completeness Theorem, prove or disprove each of the following:

a. $\vdash \forall x\varphi \to \exists x\varphi$.

b. $\vdash \forall x\forall y\varphi \to \forall y\forall x\varphi$.

c. $\vdash \exists x\exists y\varphi \to \exists y\exists x\varphi$.

Solution. a. This is true. By Deduction Theorem and the fact that $\exists x = \neg\forall x\neg$, we need to prove $\forall x\varphi \vdash \neg\forall x\neg\varphi$. By the Proof by Contradiction Therem, it is enough to prove $\Sigma = \{\forall x\varphi, \forall x\neg\varphi\}$ is inconsistent. By Substitution Axiom and the Deduction Theorem we conclude that $\forall x\varphi \vdash \varphi$, and $\forall x\neg\varphi \vdash \neg\varphi$. Therefore, $\Sigma \vdash \varphi$, and $\Sigma \vdash \neg\varphi$, and hence Σ is inconsistent, as desired.

b. This is true. By Deduction Theorem it is enough to prove $\forall x\forall y\varphi \vdash \forall y\forall x\varphi$. Since x and y are not free in $\forall x\forall y\varphi$, by the Generalization Theorem it is enough to prove $\forall x\forall y\varphi \vdash \varphi$. This is true by two applications of the Substitution Axiom and Deduction Theorem.

c. This is true. Using the tautology $\vdash (A \to B) \to (\neg B \to \neg A)$, we know

$$\vdash (\forall y\neg\exists x\varphi \to \forall x\neg\exists y\varphi) \to (\neg\forall x\neg\exists y\varphi \to \neg\forall y\neg\exists x\varphi).$$

Using the fact that $\exists$ is the same as $\neg\forall\neg$, and Modus Ponens it is enough to show

$$\vdash \forall y\neg\exists x\varphi \to \forall x\neg\exists y\varphi.$$

Using the Deduction Theorem it is enough to prove $\forall y\neg\exists x\varphi \vdash \forall x\neg\exists y\varphi$. Since x is bound on the left side, by the Generalization Theorem and the fact that $\exists = \neg\forall\neg$ it is enough to prove $\forall y\neg\exists x\varphi \vdash \neg\neg\forall y\neg\varphi$. By the Double Negation Theorem (Theorem 7.3, part (b)) it is enough to prove $\forall y\neg\exists x\varphi \vdash \forall y\neg\varphi$. This is the same as $\forall y\neg\neg\forall x\neg\varphi \vdash \forall y\neg\varphi$. By the Generalization Theorem, it is enough to prove $\forall y\neg\neg\forall x\neg\varphi \vdash \neg\varphi$. By the Proof by Contradiction Theorem, it is enough to show $\Sigma = \{\forall y\neg\neg\forall x\neg\varphi, \varphi\}$ is inconsistent. By the Substitution Axiom and the Deduction Theorem we see $\forall y\neg\neg\forall x\neg\varphi \vdash \neg\neg\forall x\neg\varphi$. Using the Double Negation Theorem we conclude that $\forall y\neg\neg\forall x\neg\varphi \vdash \forall x\neg\varphi$, and thus $\Sigma \vdash \forall x\neg\varphi$. By the Substitution Axiom we have $\vdash \forall x\neg\varphi \to \neg\varphi$. Applying Modus Ponens we obtain $\Sigma \vdash \neg\varphi$. Since $\varphi \in \Sigma$ we conclude that Σ is inconsistent, which is what we were trying to prove. □

7.4 Exercises

7.4.1 Problems for grading

Exercise 7.1 (20 pts)**.** *In this exercise you will prove Theorems 7.6, and 7.7. Suppose Σ is a consistent set of formulas, and θ is a formula.*

a. Prove that $\Sigma \cup \{\theta\}$ or $\Sigma \cup \{\neg\theta\}$ is consistent.

b. Prove that there is a maximal consistent set of formulas that contains Σ.

Hint: The proof is similar to the one in sentential logic.

Exercise 7.2 (30 pts)**.** *Prove each of the following using Axioms of logic, Modus Ponens or the Deduction Theorem. Make sure in each step you clearly specify what axiom or theorem you are using. Do not use the Completeness Theorem.*

(a) $\vdash (x = y \to (y = z \to z = x))$.

(b) $\vdash (\exists x \forall z \neg\varphi(z, y)) \to (\forall z \neg\varphi(z, y))$

(c) $\vdash (\varphi(y, x) \to (\forall x \varphi(x, y) \to \varphi(y, y)))$.

Exercise 7.3 (20 pts)**.** *Suppose* $\mathcal{L}^{nl} = \{R\}$*, where* R *is a binary relation. Let* $\theta = \exists x \forall y R(x, y) \to \exists y R(y, y)$*. Prove that*

(a) $\vDash \theta$.

(b) $\vdash \theta$. *(Do not use the Completeness Theorem.)*

Exercise 7.4 (10 pts)**.** *Let* Γ *be a set of formulas. Suppose*

$$\varphi_1, \ldots, \varphi_n \qquad (*)$$

is a deduction from Γ*, and* c *is a constant for which does not appear anywhere in the formulas of* Γ*. Prove that there is a variable* z *for which replacing all occurrences of* c *in* $(*)$ *by* z *gives a deduction from* Γ.

Hint: Use the proof of the Generalization on Constants Theorem.

Exercise 7.5 (10 pts)**.** *Suppose a set of formulas* Γ *deduces a formula of the form* $\forall x_1 \cdots \forall x_n \neg\varphi$*, where* φ *is a formula. Prove that* $\Gamma \vdash \neg\forall x_1 \cdots \forall x_n \varphi$.

8 Week 8

8.1 Some Consequences of the Completeness Theorem

Theorem 8.1 (Compactness Theorem)**.** *Let* Σ *be a set of sentences, and* θ *be a formula.*

- $\Sigma \vDash \theta$ *if and only if there is a finite subset* Σ_0 *of* Σ *for which* $\Sigma_0 \vDash \theta$.
- Σ *has a model if and only if every finite subset of* Σ *has a model.*

Example 8.1. Suppose θ is a sentence that is modeled by every structure with an infinite universe. Then, there is a positive integer n for which θ is modeled by every structure whose universe contains at least n elements.

Example 8.2. Suppose $\mathcal{L}$ is a language with only one non-logical binary relation $<$. Let $\mathcal{N} = (\mathbb{N}, <)$ be the $\mathcal{L}$-structure whose universe is $\mathbb{N}$ and whose relation is the usual "less than" relation. Prove that there is an $\mathcal{L}$-structure $\mathcal{A}$ that models all sentences θ with $\mathcal{N} \vDash \theta$, but $\mathcal{A}$ contains an "infinite" element.

8.2 Arithmetic on the Natural Numbers

Let $\mathcal{L}_{\mathbb{N}}$ be a language whose non-logical symbols are $<, s, +, \cdot$, and $\overline{0}$, where $<$ is a binary relation, $+$ and $\cdot$ are binary functions, s is a unary function, and $\overline{0}$ is a constant. For simplicity, we denote $= (x, y), \neg = (x, y), +(x, y), \cdot(x, y)$, and $< (x, y)$ by $x = y, x \neq y, x + y$, $x \cdot y$, and $x < y$, respectively.

The first collection of axioms consist of nine axioms and are called **Q**.

Q1. $\forall x(s(x) \neq \overline{0})$.
Q2. $\forall x \forall y(s(x) = s(y) \rightarrow x = y)$.
Q3. $\forall x(x + \overline{0} = x)$.
Q4. $\forall x \forall y(x + s(y) = s(x + y))$.
Q5. $\forall x(x \cdot \overline{0} = \overline{0})$.
Q6. $\forall x \forall y(x \cdot s(y) = x \cdot y + x)$.
Q7. $\forall x \neg(x < \overline{0})$.
Q8. $\forall x \forall y(x < s(y) \leftrightarrow (x < y \vee x = y))$.
Q9. $\forall x \forall y(x < y \vee y < x \vee x = y)$.

The second collection of axioms are all sentences of the form below. This collection is called **IS**.

Let $\varphi(x, z_1, \ldots, z_n)$ be a formula. For simplicity let $\mathbf{z} = (z_1, \ldots, z_n)$.

$$\forall z_1 \cdots \forall z_n((\varphi(\overline{0}, \mathbf{z}) \wedge \forall x(\varphi(x, \mathbf{z}) \rightarrow \varphi(s(x), \mathbf{z}))) \rightarrow \forall x \varphi(x, \mathbf{z})).$$

These are called **induction** axioms.

Q together with **IS** is called **Peano Arithmetic**, abbreviated as **PA**. In other words, **PA** is the collection of all sentences of the form **Q** or **IS**.

Clearly $\mathcal{N} \vDash \mathbf{PA}$, where $\mathcal{N} = (\mathbb{N}, <, s, +, \cdot, 0)$. The objective of the Incompleteness Theorem is to show there is a sentence that is true in $\mathcal{N}$ but is not deducible by **PA**. In other words, there is an $\mathcal{L}_{\mathbb{N}}$-sentence θ for which $\mathcal{N} \vDash \theta$ but $\mathbf{PA} \nvDash \theta$.

Definition 8.1. In $\mathcal{L}_{\mathbb{N}}$ we recursively define $\overline{k}$ for every natural number k, by $\overline{k+1} = s(\overline{k})$. In other words, $\overline{k} = \underbrace{s \circ s \circ \cdots \circ s}_{k \text{ times}}(\overline{0})$.

Remark. Note that since we are working in First-Order Logic, by the Completeness Theorem, we can interchange $\vdash$ and $\vDash$ as we wish. So, $\mathbf{PA} \vDash \varphi(x)$ means both $\mathbf{PA} \vdash \forall x \varphi(x)$, and that $\varphi(a)$ is true for all elements a from the universe of every structure that models **PA**.

Theorem 8.2. *Let k, ℓ, and n be natural numbers. Then,* *a.* $k = \ell$ *if and only if* $\mathbf{Q} \vDash \overline{k} = \overline{\ell}$*, and* $k \neq \ell$ *if and only if* $\mathbf{Q} \vDash \overline{k} \neq \overline{\ell}$*.*

b. $k + \ell = n$ *if and only if* $\mathbf{Q} \vDash \overline{k} + \overline{\ell} = \overline{n}$*, and* $k + \ell \neq n$ *if and only if* $\mathbf{Q} \vDash \overline{k} + \overline{\ell} \neq \overline{n}$*.*

c. $k\ell = n$ *if and only if* $\mathbf{Q} \vDash \overline{k} \cdot \overline{\ell} = \overline{n}$*, and* $k\ell \neq n$ *if and only if* $\mathbf{Q} \vDash \overline{k} \cdot \overline{\ell} \neq \overline{n}$*.*

Proof. Let $\mathcal{A}$ be a structure that models $\mathbf{Q}$. All of the discussion below is done in $\mathcal{A}$.

a. Suppose $k = \ell$, then $\overline{k} = \overline{\ell}$ by definition of $\overline{n}$. If $\overline{k} \neq \overline{\ell}$, then k and ℓ cannot be equal, otherwise $\overline{k} = \overline{\ell}$. Thus $k \neq \ell$.

Now, assume $\overline{k} = \overline{\ell}$. Since $\overline{\ell} = \overline{k}$, without loss of generality we may assume $k \leq \ell$. We will now prove $k = \ell$ by induction on k.

Basis step. Suppose $k = 0$. If $\ell > 0$, then $\overline{0} = \overline{\ell} = s(\overline{\ell - 1})$. This contradicts Q1. Thus, $\ell = 0 = k$.

Inductive Step. Suppose $k > 0$ and $\overline{k} = \overline{\ell}$. By definition of $\overline{n}$ we have $s(\overline{k-1}) = s(\overline{\ell - 1})$. By Q2, we have $\overline{k-1} = \overline{\ell - 1}$. Thus, by inductive hypothesis, $k - 1 = \ell - 1$, and hence $k = \ell$, as desired.

This completes the proof of the fact that in $\mathcal{A}$ we have $\overline{k} = \overline{\ell}$ if and only if $k = \ell$. The contrapositive of this means $\mathcal{A} \vDash \overline{k} \neq \overline{\ell}$ if and only if $k \neq \ell$. Since this is true for all structures $\mathcal{A}$ that model $\mathbf{PA}$, we conclude that $\mathbf{PA} \vDash \overline{k} = \overline{\ell}$ if and only if $k = \ell$, and that $k \neq \ell$ if and only if $\mathbf{PA} \vDash \overline{k} \neq \overline{\ell}$.

b. By part (a) we know $k + \ell = n$, iff $\mathbf{PA} \vDash \overline{n} = \overline{k + \ell}$. Therefore for the first part it is enough to show $\mathbf{PA} \vDash \overline{k} + \overline{\ell} = \overline{k + \ell}$. We will prove this by induction on ℓ.

Basis step. If $\ell = 0$, then in $\mathcal{A}$ we have $\overline{k + 0} = \overline{k} = \overline{k} + \overline{0}$ by Q3.

Inductive step. Suppose $\overline{k} + \overline{\ell} = \overline{k + \ell}$. We have $\overline{k} + \overline{\ell + 1} = \overline{k} + s(\overline{\ell})$. By Q4 this is equal to $s(\overline{k} + \overline{\ell})$. By inductive hypothesis this equals $s(\overline{k + \ell}) = \overline{k + \ell + 1}$, as desired.

Since this holds for all structures $\mathcal{A}$, we conclude that $k + \ell = n$ if and only if $\mathbf{PA} \vDash \overline{k} + \overline{\ell} = \overline{k + \ell}$.

Note that by part (a) we know $n \neq k + \ell$ if and only if $\mathbf{PA} \vDash \overline{n} \neq \overline{k} + \overline{\ell}$. Since we know $\overline{k} + \overline{\ell} = \overline{k + \ell}$ in $\mathcal{A}$, it is enough to prove $n \neq k + \ell$ if and only if $\overline{n} \neq \overline{k + \ell}$. This follows from part (a).

c. Exercise! □

Theorem 8.3 (Proof by **IS** on variable x)**.** *Suppose* $\varphi(x, z_1, \ldots, z_n)$ *is an* $\mathcal{L}_{\mathbb{N}}$*-formula. Then, we have* $\mathbf{PA} \vDash \forall x \forall z_1 \cdots \forall z_n \varphi(x, z_1, \ldots, z_n)$ *if and only if both of the following hold:*

a. (Basis step) $\mathbf{PA} \vdash \varphi(\overline{0}, z_1, \ldots, z_n)$*, and* *b. (Inductive step)* $\mathbf{PA}\cup\{\varphi(x, z_1, \ldots, z_n)\} \vdash \varphi(s(x), z_1, \ldots, z_n)$ *or* $\mathbf{PA} \vdash \varphi(x, z_1, \ldots, z_n) \to \varphi(s(x), z_1, \ldots, z_n)$.

Proof. If $\mathbf{PA} \models \forall x \forall z_1 \cdots \forall z_n \varphi(x, z_1, \ldots, z_n)$, then by substitution we see $\mathbf{PA} \vdash \varphi(\overline{0}, z_1, \ldots, z_n)$, and $\mathbf{PA} \vdash \varphi(s(x), z_1, \ldots, z_n)$, as desired.

Now, suppose (a) and (b) both hold in a structure $\mathcal{A}$ that models **PA**. By **IS** we know that in $\mathcal{A}$ we have

$$(\varphi(\overline{0}, z_1, \ldots, z_n) \wedge \forall x(\varphi(x, z_1, \ldots, z_n) \to \varphi(s(x), z_1, \ldots, z_n))) \to \forall x \varphi(x, z_1, \ldots, z_n) \qquad (*)$$

By (a), we know $\mathcal{A} \models \varphi(\overline{0}, z_1, \ldots, z_n)$. By (b) we know $\mathcal{A} \models \forall x(\varphi(x, z_1, \ldots, z_n) \to \varphi(s(x), z_1, \ldots, z_n))$. Therefore, by $(*)$ we have $\mathcal{A} \models \forall x \varphi(x, z_1, \ldots, z_n)$. Since this holds for every model of **PA** we conclude that $\mathbf{PA} \models \forall x \forall z_1 \cdots \forall z_n \varphi(x, z_1, \ldots, z_n)$. □

Theorem 8.4. *The following hold:*

a. $\mathbf{PA} \models \forall x \forall y(x + y = y + x)$.

b. $\mathbf{PA} \models \forall x \forall y \forall z((x + y) + z = x + (y + z))$.

c. $\mathbf{PA} \models \forall x \forall y(x \cdot y = y \cdot x)$.

d. $\mathbf{PA} \models \forall x \forall y \forall z(x \cdot (y + z) = x \cdot y + x \cdot z)$.

e. $\mathbf{PA} \models \forall x \forall y \forall z((x \cdot y) \cdot z) = ((y \cdot x) \cdot z)$.

Proof. Let $\mathcal{A}$ be a model of **PA**. What follows is in $\mathcal{A}$.

a. First, we will show the following by **IS** on x:

i. $\mathbf{PA} \models \forall x \forall y(x + \overline{0} = \overline{0} + x)$.

ii. $\mathbf{PA} \models \forall x \forall y(s(y + x) = s(y) + x)$.

i. **Basis step.** $\overline{0} + \overline{0} = \overline{0} + \overline{0}$ is clearly true. Thus, the formula holds for $x = \overline{0}$.
Inductive step. Suppose $x + \overline{0} = \overline{0} + x$. We have $\overline{0} + s(x) = s(\overline{0} + x)$, by Q4. By inductive hypothesis, this is equal to $s(x)$, which is the same as $s(x) + \overline{0}$, by Q3. This completes the proof of i.

ii. **Basis step.** For $x = \overline{0}$, we have $s(y + \overline{0}) = s(y) = s(y) + \overline{0}$, by two applications of Q3.
Inductive step. Suppose $s(y + x) = s(y) + x$. We have $s(y + s(x)) = s(s(y + x))$, by Q4. By **IS** hypothesis this is equal to $s(s(y) + x)$, and by Q4 this equals $s(y) + s(x)$, as desired.

Now, we will prove $x + y = y + x$ by **IS** on x.
Basis step. When $x = \overline{0}$, by (i) we know $x + \overline{0} = \overline{0} + x$.
Inductive step. Suppose $x + y = y + x$. We have $s(x) + y = s(x + y)$, by (ii). By inductive hypothesis this

equals $s(y+x)$, an application of Q4 gives us $y+s(x)$, as desired.

b. We will prove this by **IS** on x.

Basis step. $(\overline{0}+y)+z = y+z = \overline{0}+(y+z)$, by two applications of (i) and Q3.

Inductive step. Suppose $(x+y)+z = x+(y+z)$. We have $(s(x)+y)+z = s(x+y)+z$, by (ii). Another application of (ii) gives us $s((x+y)+z)$. By inductive hypothesis this is equal to $s(x+(y+z))$. Applying (ii) again we obtain $s(x)+(y+z)$, as desired.

c. Exercise!

d. We will prove this by **IS** on z.

Basis step. $x\cdot(y+\overline{0}) = x\cdot y$ by Q3. $x\cdot y + x\cdot\overline{0} = x\cdot y+\overline{0} = x\cdot y$, by Q3, and Q5.

Inductive step. Suppose $x\cdot(y+z) = x\cdot y + x\cdot z$. Then, $x\cdot(y+s(z)) = x\cdot s(y+z)$ by Q4. By Q6 this is equal to $x\cdot(y+z)+x$. By inductive hypothesis this is equal to $(x\cdot y+x\cdot z)+x$. By associativity of addition and Q6 this is equal to $x\cdot y + x\cdot s(z)$. Therefore, $x\cdot(y+s(z)) = x\cdot y+x\cdot s(z)$.

Therefore, $x\cdot(y+z) = x\cdot y + x\cdot z$ in $\mathcal{A}$, which completes the proof.

e. We will prove this by **IS** on z.

Basis step. For $z=\overline{0}$, we have $(x\cdot y)\cdot\overline{0} = \overline{0}$, by Q5. Also, $x\cdot(y\cdot\overline{0}) = x\cdot\overline{0} = \overline{0}$ by two applications of Q5. This proves the basis step.

Inductive step. Suppose $(x\cdot y)\cdot z = x\cdot(y\cdot z)$. We have $(x\cdot y)\cdot s(z) = (x\cdot y)\cdot z + x\cdot y$ by Q6. By **IS** hypothesis this equals $x\cdot(y\cdot z)+x\cdot y$. By part (d) this equals $x\cdot(y\cdot z+y)$. By Q6 this is equal to $x\cdot(y\cdot s(z))$, as desired. ☐

Theorem 8.5. *The following properties of $<$ hold:*

a. $\mathbf{PA}\vDash \forall x\forall y(x<y\rightarrow \exists z(z\neq\overline{0}\wedge y=z+x))$.

b. $\mathbf{PA}\vDash \forall x\neg(x<x)$

c. $\mathbf{PA}\vDash \forall x\forall y\forall z((x<y\wedge y<z)\rightarrow x<z)$.

d. $\mathbf{PA}\vDash \forall x\forall y(x<y\rightarrow\neg(y<x))$.

e. For every positive natural number n we have $\mathbf{Q}\vDash\forall x\left(x<\overline{n}\leftrightarrow(x=\overline{0}\vee x=\overline{1}\vee\cdots\vee x=\overline{n-1})\right)$.

Proof. Let $\mathcal{A}$ be a model of $\mathbf{PA}$. What follows in parts (a)-(d) is in $\mathcal{A}$.

a. We will prove this by **IS** on y.

Basis step. By Q7, $\neg(x<\overline{0})$ and thus the sentence $x<\overline{0}\rightarrow\exists z(z\neq\overline{0}\wedge\overline{0}=z+x)$ is true by default.

Inductive step. Suppose $x<s(y)$. By Q8, either $x=y$ or $x<y$. If $x=y$, then $s(y)=s(x+\overline{0})=x+s(\overline{0})$.

We know by Q1 that $s(\overline{0}) \neq \overline{0}$. This completes the proof in this case. Suppose $x < y$. Thus, $y = x + z$ for some $z \neq \overline{0}$. Thus, $s(y) = x + s(z)$ by Q4. Again $s(z) \neq \overline{0}$ by Q1. This completes the proof.

b. By part (a) it is enough to prove that $\mathcal{A} \vDash \neg(\exists z(z \neq \overline{0} \wedge x = z + x))$. This is equivalent to $\forall z(z = \overline{0} \vee x \neq z + x)$. We will prove this by **IS** on x.
Basis step. For $x = \overline{0}$, we have $\forall z(z = \overline{0} \vee \overline{0} \neq z + \overline{0})$, which is the same as $\forall z(z = \overline{0} \vee z \neq \overline{0})$, which clearly holds.

Inductive step. Suppose $\forall z(z = \overline{0} \vee x \neq z + x)$ holds for x. We need to show $\forall z(z = \overline{0} \vee s(x) \neq s(x) + z)$. If $s(x) = s(x) + z$ for some x, z, then by Q4, $s(x) = s(x + z)$ which means $x = x + z$ by Q2. By **IS** hypothesis $z = \overline{0}$, as desired.

c. We will prove this by **IS** on z.

Basis step. For $z = \overline{0}$, note that $y < \overline{0}$ does not hold in $\mathcal{A}$ by Q7. This implies $x < y \wedge y < \overline{0}$ is false and thus the implication holds.

Inductive step. Suppose $(x < y \wedge y < z) \rightarrow x < z$ holds in $\mathcal{A}$. If $x < y \wedge y < s(z)$, then by Q8 we have $y < z$ or $y = z$. If $y < z$ by the **IS** hypothesis $x < z$. If $y = z$, since $x < y$, we have $x < z$, as desired.

d. Suppose to the contrary in $\mathcal{A}$ there are x and y for which $x < y$ and $y < x$. By part (c) we have $x < x$, which contradicts part (b).

e. We will prove that by induction on n. (Note that we are NOT using **IS**. We are using mathematical induction in $\mathbb{N}$.

Basis step. When $n = 1$, we have $x < \overline{1}$ if and only if $x < s(\overline{0})$. By Q8, this holds if and only if $x < \overline{0}$ or $x = \overline{0}$. By Q1, $x < \overline{0}$ does not hold. Therefore, $x < \overline{1}$ if and only if $x = \overline{0}$.

Inductive step. We know $x < \overline{n+1}$ if and only if $x < s(\overline{n})$. By Q8, this holds if and only if $x < \overline{n}$ or $x = \overline{n}$. By inductive hypothesis $x < \overline{n}$ if and only if $x = \overline{0} \vee \cdots \vee x = \overline{n-1}$. This completes the proof. □

8.3 More Examples

Example 8.3. Is it true that if $\mathcal{A}$ is an $\mathcal{L}_{\mathbb{N}}$-structure that models **PA**, then every element of the universe of $\mathcal{A}$ is of form $\overline{n}^{\mathcal{A}}$ for some natural number n?

Solution. The answer is no. Let $\mathcal{L} = \mathcal{L}_{\mathbb{N}} \cup \{c\}$, where c is a new constant symbol. Let Σ be **PA** along with sentence $\psi_n(c)$ that say c is larger than n distinct elements. Show this set is finitely satisfiable and conclude that there is a model of **PA** that has an "infinite" element. Using Theorem 8.5 part (e) show $c^{\mathcal{A}}$ cannot be $\overline{n}$ for any natural number n. (You should complete this solution by referring to the solution to Example 8.2.) □

Example 8.4. Prove that there is no unary function s that turns the $\mathcal{L}_{\mathbb{N}}$-structure $\mathcal{A} = ([0,\infty), <, s, +, \cdot, 0)$ into a model of **PA**, where $<, +, \cdot, 0$ are the usual relation, functions and constant of real numbers.

Solution. Suppose there is such a successor function s. By Q1, we know $s(0) \neq 0$, and thus $0 < s(0)$. Note that $s(0)/2 < s(0)$, which by Q8 we conclude $s(0)/2 \leq 0$. This contradicts the fact that $s(0) > 0$. □

Example 8.5. By an example show that $\Sigma \nvDash \theta$ and $\Sigma \vDash \neg\theta$ are not equivalent. Does either of these two imply the other?

Solution. Let θ be $\forall x P(x)$, where P is a unary relation symbol.

Let the universe of a structure $\mathcal{A}$ be $\{1,2\}$ and $P^{\mathcal{A}} = \{1\}$. Since $P(2)$ does not hold, we have $\mathcal{A} \nvDash \theta$. Thus, $\nvDash \theta$.

Now, if we let the universe of a structure $\mathcal{B}$ be $\{1,2\}$ and $P^{\mathcal{B}} = \{1,2\}$, then $\mathcal{B} \nvDash \neg\theta$. This means $\nvDash \neg\theta$.

This example shows that we might have examples that $\Sigma \nvDash \theta$ is true but $\Sigma \vDash \neg\theta$ is false.

Now, suppose $\Sigma \vDash \neg\theta$. This means every structure that models Σ also models $\neg\theta$, which means θ is false in every structure that models Σ. Therefore, θ is not a logical consequence of Σ. This means if $\Sigma \vDash \neg\theta$, then $\Sigma \nvDash \theta$. □

8.4 Exercises

8.4.1 Problems for grading

Exercise 8.1 (5 pts)**.** *We know for every positive integer n there are sentences ψ_n that determine if the universe has at least n elements. Prove that there is no sentence θ that is true if and only if the universe is infinite.*

Hint: Use one of the examples done after the Compactness Theorem.

Exercise 8.2 (30 pts)**.** *In this problem you will prove there is a structure that models all sentences that are true in $\mathcal{N} = (\mathbb{N}, <)$, and this structure has infinitely many "infinite" elements.*

Let $\mathcal{L}^{nl} = \{<\}$, where $<$ is a binary relation symbol (called "less than"). Note that "$<$" is just a relation symbol whose interpretation in natural numbers is the usual "less than" relation.

For every positive integer n let

$$\varphi_n(x) = \exists x_1 \cdots \exists x_n \left(\bigwedge_{1 \le i < j \le n} (x_i \neq x_j) \wedge \bigwedge_{1 \le i \le n} (x_i < x) \right)$$

be a formula which says "there are at least n different elements less than x." Let $\mathcal{L}' = \mathcal{L} \cup \{c_1, c_2, \ldots\}$, where c_i's are constant symbols. Consider the following set of $\mathcal{L}'$-sentences.

$$\Sigma = \{\theta \mid \theta \text{ is an } \mathcal{L}\text{-sentence and } \mathcal{N} \models \theta\} \cup \bigcup_{i=1}^{\infty} \{\psi_1(c_i), \psi_2(c_i), \psi_3(c_i), \ldots\}.$$

(a) Show Σ is finitely satisfiable (i.e. for every finite $\Sigma_0 \subseteq \Sigma$, Σ_0 is satisfiable.)

(b) Now consider $\Gamma = \Sigma \cup \{ \bigwedge_{i=1}^{n} (c_i \neq c_{n+1}) \mid n \text{ is a positive integer}\}$. Prove that Γ is finitely satisfiable.

(c) Conclude using the Compactness Theorem that Γ is satisfiable. Use this to show there is an $\mathcal{L}$-structure $\mathcal{A}$ such that $\mathcal{A} \vDash \theta$ for every $\mathcal{L}$-sentence θ that is true in $\mathcal{N}$ and that there are infinitely many "infinite" elements in the universe of $\mathcal{A}$.

Definition 8.2. A set of $\mathcal{L}$-sentences Γ is called **complete** if for every $\mathcal{L}$-sentence φ either $\Gamma \models \varphi$ or $\Gamma \models \neg\varphi$.

Exercise 8.3 (10 pts)**.** *Let $\mathcal{L}^{nl} = \{F, c\}$ where F is a binary function symbol and c is a constant symbol. Let T_G be the set consisting of all of the following sentences.*

1. *(Identity axiom)* $\forall x(F(x, c) = x \wedge F(c, x) = x)$.
2. *(Inverse axiom)* $\forall x \exists y(F(x, y) = c \wedge F(y, x) = c)$.
3. *(Associativity axiom)* $\forall x \forall y \forall z\, (F(F(x, y), z) = F(x, F(y, z)))$.

A model of T_G is called a **group**. *Show T_G is satisfiable but is not complete.*

Hint: To show T_G is satisfiable give an example of a model that satisfies all of the above properties. To show T_G is not complete find two models that are fundamentally different. In other words, find two models $\mathcal{A}$ and $\mathcal{B}$ and a sentence θ for which $\mathcal{A} \vDash \theta$ and $\mathcal{B} \vDash \neg\theta$. For instance, you could find a model whose universe has one element and a model whose universe has more than one elements.

Exercise 8.4 (10 pts)**.** *Prove that if $\mathcal{A}$ is a model of* **Q** *then for every natural numbers k, ℓ, n we have $k\ell = n$ if and only if $\mathcal{A} \vDash (\overline{k} \cdot \overline{\ell} = \overline{n})$. Deduce that $k\ell = n$ if and only if $\mathbf{Q} \vDash (\overline{k} \cdot \overline{\ell} = \overline{n})$, and that $k\ell \neq n$ if and only if $\mathbf{Q} \vDash (\overline{k} \cdot \overline{\ell} \neq \overline{n})$.*

Exercise 8.5 (10 pts)**.** *Prove the following part of Theorem 8.4:* $\mathbf{PA} \vDash \forall x \forall y(x \cdot y = y \cdot x)$.

8.4.2 Problems for practice

Exercise 8.6. *Prove each of the following:*

a. $\mathbf{Q} \vDash \forall x(x = x \cdot \overline{1})$.

b. $\mathbf{PA} \models \forall x((x \neq \overline{0}) \rightarrow \exists y(x = s(y)))$.

c. $\mathbf{PA} \models \forall x \forall y \forall z((x + z = y + z) \rightarrow (x = y))$.

d. $\mathbf{PA} \models \forall x \forall y(x + y = \overline{0} \rightarrow (x = \overline{0} \wedge y = \overline{0}))$

e. $\mathbf{PA} \models \forall x \forall y(\exists z((z \neq \overline{0}) \wedge (y = z + x)) \rightarrow x < y)$.

f. $\mathbf{PA} \models \forall x \forall y(((x \neq \overline{0}) \wedge (\overline{1} < y)) \rightarrow (x < x \cdot y))$.

Exercise 8.7. *Let Σ be a set of sentences, and $\varphi_1, \varphi_2, \ldots$ be a sequence of sentences. Suppose for every natural number n we have*

$$\Sigma \models \varphi_{n+1} \rightarrow \varphi_n, \text{ and } \Sigma \not\models \varphi_n \rightarrow \varphi_{n+1}.$$

Prove that the set $\Sigma \cup \{\varphi_1, \varphi_2, \ldots\}$ is satisfiable.

9 Week 9

9.1 Defining Relations and Functions in $\mathcal{N}$ and PA

Definition 9.1. An n-ary relation R is **definable in $\mathcal{N}$** provided there is an $\mathcal{L}_{\mathbb{N}}$-formula $\varphi(x_1, \ldots, x_n)$ such that for every $k_1, \ldots, k_n \in \mathbb{N}$ we have $R(k_1, \ldots, k_n)$ holds if and only if $\mathcal{N} \models \varphi(\overline{k}_1, \ldots, \overline{k}_n)$.

An n-ary function F is **definable in $\mathcal{N}$** provided there is an $\mathcal{L}_{\mathbb{N}}$-formula $\varphi(x_1, \ldots, x_n, y)$ such that for all natural numbers $k_1, \ldots, k_n, \ell$ we have $F(k_1, \ldots, k_n) = \ell$ if and only if $\mathcal{N} \models \varphi(\overline{k}_1, \ldots, \overline{k}_n, \overline{\ell})$.

Example 9.1. Show that the relation "n is a perfect square" is definable in $\mathcal{N}$.

Definition 9.2. An n-ary relation R in $\mathcal{N}$ is called **definable in PA** provided there is an $\mathcal{L}_{\mathbb{N}}$-formula for which for every $k_1, \ldots, k_n \in \mathbb{N}$ the following are equivalent:

i. $R(k_1, \ldots, k_n)$ holds.

ii. $\mathcal{N} \models \varphi(\overline{k}_1, \ldots, \overline{k}_n)$.

iii. $\mathbf{PA} \models \varphi(\overline{k}_1, \ldots, \overline{k}_n)$.

If this holds we say R is definable in **PA** by φ.
Similarly, we say an n-ary function F in $\mathcal{N}$ is definable in **PA** if the relation $F(x_1, \ldots, x_n) = y$ is definable in **PA**.

Theorem 9.1. *An n-ary relation R in $\mathcal{N}$ is definable in* **PA** *by a formula $\varphi(x_1, \ldots, x_n)$ if and only if for every $k_1, \ldots, k_n \in \mathbb{N}$ the following hold:*

a. If $R(k_1, \ldots, k_n)$ holds, then $\mathbf{PA} \models \varphi(\overline{k}_1, \ldots, \overline{k}_n)$, and

b. If $\mathcal{N} \models \varphi(\overline{k}_1, \ldots, \overline{k}_n)$, then $R(k_1, \ldots, k_n)$ holds.

Example 9.2. Prove that addition and multiplication functions, and the "less than" relation are all definable in **PA**.

Example 9.3. Show that there is an $\mathcal{L}_{\mathbb{N}}$-formula $\delta(x, y)$ that defines the divisibility relation in **PA**.

9.2 Recursive (or Computable) Functions

Informally, we can say an n-ary function in $\mathcal{N}$ is computable, provided there is an algorithm consisting of a finite list of "instructions" that given inputs $k_1, \ldots, k_n$ the output $F(k_1, \ldots, k_n)$ can be evaluated by carrying out this finite set of instructions. We will formally define this later, but to get an idea of how this might be useful note that each instruction uses symbols of **PA** and thus there are a countably many possible instructions. Since we require algorithms to be a finite list of instructions, we have countably many possible algorithms and thus we have countably many computable functions. However there are uncountably many functions $F : \mathbb{N} \to \mathbb{N}$. This means there are uncountably many functions that are not computable. Therefore, most functions are not computable.

Theorem 9.2. *There are uncountably many functions* $f : \mathbb{N} \to \mathbb{N}$.

Proof. Suppose to the contrary all functions $f : \mathbb{N} \to \mathbb{N}$ can be listed as $f_1, f_2, \ldots$. Define a function $g : \mathbb{N} \to \mathbb{N}$ by $g(n) = f_n(n) + 1$. Clearly $g(n) \neq f_n(n)$ for every n, and thus $g \neq f_n$, which means g is a function that is not listed. □

We will later see that the existence of functions that are not computable allows us to prove the Incompleteness Theorem.

To every relation we can assign a function that allows us to define "decidability" (i.e. computability for relations) as well.

Definition 9.3. Let R be an n-ary relation in $\mathcal{N}$. The **characteristic function of** R is the n-ary function K_R defined by

$$K_R(k_1, \ldots, k_n) = \begin{cases} 1 & \text{if } R(k_1, \ldots, k_n) \text{ holds} \\ 0 & \text{otherwise} \end{cases}$$

To define computable functions we start with the known functions $s, +, \cdot, K_<$, and the constant function 0 over $\mathbb{N}$, and allow three different rules: composition, primitive recursions, and unbound search.

Definition 9.4. Let $i \leq n$ be positive integers. The function $\pi_{in} : \underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{n \text{ times}} \to \mathbb{N}$ defined by $\pi_{in}(a_1, \ldots, a_n) = a_i$ is called the **projection onto the i-th component**.

The projection function just ignores all but one of the variables. For simplicity we denote all projection functions by π_i without having their arity.

Definition 9.5. Given an n-ary function F and k-ary functions $G_1, \ldots, G_n$ we define the **composition** function $F \circ (G_1, \ldots, G_n)$ to be the function H defined by

$$H(a_1, \ldots, a_k) = F(G_1(a_1, \ldots, a_k), \ldots, G_n(a_1, \ldots, a_k)).$$

Example 9.4. The function $F(a, b, c) = a \cdot b + c \cdot (a + b)$ can be obtained from $+, \cdot$, and the projection functions by repeatedly applying the composition.

Definition 9.6. Let G be an n-ary and H be an $(n+2)$-ary function on $\mathbb{N}$. We say the $(n+1)$-ary function F is obtained by **Primitive Recursion** from G and H, if F is defined by the following:

- $F(0, b_1, \ldots, b_n) = G(b_1, \ldots, b_n)$, and
- $F(a+1, b_1, \ldots, b_n) = H(a, F(a, b_1, \ldots, b_n), b_1, \ldots, b_n)$.

Note that when $n = 0$ we consider G to be a constant. In other words, a nullary (i.e. 0-ary) function is just a constant.

Example 9.5. The functions $F(n) = n!$ and $F(n, m) = n^m$ are obtained using Primitive Recursion. (Here we define $0^0 = 1$.)

Definition 9.7. Let R be an $(n+1)$-ary relation on $\mathbb{N}$ such that for all $a_1, \ldots, a_n \in \mathbb{N}$ there is some $b \in \mathbb{N}$ for which $R(a_1, \ldots, a_n, b)$ holds. Then, the n-ary function $F(a_1, \ldots, a_n) = (\mu b)[R(a_1, \ldots, a_n, b)]$ is defined to be the least natural number b for which $R(a_1, \ldots, a_n, b)$ holds.

Example 9.6. Using the above definition define a function that assigns to each $n \in \mathbb{N}$ the first prime more than n.

Definition 9.8. Let G be an $(n+1)$-ary function on $\mathbb{N}$ such that for every $a_1, \ldots, a_n \in \mathbb{N}$, there is a natural number b for which $G(a_1, \ldots, a_n, b) = 0$. Then the n-ary function F defined by

$$F(a_1, \ldots, a_n) = (\mu b)[G(a_1, \ldots, a_n, b) = 0]$$

is said to be a function obtained from G by **μ-recursion** or **unbound search**.

Definition 9.9. A function F on $\mathbb{N}$ is said to be **computable** or **recursive** if it can be obtained using $s, +, \cdot, K_<$, the constant function 0, and the projection functions, (as starting functions) along with a finite number of applications of the three rules of composition, primitive recursion, and unbound search.

Note that if F is a recursive n-ary function, and $k > n$, then the k-ary function

$$G(x_1, \ldots, x_k) = F(x_1, \ldots, x_n)$$

can be written as $F(\pi_1(x_1, \ldots, x_n), \ldots, \pi_n(x_1, \ldots, x_n))$ and thus it is also recursive. This is essentially taking an n-ary function F and treating it as a k-ary function for some $k > n$ by disregarding the extra variables.

Example 9.7. Each of the following functions are recursive.

a. Every constant function.

b. The characteristic function of $\{0\}$. In other words, the function K_0 defined by $K_0(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$

c. Any polynomial on any number of variables with coefficients in $\mathbb{N}$.

Solution. a. Suppose $f(a) = n$ for every $a \in \mathbb{N}$. Then f is a composition of n copies of s and 0. In other words $f(a) = s \circ \cdots \circ s(0)$.

b. This can be done by Primitive Recursion. $K_0(0) = 1$, where 1 is the constant function. $K_0(n+1) = 0$, where 0 is the constant zero function.

c. Note that if f and g are recursive, then so are $f + g$ and $f \cdot g$ by composition. Since every polynomial is obtained by adding and multiplying functions $\pi_i(x_1, \dots, x_n) = x_i$, and constant c we only need to show π_i and c are recursive. We have already see that before. □

Definition 9.10. A relation R is said to be **recursive** if its characteristic function K_R is recursive.

Definition 9.11. For n-ary relations R and S, the relation $R \vee S$ is a relation that holds at $(a_1, \dots, a_n)$ if $R(a_1, \dots, a_n)$ or $S(a_1, \dots, a_n)$ hold. The relation $R \wedge S$ is a relation that holds at $(a_1, \dots, a_n)$ if $R(a_1, \dots, a_n)$ and $S(a_1, \dots, a_n)$ both hold.
If $F_1, \dots, F_n$ are k-ary functions, then the relation $R \circ (F_1, \dots, F_n)$ is a k-ary relation S such that for all $a_1, \dots, a_k \in \mathbb{N}$ we have $S(a_1, \dots, a_k)$ holds iff $R(F_1(a_1, \dots, a_k), \dots, F_n(a_1, \dots, a_k))$ holds.

Theorem 9.3. *Let R and S be n-ary recursive relations on $\mathbb{N}$. Then,*

a. $\neg R$ is recursive.

b. $R \wedge S$ and $R \vee S$ are recursive.

c. Suppose for very $a_1, \dots, a_{n-1} \in \mathbb{N}$, there is $b \in \mathbb{N}$ for which $R(a_1, \dots, a_{n-1}, b)$ holds. Then, the function F defines by $F(a_1, \dots, a_{n-1}) = (\mu b)[R(a_1, \dots, a_{n-1}, b)$ holds$]$ is recursive.

d. The relations $<, >$, and $=$ are all recursive.

e. If $F_1, \dots, F_n$ are k-ary recursive functions, then the relation $R \circ (F_1, \dots, F_n)$ is recursive.

Example 9.8. The following are all recursive.

a. The divisibility relation.

b. The set of all primes.

c. The function enumerating prime numbers.

Solution. a. Consider the relation $R(a, b, c)$ given by $(bc = a) \vee (a < c)$. Note that since multiplication, equality and $<$ are all recursive, by a theorem this relation is recursive. Also, note that if $bc = a$, then b divides a, and if the first number that satisfies this relation is more than a, then $bc = a$ cannot be true for any natural number c. Thus, if the smallest natural number c satisfying $R(a, b, c)$ does not exceed a, then b

divides a. Note that $a < a+1$, and thus the relation $R(a, b, a+1)$ is satisfied. So, we can define a function $F(b, a)$ by $(\mu c)[bc = a \vee a < c]$. We know F is recursive. Therefore, the relation $F(b, a) \leq a$ is also recursive. Based on what we discussed before this is equivalent to saying "b divides a", as desired.

b. Exercise!

c. We will use primitive recursion along with unbound search. Define $p(0) = 2$. Since constants are recursive, 2 is recursive. Also, define $p(n+1) = (\mu a)[(a \text{ is prime}) \wedge (p(n) < a)]$. Note that the relations "a is prime", and $b < a$ are recursive. Therefore, the relation "$(a \text{ is prime}) \wedge (b < a)$" is recursive, which means $(\mu a)[(a \text{ is prime}) \wedge (b < a)]$ is recursive. Thus, the enumeration function p is obtained from a primitive recursion and thus it is a recursive function, as desired. □

Definition 9.12. p_n denoted the value of the prime enumerating function in the previous example. In other words, $p_0 = 2, p_1 = 3$, and p_n is the $(n+1)$-th prime.

9.3 More Examples

Example 9.9. Let F be a recursive n-ary function and k be a natural number. Prove that the n-ary relation $F(a_1, \ldots, a_n) = k$ is recursive.

Solution. We know the equality is a recursive relation. We know the constant k is recursive and F is recursive. Therefore, the composition relation $F(a_2, \ldots, a_n) = k$ is recursive. □

Example 9.10. Suppose F and G are recursive n-ary functions for which $F(a_1, \ldots, a_n) \leq G(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in \mathbb{N}$. Prove that the difference function $G - F$ is also recursive.

Solution. Note that the $(n+1)$-ary function $F(a_1, \ldots, a_n) + b$ is recursive since F and $h(b) = b$ are recursive. Therefore the $(n+1)$-ary relation $G(a_1, \ldots, a_n) = F(a_1, \ldots, a_n) + b$ is recursive. Note that since $F(a_1, \ldots, a_n) \leq G(a_1, \ldots, a_n)$, there is a natural number c for which $G(a_1, \ldots, a_n) = F(a_1, \ldots, a_n) + c$. Therefore, the function

$$H(a_1, \ldots, a_n) = (\mu b)[G(a_1, \ldots, a_n) = F(a_1, \ldots, a_n) + b]$$

is recursive. By definition $H = G - F$, as desired. □

Example 9.11. Prove that every finite subset of $\mathbb{N}^n$ is recursive.

Solution. First denote by $\mathbf{a}$ an element $(a_1, \ldots, a_n)$ of $\mathbb{N}^n$. Note that the empty set can be interpreted at $\neg(\pi_1(\mathbf{a}) = \pi_1(\mathbf{a}))$, and thus it is recursive. Now, by Theorem 9.3 the union of every two recursive relations is recursive. Therefore, it is enough to show every relation with one element is recursive. We will show $R = \{(b_1, \ldots, b_n)\}$ is recursive, for every $b_1, \ldots, b_n \in \mathbb{N}$. Note that $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ if and only if

$\pi_i(a_1, \ldots, a_n) = \pi_i(b_1, \ldots, b_n)$ for every i. Thus, the relation R is the same as $R_1 \wedge \cdots \wedge R_n$, where R_i is defined by $\pi_i(\mathbf{a}) = b_i$. Note that since b_i is constant, b_i is recursive. Since $=$ and π_i are also recursive, R_i is recursive, as desired. □

Example 9.12. Prove that the predecessor function $pred : \mathbb{N} \to \mathbb{N}$ defined by $pred(n) = n - 1$ for all $n \geq 1$ and $pred(0) = 0$ is recursive.

Solution. Note that the relation $R(m, n)$ given by $(m = n) \vee (s(m) = n)$ is recursive, since $s, =$ are recursive and the disjunction of two recursive relations is recursive. Therefore, the function $f(n) = (\mu m)[R(m, n) \text{ holds}]$ is recursive. Note that for every n, $R(n, n)$ holds and thus this is a valid unbound search. Also, since $R(0, 0)$ holds, $f(0) = 0$. For every $n > 0$, we know $s(n-1) = n$ and thus $R(n-1, n)$ holds. If $k < n-1$, then $k \neq n$, and $s(k) \neq n$. Thus, $f(n) = n - 1$ for all $n > 1$. This means f is the predecessor function given above. □

9.4 Exercises

9.4.1 Problems for grading

Exercise 9.1 (10 pts)**.** *Prove that if A is a countable set, then its power set defined by $\mathcal{P}(A) = \{B \mid B \subseteq A\}$ is uncountable.*

Hint: Let $A = \{a_1, a_2, \ldots\}$. Suppose on the contrary that $S_1, S_2, \ldots$ is a list of all subsets of A. Show that the set $\{a_n \mid a_n \notin S_n\}$ cannot appear in this list of S_i's.

Exercise 9.2 (10 pts)**.** *Using the fact that the set of all $\mathcal{L}_{\mathbb{N}}$-formuals is countable, prove that for every positive integer n, there are countably many n-ary relations that are definable in* **PA***. Use that and the previous exercise to show for every n there are uncountably many n-ary relations that are not definable in* **PA***.*

Exercise 9.3 (30 pts)**.** *Prove that the relations and functions below are definable in* **PA***:*

a. The unary relation consisting of all prime numbers.

b. The binary function given by $F(m, n) = m + 2n$.

c. The binary relation R given by: "$R(m, n)$ holds if and only if $m = 0 + 1 + \cdots + n$."

Hint: For the last one first find a formula for the right hand side.

Exercise 9.4 (10 pts)**.** *Prove that the unary relation P for which $P(n)$ holds iff n is a prime is recursive.*

Hint: Use Example 9.8, part (a).

Exercise 9.5 (10 pts)**.** *Suppose $a_0, a_1, a_2, \ldots$ is a strictly increasing sequence of natural numbers for which the unary relation $\{a_0, a_1, a_2, \ldots\}$ is recursive. Prove that there is a recursive function f for which $f(n) = a_n$.*

Hint: See Example 9.8, part (c).

9.4.2 Problems for practice

Exercise 9.6. *Prove that if $f : \mathbb{N} \to \mathbb{N}$ is a bijective (i.e. one-to-one and onto) recursive function, then its inverse function f^{-1} is also recursive.*

Solution. We will use unbound search. Note that the binary relation $f(a) = b$ is recursive, as equality and f are both recursive. We also know that for every b, there is a natural number a such that $f(a) = b$, since f is onto. Thus, the function $g(b) = (\mu a)[f(a) = b]$ is recursive. However, this means $f(g(b)) = b$, and since f is a bijection, this function g is the inverse of f. □

Exercise 9.7. *Show that the function f that assigns to every $n \in \mathbb{N}$ the least natural number more than n^2 is recursive:*

- *using μ-recursion.*
- *without using any recursions.*

Exercise 9.8. *Suppose $f_1, \ldots, f_n$ are recursive unary functions. Prove that the functions* $\operatorname{lcm}(f_1, \ldots, f_n)$ *and* $\gcd(f_1, \ldots, f_n)$ *whose outputs at every natural number a are the least common multiple, and the greatest common divisor of $f_1(a), \ldots, f_n(a)$ is recursive.*

10 Week 10

Theorem 10.1. *Suppose $R(a_1, \ldots, a_n, b, c)$ is a recursive relation on $\mathbb{N}$. Then the relation*

$$\exists x \, (x \leq c \wedge R(a_1, \ldots, a_n, x, c))$$

is recursive.

Solution. Let $F(a_1, \ldots, a_n, c) = (\mu b)[R(a_1, \ldots, a_n, b, c) \vee (c < b)]$. Note that since R and $c < b$ are recursive, F is recursive. Also note that there is a natural number b for which $b \leq c$ and $R(a_1, \ldots, a_n, b, c)$ holds iff the smallest natural number b that satisfies $R(a_1, \ldots, a_n, b, c)$ does not exceed c. In other words, the given relation is equivalent to $F(a_1, \ldots, a_n, c) \leq c$, which is recursive. □

Definition 10.1. For every two integers m, n the number $\operatorname{rem}(m, n)$ is the remainder when m is divided by n, if $n \neq 0$. Otherwise, $\operatorname{rem}(m, 0) = m$.

Theorem 10.2. *The function* rem *is recursive and can be defined without using primitive recursion.*

Proof. Note that $\operatorname{rem}(m, n) = r$ iff $m = nq + r$ and $r < n$ for some natural number q, unless $n = 0$ which we set $r = m$. Since $q \leq m$ we will see if such a q exists using Theorem 10.1. Consider the following relation:

$$\exists q((q \leq m) \wedge (m = nq + r) \wedge (r \leq n)) \vee (n = 0 \wedge r = m)$$

By Theorem 10.1 this relation is recursive. Denote the above relation by $R(m, n, r)$. We note that $\text{rem}(m, n) = (\mu r)[R(m, n, r)$ holds]. Thus, $\text{rem}(m, n)$ is recursive. □

Recall that the sequence of primes given by $p_0 = 2, p_1 = 3, p_2 = 5, \ldots, p_n, \ldots$ is a recursive function of n.

Definition 10.2. Let $(k_0, \ldots, k_{n-1})$ be a sequence of natural numbers. The **sequence number** of this sequence is

$$\langle k_0, \ldots, k_{n-1} \rangle = 2^{k_0+1} \cdots p_{n-1}^{k_{n-1}+1}$$

The set of all sequence numbers is denoted by Seq.

Theorem 10.3. *If* $\langle k_0, \ldots, k_{n-1} \rangle = \langle \ell_0, \ldots, \ell_{m-1} \rangle$*, then* $m = n$*, and* $k_i = \ell_i$ *for all* i*.*

Definition 10.3. Given two sequences of natural numbers $\mathbf{k} = (k_0, \ldots, k_{n-1})$ and $\mathbf{l} = (\ell_0, \ldots, \ell_{m-1})$ the **concatenation** of **k** and **l** is the sequence $(k_0, \ldots, k_{n-1}, \ell_0, \ldots, \ell_{m-1})$.

Theorem 10.4. *a.* Seq *is recursive.*

b. There is a recursive unary function Ln*, called the* **length** *function, for which* $\text{Ln}(k) = n$ *for every sequence number* $k = \langle k_0, \ldots, k_{n-1} \rangle$*.*

c. There is a binary recursive function C *such that for every sequence number* $k = \langle k_0, \ldots, k_{n-1} \rangle$ *and every* $i < n$*,* $C(k, i) = k_i$*.*

d. There is a binary function In *such that for every sequence number* $k = \langle k_0, \ldots, k_{n-1} \rangle$ *and every* $i < n$*,* $\text{In}(k, i) = \langle k_0, \ldots, k_{i-1} \rangle$*, the sequence number of the initial segment of length* i *of the sequence* $(k_0, \ldots, k_{n-1})$*.*

e. There is a binary recursive function $\star$ *for which for every two sequence numbers* $k = \langle k_0, \ldots, k_{n-1} \rangle$*, and* $\ell = \langle \ell_0, \ldots, \ell_{m-1} \rangle$ *we have* $k \star \ell = \langle k_0, \ldots, k_{n-1}, \ell_0, \ldots, \ell_{m-1} \rangle$*, the sequence number of the concatenation.*

Proof. a. A natural number k is in Seq iff $k \neq 0, 1$, and if p_{n+1} divides a then the previous prime p_n also divides a. Also note that if p_n divides k, then $p_n \leq k$ and thus $n < k$. Therefore, a sequence number is a natural number k that does not satisfy the following:

$$\exists n((n \leq k) \wedge (p_{n+1} \text{ divides } k) \wedge (p_n \text{ does not divide } k)) \vee (k = 0) \vee (k = 1)$$

Note that by Theorem 10.1 this relation is recursive. Therefore, the complement of Seq and thus Seq is recursive.

b. The length of a sequence number k is the least natural number n for which p_n does not divide k. The only issue is that 0 is divisible by all primes, which is problematic. So we will define Ln as follows:

$$\text{Ln}(k) = (\mu n)[(p_n \text{ does not divide } k) \vee (k = 0)].$$

Note that p_n and dividing relation are both recursive. Also, negation of a recursive relation is recursive. Therefore, Ln is recursive.

c. Note that k_i is the least natural for which $p_i^{k_i+2}$ does not divide k. So, we can define k_i by

$$C(k, i) = (\mu n)[(p_i^{n+2} \text{ does not divide } k) \vee (k = 0)].$$

Note that $n+2 = s(s(n))$ is recursive, so is p_i and the dividing relation. Also, note that for every $k \neq 0$, there always is a natural number r for which p_i^r does not divide k. Since, the negation of a recursive relation is recursive. Thus, $C(k, i)$ is recursive.

d. We will prove this by Primitive Recursion as follows:

$\text{In}(k, 0) = 1$. Note that 1 as a constant function is recursive.

$\text{In}(k, i+1) = In(k, i) \cdot p_i^{C(k,i)+1}$. Note that p_i, $C(k, i)$, a^{b+1}, and multiplication are all recursive functions.

e. Exercise! ☐

Notation: We will denote $C(k, i)$ by $(k)_i$.

Definition 10.4. Let F be a unary function, the **course-of-function of** F is the function $\overline{F} : \mathbb{N} \to \mathbb{N}$ given by $\overline{F}(0) = 1$, and $\overline{F}(n) = \langle F(0), \ldots, F(n-1) \rangle$ for all $n > 0$.

Theorem 10.5. *A unary function F is recursive if and only if $\overline{F}$ is recursive.*

Theorem 10.6. *[Course-of-Value Recursion] Assume H is a unary recursive function. Then so is the function the function F defined by*

- *$F(0) = H(1)$.*
- *$F(n) = H(\langle F(0), \ldots, F(n-1) \rangle)$.*

Theorem 10.7. *Assume S is a unary recursive relation. Then the unary relation R defined by*

- *$R(0)$ holds if and only if $S(1)$ holds, and*
- *For all $n > 0$ we have $R(n)$ holds if and only if $S(\langle K_R(0), \ldots, K_R(n-1) \rangle)$ holds.*

is recursive.

10.1 Definability of Recursive Relations in PA

In this section we will prove every recursive function is definable in **PA**. First, note that the following theorem relates definability of relations and their characteristic function.

Theorem 10.8. *A relation R is definable in* **PA** *if and only if the function K_R is definable in* **PA**.

Proof. Suppose $R(a_1, \ldots, a_n)$ is definable by a formula $\varphi(x_1, \ldots, x_n)$. We will prove that K_R is definable by

$$(\varphi(x_1, \ldots, x_n) \wedge (x_{n+1} = \overline{1})) \vee (\neg\varphi(x_1, \ldots, x_n) \wedge (x_{n+1} \neq \overline{1})).$$

☐

To prove all recursive functions are definable in **PA** it is enough to show all starting functions are definable in **PA**, and the three rules of composition, , μ-recursion, and Primitive Recursion preserve definability in **PA**. We have already shown that $+, \cdot$, and $<$ (and thus $K_<$) are definable in **PA**. Also note that the projection functions are definable in **PA**. Therefore, it is left to prove the three rules above preserve definability in **PA**.

Theorem 10.9. *The following hold:*

a. All projection functions are definable in **PA**.

b. Suppose $F_1, \dots, F_n$ are k-ary functions definable in **PA**, *and G is an n-ary function that is definable in* **PA**. *Then the composition function $H = G \circ (F_1, \dots, F_n)$ is definable in* **PA**.

c. Suppose G is an $(n+1)$-ary function that is definable in **PA**. *Then the n-ary function F defined by $F(a_1, \dots, a_n) = (\mu b)[G(a_1, \dots, a_n, b) = 0]$ is definable in* **PA**.

In order to prove Primitive Recursions preserve definability in **PA** we need a recursive function that labels all terms of all finite sequences of natural numbers. This requires a tool from Number Theory called the Chinese Remainder Theorem.

Theorem 10.10 (Chinese Remainder)**.** *Suppose $m_0, \dots, m_n$ are pairwise relatively prime positive integers, and $a_0, \dots, a_n$ are integers for which $a_i < m_i$ for all i. Then, there exists a natural number b for which* $\mathrm{rem}(a, m_i) = a_i$ *for all i.*

Example 10.1. There is a natural number a for which $\mathrm{rem}(a, 5) = 0, \mathrm{rem}(a, 7) = 2$, and $\mathrm{rem}(a, 9) = 1$.

10.2 More Examples

Example 10.2. Suppose R is a recursive n-ary relation, and f and g are recursive n-ary functions. Prove that the following n-ary function is recursive:

$$h(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \in R \\ g(\mathbf{x}) & \text{if } \mathbf{x} \notin R \end{cases}$$

Solution. We will show that $h(\mathbf{x}) = f(\mathbf{x}) \cdot K_R(\mathbf{x}) + g(\mathbf{x}) \cdot K_{\neg R}(\mathbf{x})$. If $R(\mathbf{x})$ holds, then $K_R(\mathbf{x}) = 1$, and $K_{\neg R}(\mathbf{x}) = 0$, thus the equality holds. Similarly when $R(\mathbf{x})$ does not hold the equality holds. Therefore, $h = f \cdot K_R + g \cdot K_{\neg R}$. Since f, g, R, and $\neg R$ are recursive, h is recursive. □

Example 10.3. Find all integers n for which $\mathrm{Ln}(n) = 0$, where Ln is the length function given in the proof of Theorem 10.4.

Solution. If $n \neq 0$ is even, then 2 divides n and thus $(\mu k)[(p_k$ does not divide $n) \vee (n = 0)]$ produces a number more than 0. If n is odd or $n = 0$, then 2 does not divide n or $n = 0$, which means $(\mu k)[(p_k$ does not divide $n) \vee (n = 0)] = 0$. Thus, $\mathrm{Ln}(n) = 0$ if and only if n is odd or $n = 0$. □

10.3 Exercises

10.3.1 Problems for grading

Exercise 10.1 (20 pts)**.** *For each of the following natural numbers k answer these questions: Is k a sequence number? What is $Ln(k)$? What is* $\mathrm{In}(k, 3)$*? What is $C(k, 2)$? What is $k \star k$? If k is not a sequence number* *use the proof of Theorem 10.4 to find the values of these functions.*

a. $k = 11550$.

b. $k = 15288$.

Exercise 10.2 (10 pts)**.** *Prove the last part of Theorem 10.4: There is a binary recursive function* $\star$ *for which for every two sequence numbers* $k = \langle k_0, \ldots, k_{n-1} \rangle$*, and* $\ell = \langle \ell_0, \ldots, \ell_{m-1} \rangle$ *we have* $k \star \ell = \langle k_0, \ldots, k_{n-1}, \ell_0, \ldots, \ell_{m-1} \rangle$*, the sequence number of the concatenation.*

Hint: One way of proving this would be to define a function $f : \mathbb{N}^3 \to \mathbb{N}$ using Primitive Recursion in such a way that $f(k, \ell, 0) = k \cdot p_{Ln(k)}^{(\ell)_0+1}$, and $f(k, \ell, Ln(\ell) - 1)$ ends up being $k \star \ell$.

Exercise 10.3 (10 pts)**.** *Using the Course-of-Value Recursion Theorem show that the function given by* $F(0) = 0$*,* $F(1) = 1$*,* $F(n) = F(n-1) + F(n-2)$ *is recursive.*

Exercise 10.4 (10 pts)**.** *Prove the Theorem: Suppose* F_1*, and* F_2 *are unary functions definable in* **PA***, and* G *is an binary function that is definable in* **PA***. Prove that the composition function* $H = G \circ (F_1, F_2)$ *is definable in* **PA***.*

Hint: We discussed this in class. You would have to turn what we discussed into a rigorous proof.

Exercise 10.5 (10 pts)**.** *Prove the Theorem: Suppose* G *is an* $(n+1)$*-ary function that is definable in* **PA***. Then the* n*-ary function* F *defined by* $F(a_1, \ldots, a_n) = (\mu b)[G(a_1, \ldots, a_n, b) = 0]$ *is definable in* **PA***.*

Hint: We discussed this in class. You would have to turn what we discussed into a rigorous proof.

10.3.2 Problems for practice

Exercise 10.6. *Prove that for every natural number* n *the* $(n+1)$*-ary function that assigns the sequence number* $\langle a_0, \ldots, a_n \rangle$ *to every finite sequence* $a_0, \ldots, a_n$ *is recursive.*

11 Week 11

11.1 Primitive Recursions

Lemma 11.1. *Let* n *be a positive integer, and let* m *be a natural number that is divisible by all integers* $1, 2, \ldots, n$*. Then the natural numbers* $1 + (1+i)m$ *where* $i = 0, 1, \ldots, n$ *are pairwise relatively prime.*

Theorem 11.1. *There is a recursive function* $\alpha(w, x, y)$ *defined without Primitive Recursion, such that for every natural number* n *and every sequence* $a_0, \ldots, a_n$ *of natural numbers, there are natural numbers* m *and* a *for which* $\alpha(m, a, i) = a_i$ *for all* $i \leq n$*.*

Theorem 11.2. *There is a recursive function* $\beta(x, y)$ *defined without Primitive Recursion, such that for every natural number* n *and every sequence* $a_0, \ldots, a_n$ *of natural numbers, there are natural numbers* m *and* a *for which* $\beta(a, i) = a_i$ *for all* $i \leq n$*.*

Theorem 11.3. *Primitive Recursions preserve definability in* **PA**.

Theorem 11.4. *Every recursive function and relation is definable in* **PA**.

11.2 Gödel Numbering

We will assign a natural number to each symbol of $\mathcal{L}_{\mathbb{N}}$ as follows:

symbol	g(symbol)	symbol	g(symbol)	symbol	g(symbol)
$\neg$	1	$\to$	3	$\forall$	5
$=$	7	(	9	)	11
,	13	s	15	$+$	17
$\cdot$	19	$\overline{0}$	21	$<$	23

Table 1

Finally we assign $2n$ to the variable v_n. In other words, we will use $g(v_n) = 2n$.

The choice of function g is of no importance as long as it satisfies the following two conditions:

- g is one-to-one. In other words, each two distinct symbols have distinct codes, and
- The function Var defined by $\mathrm{Var}(n) = g(v_n)$ is recursive.

Remark. For the purpose of Gödel Numbering we will we will use $+(v_1, v_2)$ instead of $v_1 + v_2$; $\cdot(v_1, v_2)$ instead of $v_1 \cdot v_2$; $<(v_1, v_2)$ instead of $v_1 < v_2$; and $=(v_1, v_2)$ instead of $v_1 = v_2$.

Definition 11.1. Let $\epsilon_0 \cdots \epsilon_n$ be a sequence of symbols. The Gödel number of this sequence is given by $\langle g(\epsilon_0), \ldots, g(\epsilon_n)\rangle$, and is denoted by $\ulcorner \epsilon_0, \ldots, \epsilon_n \urcorner$.

Remark. Note that the Gödel numbers of two sequence of symbols are the same if and only if the sequences are the same. Also, Gödel number of a proper subsequence of symbols is smaller than the Gödel number of a sequence of symbols.

Example 11.1. Find the Gödel number of each formula.

a. $\forall v_2 \neg v_2 + v_3 = \overline{0}$.

b. $\overline{0}, \overline{1}$, and $\overline{2}$.

Definition 11.2. Each term of form $\overline{n}$, where n is a natural number is called a **numeral**. The function $\mathrm{Num} : \mathbb{N} \to \mathbb{N}$ is defined by $\mathrm{Num}(n) = \ulcorner \overline{n} \urcorner$.

Theorem 11.5. *The following are true about the numerals.*

a. Num *is recursive.*

b. $n < \mathrm{Num}(n)$ *for all* $n \in \mathbb{N}$.

c. The set of all Gödel numbers of numerals (i.e. the image of Num*) is recursive.*

Remark. The set of all Gödel numbers of numerals is denoted by N.

Theorem 11.6. *The set Tm of all Gödel numbers of terms of $\mathcal{L}_{\mathbb{N}}$ is recursive.*

11.3 More Examples

Example 11.2. Prove that the Gödel number of a sequence of symbols is a perfect square if and only if the sequence contains no variables.

Solution. Gödel number of the sequence $\epsilon_0 \cdots \epsilon_n$ is a perfect square if and only if the exponent of each p_i in the prime factorization of this Gödel number is even. This is equivalent to $g(\epsilon_i) + 1$ being even or $g(\epsilon_i)$ being odd, which is true if and only if ϵ_i is not a variable, as desired. □

11.4 Exercises

11.4.1 Problems for grading

Exercise 11.1 (10 pts)**.** *Prove that there is a bijective function $f = (f_1, f_2) : \mathbb{N} \to \mathbb{N}^2$ for which both f_1, and f_2 are recursive and do not use Primitive Recursion.*

Hint: One possible such function can be obtained as follows: Let c be the natural number satisfying $\frac{c(c+1)}{2} \leq n < \frac{(c+1)(c+2)}{2}$. Define $f(n) = (a, b)$, where a, b are natural numbers satisfying, $b = n - \frac{c(c+1)}{2}$, and $a = c - b$. Show c, b, and a are obtained recursively without the use of Primitive Recursions.

Exercise 11.2 (10 pts)**.** *Find the Gödel number of the following formula. (First, make sure you write the formula in the correct format.)* $\forall v_3((v_1 \neq v_2) \to (s(v_3) < v_1))$.

Exercise 11.3 (10 pts)**.** *Which of the following functions can be used to define Gödel numbers? Assume the values of g at non-variable symbols are given in Table 1.*

a. $g(v_n) = 5n$.

b. $g(v_n) = (2n + 3)!$

12 Week 12

12.1 Gödel Numbers (Continued)

Lemma 12.1. *Let A be a unary relation on $\mathbb{N}$, and R be a recursive $(n+1)$-ary relation. Suppose $k \in A$ if and only if*

$$\exists \ell_1 \cdots \exists \ell_n \left(\bigwedge_{i=1}^{n} ((\ell_i < k) \wedge (\ell_i \in A)) \wedge (R(\ell_1, \ldots, \ell_n, k) \text{ holds }) \right).$$

Then A is recursive.

Sketch of Proof. By repeatedly using Theorem 10.1 the following relation $S(a,k)$ is recursive.

$$\exists \ell_1 \cdots \exists \ell_n \left(\bigwedge_{i=1}^{n} (\ell_i < k) \wedge (a)_{\ell_i} = 1 \wedge (R(\ell_1, \ldots, \ell_n, k) \text{ holds }) \right).$$

By a modified version of Theorem 10.7 (See Exercise 12.1) the relation A defined by $k \in A$ if and only if

$$\exists \ell_1 \cdots \exists \ell_n \left(\bigwedge_{i=1}^{n} ((\ell_i < k) \wedge (\langle K_A(0), \ldots, K_A(k-1) \rangle)_{\ell_i} = 1)) \wedge (R(\ell_1, \ldots, \ell_n, k) \text{ holds }) \right)$$

is recursive. □

Theorem 12.1. *The set of Gödel numbers of all formulas is recursive.*

Proof. First we will show the set of Gödel numbers of all atomic formulas is recursive. Let At be this set. By definition, $k \in At$ if and only if k is the Gödel number of a formula of the form $R(t_1, t_2)$, where R is either $=$ or $<$. Thus, $k = \ulcorner R(\urcorner \star \ell_1 \star \ulcorner, \urcorner \star \ell_2 \star \ulcorner) \urcorner$, where $\ell_1, \ell_2 \in Tm$. Since Tm is recursive, concatenation and equality are recursive, At is a recursive relation.

Let Fm be the set of Gödel numbers of all formulas. By definition of a formula, k is the Gödel number of a formula φ if and only if one of the following occurs:

- $k \in At$. This happens if and only if φ is atomic.
- $k = \ulcorner \neg \urcorner \star \ell$ for some $\ell \in Fm$ that is less than k. This happens if and only if , $\varphi = \neg\psi$, where ψ is a formula. Let's call this relation $R_1(\ell, k)$.
- $k = \ulcorner (\urcorner \star \ell_1 \star \ulcorner \rightarrow \urcorner \star \ell_2 \star \ulcorner) \urcorner$ for some ℓ_1, ℓ_2 in Fm that are less than k. This happens if and only if φ is $\neg\psi$ for some formula ψ. Let's call this relation $R_2(\ell_1, \ell_2, k)$.
- $k = \ulcorner \forall \urcorner \star \langle \mathrm{Var}(n) \rangle \star \ell$, for some $\ell \in Fm$ that is less than k and some $n < k$. Let's call this relation $R_3(\ell, n, k)$.

Therefore, $k \in Fm$ if and only if $k \in At$ or the following holds:

$$\exists \ell_1 \exists \ell_2 \exists n \, (n < k \wedge \ell_1 < k \wedge \ell_2 < k \wedge (R_1(\ell_1, k) \vee R_2(\ell_1, \ell_2, k) \vee R_3(\ell_1, n, k)))$$

By the previous Lemma, Fm is recursive. □

Definition 12.1. Let $\varphi_0, \ldots, \varphi_n$ be a deduction from a set of formulas. The Gödel number of this deduction is given by $\langle \ulcorner \varphi_0 \urcorner, \ldots, \ulcorner \varphi_n \urcorner \rangle$.

Theorem 12.2. *The set of Gödel numbers of all logical axioms of $\mathcal{L}_{\mathbb{N}}$ is recursive.*

Sketch of Proof. The proof is done in multiple steps:
Step 1. The set Ax of all Gödel numbers of all logical axioms in Λ_0 is recursive.
Step 2. The set of Gödel numbers of all deducible $\mathcal{L}_{\mathbb{N}}$-formulas is recursive. Therefore, the set of Gödel numbers of all tautologies is recursive.

Step 3. The set of Gödel numbers of all formulas in the form of the Substitution Axiom is recursive.

Step 4. The set of Gödel numbers of all formulas of the form $\forall v_n(\varphi \to \psi) \to (\forall v_n \varphi \to \forall v_n \psi)$ is recursive,

Step 5. The set of Gödel numbers of all formulas of the form Generalization Axiom is recursive.

Step 6. The set of all formulas of the form Equality Axioms is recursive. □

Theorem 12.3. *The set of all Gödel numbers of all deductions from* **PA** *is recursive.*

12.2 More Examples

Example 12.1. Find a sequence of symbols whose Gödel number is $2^6 \cdot 3^5 \cdot 5^8 \cdot 7^{10} \cdot 11^5 \cdot 13^{14} \cdot 17^5 \cdot 19^{12}$. Is this sequence a formula?

Solution. This number is the sequence number of the sequence $5, 4, 7, 9, 4, 13, 4, 11$. Using the definition of g we conclude that this sequence corresponds to the sequence $\forall v_2 = (v_2, v_2)$. □

Example 12.2. What is the smallest Gödel number of a formula and what is its corresponding formula?

Solution. First, if a formula φ is a subsequence of another formula ψ, then the Gödel number of φ does not exceed the Gödel number of ψ. Thus, the formula with the smallest Gödel number must be an atomic formula. An atomic formula is of one of the forms $R(t_1, t_2)$, where R is one of the relations $<$ or $=$, and t_1, t_2 are two terms. We will find the terms with the smallest Gödel numbers. With the same argument the term with the smallest Gödel number is either $\overline{0}$ or v_0. Since the Gödel number of v_0 is 2 which is the smallest possible Gödel number. Thus, the formula with the smallest Gödel number is $= (v_0, v_0)$. The Gödel number of this formula is $2^8 \cdot 3^{10} \cdot 5^1 \cdot 7^{14} \cdot 11^1 \cdot 13^{12}$. □

Example 12.3. Show that the Gödel number of a deduction from **PA** can never be equal to the Gödel number of a formula.

Solution. Suppose $n = \langle k_0, \ldots, k_n \rangle$ is the Gödel number of a deduction from **PA**. Thus, each k_i is a Gödel number of a formula. This implies each k_i is even and thus the sequence of symbols whose Gödel number is n consists of variables only, and thus it is not a formula! □

Example 12.4. Show that the set of all Gödel numbers of formulas that none of the variables $v_0, v_2, v_4, \ldots$ occur is recursive.

Solution. Since $g(v_n) = 2n$, the natural number n is even if and only if $g(n)$ is a multiple of 4. Thus a formula φ does not have any occurances of v_n with n even if and only if $(\ulcorner \varphi \urcorner)_\ell$ is never a multiple of 4 for any ℓ less that the length of $\ulcorner \varphi \urcorner$. Therefore the following relation holds for a Gödel number k associated to a formula φ if and only if φ contains no variable v_n with n being even.

$$\neg \exists \ell (\ell < \mathrm{Ln}(k) \wedge (k)_\ell \text{ is a multiple of 4}) \wedge k \in Fm.$$

Since $<, \mathrm{Ln}, 4, (k)_\ell, Fm$, and divisibility are all recursive, by a problem from Exam 2 this relation is recursive (its proof is similar to that of Theorem 10.1.) $\square$

12.3 Exercises

12.3.1 Problems for grading

The following problems must be submitted on Monday 11/30/2020 before the beginning of class. The submission will be on Gradescope via Elms. **Late submission will not be accepted.**

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

Exercise 12.1 (10pts). *Let $R(a, k)$ be a recursive binary relation. Suppose $S(n)$ is a unary relation such that for every positive natural number n, $S(n)$ holds if and only if $R(\langle K_S(0), \ldots, K_S(n-1)\rangle, n)$ holds. Prove that S is recursive.*

Hint: Similar to Theorem 10.7 prove that $\overline{K_S}$ is recursive.

Exercise 12.2 (10 pts). *Show that the set of all Gödel numbers of axioms of* **PA** *(i.e.* $\mathbf{Q} \cup \mathbf{IS}$*) is recursive. Note that the variables in these axioms can be any of the v_n's.*

Exercise 12.3 (10 pts). *Prove that the set of all Gödel numbers of all formulas of the form*

$$\forall v_n(\varphi \to \psi) \to (\forall v_n \varphi \to \forall v_n \psi)$$

is recursive.

Exercise 12.4 (10 pts). *Prove that the set of Gödel numbers of all formulas without any quantifiers is recursive.*

13 Week 13

13.1 Proof of the Incompleteness Theorem

Theorem 13.1. *a. There is a recursive binary function S such that whenever $\ell = \ulcorner\varphi\urcorner$ for some formula $\varphi(v_0)$, we have $S(\ell, k) = \ulcorner\varphi(\overline{k})\urcorner$.*

b. Let Pf *be a binary relation for which* $\mathrm{Pf}(n, m)$ *holds if and only if $m = \ulcorner\psi\urcorner$ for some formula ψ and n is the Gödel number of a deduction of ψ from* **PA**. *Then* Pf *is recursive.*

Proof. a. Exercise!

b. Let Fm and De be the sets of all Gödel numbers of formulas and deductions from **PA**, respectively. $\text{Pf}(m, n)$ holds if and only if $m \in Fm$, $n \in De$, and $(n)_{pred(Ln(n))} = m$. Since $Fm, De, Ln, pred, C(n, i)$, and $=$ are all recursive, Pf is recursive. □

Theorem 13.2 (Gödel Incompleteness Theorem)**.** *There is an $\mathcal{L}_{\mathbb{N}}$-sentence σ for which $\mathcal{N} \models \sigma$, but* **PA** $\not\models \sigma$.

The following theorem which is a more general form of the Incompleteness Theorem stated above can be proved in a similar manner to the Gödel Incompleteness Theorem.

Theorem 13.3. *Suppose Σ is a set of sentences for which $\{\ulcorner\varphi\urcorner \mid \varphi \in \Sigma\}$ is recursive and that $\mathcal{N} \models \Sigma$. Then, there is a sentence σ for which $\mathcal{N} \models \sigma$ but $\Sigma \not\models \sigma$.*

13.2 Exercises

The following problems must be submitted on Monday 12/7/2020 before the beginning of class. The submission will be on Gradescope via Elms. **Late submission will not be accepted.**

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

Definition 13.1. Given a formula φ, a variable x, and a term t, we say t is **substitutable** for x in φ if no occurance of any variable y is t is bound by a $\forall y$ in φ when x is replaced by t. In other words, the definition can be formalized as follows:

- If φ is an atomic formula, then t is substitutable for x in φ.
- t is subsititutable for x in $\neg\varphi$ if and only if t is substitutable for x in φ. t is substitutable for x in $\varphi \to \psi$ if and only if t is substitutable for x in φ, and ψ.
- t is substitutable for x in $\forall y\varphi$ if and only if one of the following occurs:
  - y does not occur in t, and t is substitutable for x in φ, or
  - x is not a free variable of $\forall y\varphi$.

Given any term φ, every term t is substitutable for every variable x.

This definition formalizes the Substitution Axiom.

Exercise 13.1 (10 pts)**.** *Prove there is a recursive function $f(a, b, c)$ that satisfies the following:*
If $a = \ulcorner\varphi\urcorner$, where φ is a formula or a term, $b = \ulcorner x\urcorner$, with x a variable, and $c = \ulcorner t\urcorner$, where t is term that is substitutable for x in φ, and φ_x^t is the formula or term obtained by substiuting x by t in φ, then $\ulcorner\varphi_x^t\urcorner \leq f(a, b, c)$.

Exercise 13.2 (10 pts)**.** *Prove that there is a recursive relation $V(n, c)$ that holds if and only if $c = \ulcorner t\urcorner$, where t is a term that does not contain v_n.*

For the following exercises you may use the following form of Course-of-Value Recursion:

Theorem 13.4. *Suppose $S(a, b, c, d)$, and $T(a, b, c, d, \ell_1, \ell_2, e_1, e_2, n)$ are recursive relations, $f(a, b, c)$ is a recursive function. Then the relation $R(a, b, c, d)$ defined as follows is recursive.*

$$S(a, b, c, d) \vee \exists \ell_1 \exists \ell_2 \exists e_1 \exists e_2 \exists n(\ell_1 < a \wedge \ell_2 < a \wedge n < a \wedge e_1 < f(\ell_1, b, c) \wedge e_2 < f(\ell_2, b, c)$$
$$\wedge(\ell_1, b, c, e_1) \in R \wedge (\ell_2, b, c, e_2) \in R \wedge T(a, b, c, d, \ell_1, \ell_2, e_1, e_2, n)).$$

This theorem also holds if you reduce the number of variables, e.g if you remove $\exists n$. Feel free to use slight modifications of this theorem as needed.

Exercise 13.3 (10 pts)**.** *Prove that there is a relation $R(a, b, c, d)$ for which it holds if and only if a is the Gödel number of a term t, b is the Gödel number of a variable x, and c is the Gödel number of a term t_0, and d is the Gödel number of the term obtained by substituting x by t_0 into t.*

Hint: Define the relation as follows.

- If $a = b = \langle \mathrm{Var}(n) \rangle$, then $d = c$.
- If $b = \langle \mathrm{Var}(n) \rangle$, and $V(n, a)$ holds, then let $d = a$.
- Suppose a is a Gödel number of a term of the form $F(t_1, t_2)$, or $s(t_1)$, where F is $+$ or $\cdot$, and t_1, t_2 are terms. Then choose e_1 and e_2 for which $R(\ulcorner t_1 \urcorner, b, c, e_1)$ and $R(\ulcorner t_2 \urcorner, b, c, e_2)$ both hold. Then define d by $\ulcorner F(\urcorner \star e_1 \star \ulcorner, \urcorner \star e_2 \star \ulcorner) \urcorner$ or $\ulcorner s(\urcorner \star e_1 \star \ulcorner) \urcorner$.
- You may need to use the recursive function found in the first exercise.

Exercise 13.4 (20 pts)**.** *Prove that there is a recursive relation $R(a, b, c, d)$ that holds if and only if $a = \ulcorner \varphi \urcorner$ for a term or a formula φ, $b = \ulcorner v_n \urcorner$ for some variable v_n, and $c = \ulcorner t \urcorner$ for some term t that is substitutable for v_n in φ, and d is the Gödel number of the formula or term obtained by substituting t for v_n in φ.*

Hint: Use the previous theorem as follows:

- First, use the relation in the previous exercise to cover all the terms.
- If a is the Gödel number of an atomic formula $L(t_1, t_2)$, where L is $=$ or $<$, then we let $R(a, b, c, d)$ to hold as long as $a = \ulcorner L(\urcorner \star \ell_1 \star \ulcorner, \urcorner \star \ell_2 \star \ulcorner) \urcorner$, and $R(\ell_1, b, c, e_1)$, and $R(\ell_2, b, c, e_2)$ both hold for some e_1, e_2, and we define d accordingly.
- If a is the Gödel number of a formula of the form $\neg\varphi$, then let $R(a, b, c, d)$ to hold if and only if there is $\ell_1 < a$ and e_1 for which $R(\ell_1, b, c, e_1)$ holds for some e_1, and we define d accordingly.
- If a is the Gödel number of a formula $(\varphi \to \psi)$, then let $R(a, b, c, d)$ to hold if and only if $R(\ell_1, b, c, e_1)$, and $R(\ell_2, b, c, e_2)$ hold for appropriate ℓ_1, ℓ_2, e_1, e_2, and d.
- If a is the Gödel number of a formula of the form $\forall x \varphi$, where x is a variable that does not appear in t (you may want to use the relation $V(n, c)$ from a previous exercise), then $R(a, b, c, d)$ holds if and only if $R(\ell_1, b, c, e_1)$ holds for appropriate ℓ_1, e_1, n and d.

- If a is the Gödel number of a formula of the form $\forall x\varphi$, where x appears in t, and b is the Gödel number of x, then we set $d = c$.
- If a is the Gödel number of a formula of form $\forall x\varphi$, where x appears in t, and b is the Gödel number of a variable $v_n \neq x$, and v_n does not appear free in φ, then set $d = c$.
- To check if v_n is free in φ you would need to check if $R(\ulcorner\varphi\urcorner, \ulcorner v_n\urcorner, c, \ulcorner\varphi\urcorner)$ holds. Note that $\ulcorner\varphi\urcorner < a$, which means this can be achieved using Course-of-Value recursion.

Exercise 13.5 (10 pts)**.** *Use the previous exercise to prove that there is a recursive function $Sub(a, b, c)$ for which whenever a is the Gödel number of a formula or term φ, b is the Gödel number of a variable v_n, and c is the Gödel number of a term t, where t is substitutable for v_n in φ, then $Sub(a, b, c)$ is the Gödel number of the formula or term obtained by substituting t for v_n in φ.*

14 Week 14

14.1 Some Consequences of the Incompleteness Theorem

Theorem 14.1. *The set $\{\ulcorner\theta\urcorner \mid \theta$ is an $\mathcal{L}_{\mathbb{N}}$-sentence, and $\mathcal{N} \models \theta\}$ is not recursive.*

Proof. Suppose on the contrary that this set is recursive, and let $\Sigma = \{\theta \mid \theta$ is an $\mathcal{L}_{\mathbb{N}}$-sentence, and $\mathcal{N} \models \theta\}$. By Theorem 13.3, there is a sentence σ for which $\mathcal{N} \models \sigma$, and $\Sigma \not\models \sigma$. Since $\mathcal{N} \models \sigma$, by definition $\sigma \in \Sigma$. Therefore, $\Sigma \vdash \sigma$, and thus $\Sigma \models \sigma$, which is a contradiction. □

Theorem 14.2. *Every relation and function definable in* **PA** *is recursive.*

Proof. Suppose $F(a_1, \ldots, a_n)$ is a function definable in **PA**. By definition, there is a formula $\varphi(x_1, \ldots, x_n, y)$ for which $F(a_1, \ldots, a_n) = b$ for natural numbers $a_1, \ldots, a_n, b$ if and only if $\mathbf{PA} \models \varphi(\overline{a_1}, \ldots, \overline{a_n}, \overline{b})$. By definition of Pf, $\mathbf{PA} \models \varphi(\overline{a_1}, \ldots, \overline{a_n}, \overline{b})$ if and only if there is a natural number k for which $Pf(k, \ulcorner\varphi(\overline{a_1}, \ldots, \overline{a_n}, \overline{b})\urcorner)$ holds.

Note that Pf is recursive. We will show the function $\ulcorner\varphi(\overline{a_1}, \ldots, \overline{a_n}, \overline{b})\urcorner$ is recursive for any given formula φ. By Theorem 11.5 the function $Num(n) = \ulcorner\overline{n}\urcorner$ is recursive. We can see that the function $\ulcorner\varphi(\overline{a_1}, \ldots, \overline{a_n}, \overline{b})\urcorner$ is the concatenation of some constant Gödel numbers and the functions $Num(a_i)$'s and $Num(b)$. Thus, this function is recursive. Therefore, the relation $Pf(k, \ulcorner\varphi(\overline{a_1}, \ldots, \overline{a_n}, \overline{b})\urcorner)$ is a recursive $(n+2)$-ary relation. Taking $w = \langle k, b\rangle$, we see that $F(a_1, \ldots, a_n) = b$ if and only if $Pf((w)_0, \ulcorner\varphi(\overline{a_1}, \ldots, \overline{a_n}, \overline{(w)_1})\urcorner)$ holds for some w with $(w)_1 = b$. Therefore, we can say that $F(a_1, \ldots, a_n) = ((\mu w)[Pf((w)_0, \ulcorner\varphi(\overline{a_1}, \ldots, \overline{a_n}, \overline{(w)_1})\urcorner)_1$, which means $F(a_1, \ldots, a_n)$ is recursive.

Note that if a relation R is definable in **PA**, then by Theorem 10.8, K_R is a function that is definable in **PA**. Therefore, by the above argument K_R is recursive. Thus, by definition, R is a recursive relation. □

14.2 More Examples

Example 14.1. Prove that the n-ary function $\langle a_0, \ldots, a_{n-1} \rangle$ that assigns to every sequence of length n its sequence number is recursive,

The following example allows us to turn all n-ary functions and relations into unary functions and relations.

Example 14.2. Let F be an n-ary function, and G be a unary function given by $G(a) = F((a)_0, \ldots, (a)_{n-1})$. Prove that F is recursive if and only if G is recursive. Similarly let R be an n-ary relation and S be a unary relation defined by

$$S(a) \text{ holds iff } R((a)_0, \ldots, (a)_{n-1}) \text{ holds.}$$

Then, R is recursive iff S is recursive.

Example 14.3. Suppose S and T are n-ary relations. Define an n-ary relation R by: $R(a_1, \ldots, a_n)$ holds if and only if

$$S(a_1, \ldots, a_n) \vee \exists x_1 \cdots \exists x_n \left(\bigwedge_{i=1}^{n} (x_i < a_i) \wedge \bigwedge_{i=1}^{n} R(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n) \wedge T(x_1, \ldots, x_n) \right)$$

Prove R is recursive.

14.3 Exercises

14.3.1 Problems for grading

The following problems must be submitted on Monday 12/14/2020 before the beginning of class. The submission will be on Gradescope via Elms. **Late submission will not be accepted.**

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

Exercise 14.1 (10 pts)**.** *Using the recursive function Sub obtained in Exercise 13.5, Prove part (a) of Theorem 13.1.*

Exercise 14.2 (10 pts)**.** *Using the function Sub in Exercise 13.5, prove that there is a relation $Fr(c, n)$ that holds if and only if c is the Gödel number of a formula φ, and v_n is a free variable of φ.*

Exercise 14.3 (10 pts)**.** *Prove that the set of Gödel numbers of all formulas of the form $\forall x \varphi \to \varphi^t_x$ is recursive. Here φ is a formula, t is a term that is substitutable for variable x in φ, and φ^t_x is the formula obtained when x is substituted by t in φ.*

Hint: Use the relation $R(a, b, c, d)$ defined in Exercise 13.3. Choose all natural numbers n for which $R((n)_0, (n)_1, (n)_2, (n)_3)$ holds, and that $(n)_0$ starts with $g(\forall)$. Then use that to form the set of all Gödel numbers of formulas of the given form.

Exercise 14.4 (10 pts)**.** *Define a unary function G for which $G(a) = b$, if whenever $a = \ulcorner \varphi \urcorner$ for some formula φ, we have $b = \langle i_0, \ldots, i_{k-1} \rangle$, where $v_{i_0}, \ldots, v_{i_{k-1}}$ are all free variables of φ with $i_0 < \cdots < i_{k-1}$.*

Hint: First define $G(a,n)$ by primitive recursion. $G(a,0) = 2^{K_{Fr}(a,0)}$, and $G(a,n+1) = (G(a,n) \star \langle n+1 \rangle) \cdot K_{Fr}(a,n+1) + G(a,n) \cdot (1 - K_{Fr}(a,n+1))$. Note that here $1 \star k = k \star 1 = k$.

Exercise 14.5 (10 pts)**.** *Prove that the set of Gödel numbers of formulas of the form $\varphi \to \forall x \varphi$, where x does not occur free in the formula φ is recursive. (Note that these are all formulas that appear in the Generalization Axiom.)*

15 Week 15

Recall that if R is a recursive $(n+1)$-ary relation, and f is an n-ary recursive function, then the relation define by

$$\exists x (x < f(a_1, \ldots, a_n) \wedge S(x, a_1, \ldots, a_n))$$

is recursive. A natural question is if we can remove the condition $x < f(a_1, \ldots, a_n)$ and obtain a recursive function. The following example answers this question.

Example 15.1. Let $S(k, \ell)$ and $Pf(n,m)$ be the function and relation defined in Theorem 13.1. Prove that the relation $\exists x Pf(x, S(k,k))$ is not recursive.

Solution. Suppose on the contrary $\exists x Pf(x, S(k,k))$, and thus $\neg \exists x Pf(x, S(k,k))$ is a recursive relation. By Theorem 11.4, this relation is definable. Let $R(k)$ be the relation $\neg \exists x Pf(x, S(k,k))$, and assume R is definable by a formula $\varphi(v_0)$ in **PA**. We let $k = \ulcorner \varphi(v_0) \urcorner$. By definition $S(k,k) = \ulcorner \varphi(\overline{k}) \urcorner$. We can see that $\mathbf{PA} \models \varphi(\overline{k})$ if and only if there is no natural number n for which $Pf(n, S(k,k))$ holds. By definition of Pf this is equivalent to $\mathbf{PA} \not\models S(k,k)$, which is the same as $\mathbf{PA} \not\models \varphi(\overline{k})$. This contradiction shows that $R(k)$ and thus $\exists x Pf(x, S(k,k))$ is not recursive. □

15.1 Hilbert's Tenth Problem (optional)

Hilbert's Tenth Problem. Is there an effective procedure which, given any Diophantine equation $P(x_1, \ldots, x_n) = 0$, where P is a polynomial, we can see whether or not it has a solution in integers?

Some examples of Diophantine equations:

- Pythagorean Triples: Positive integers satisfying $x^2 + y^2 = z^2$.
- Fermat's Last theorem: If for an integer $n \geq 3$ and integers x, y, z we have $x^n + y^n = z^n$, then $xyz = 0$.
- Linear Diophantine equations: Solving equations of form $a_1 x_1 + \cdots + a_n x_n = b$, where $a_1, \ldots, a_n, b$ are constant integers.

We are only working with natural numbers. So, we can move the terms with negative coefficients to the other side and obtaine polynomials with coefficients in $\mathbb{N}$.

Definition 15.1. An n-ary relation R is said to be **recursively enumerable** (r.e. for short) relation if there is an $(n+1)$-ary recursive relation S for which $R(a_1, \ldots, a_n)$ holds if and only if $\exists x S(x, a_1, \ldots, a_n)$.

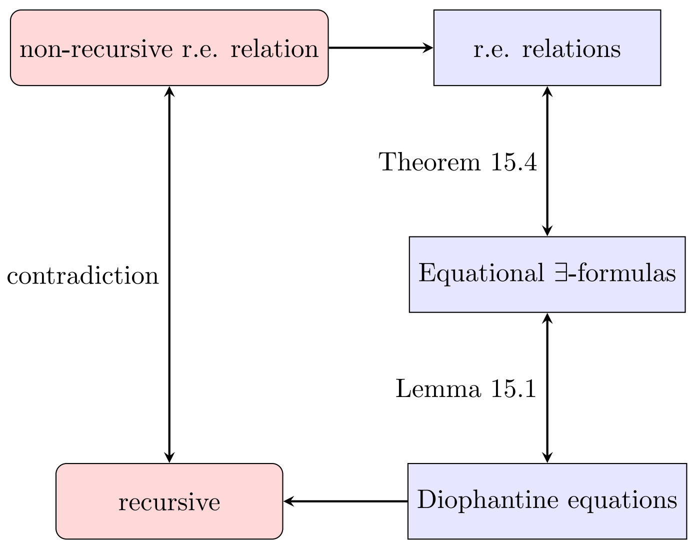
Theorem 15.1. *A relation R is recursive if and only if R and $\neg R$ are both r.e.*

Theorem 15.2. *Let $R(x_0, \ldots, x_{n-1}, a_0, \ldots, a_{m-1})$ be a recursive relation. Then the relation*

$$\exists x_0 \cdots \exists x_{n-1} R(x_0, \ldots, x_{n-1}, a_0, \ldots, a_{m-1})$$

is r.e. In other words, if S is r.e. then, so is the relation $\exists x S$.

The strategy of resolving Hilbert's Tenth Problem is to relates r.e. relations with specific formulas involving equations of form $t_1 = t_2$, where t_1 and t_2 are terms, then relate these equations with Diophantine equations over integers. We will then use the fact that there are r.e. relations that are not recursive and answer Hilbert's Tenth Problem in negative.



First, we need the following fascinating theorem from number theory. We will not prove this theorem:

Theorem 15.3 (Lagrange's Four Square Theorem)**.** *Every natural number can be written as a sum of four perfect squares.*

We will now state the main theorem that will be used in solving Hilbert's Tenth Problem in negative.

Theorem 15.4. *If R is a r. e. n-ary relation. Then, there is a formula $\varphi(x_1, \ldots, x_n)$ of the form*

$$\exists y_1 \cdots \exists y_m (t_1(x_1, \ldots, x_n, y_1, \ldots, y_m) = t_2(x_1, \ldots, x_n, y_1, \ldots, y_m)),$$

where t_1, t_2 are terms, and that φ defines R in $\mathcal{N}$.

Before proving this theorem, we will use it and provide a proof that Hilbert's Tenth Problem is unsolvable. In other words, there is no effective way that we can determine if Diophantine equations have solutions over integers.

Lemma 15.1. *Let $\varphi(x)$ be a formula of the form*

$$\exists y_1 \cdots \exists y_n (t_1(x, y_1, \ldots, y_n) = t_2(x, y_1, \ldots, y_n)).$$

Then, there is a polynomial $P(x, y_1, \ldots, y_m)$ with integer coefficients for which for every natural number k we have $\mathcal{N} \vDash \varphi(\overline{k})$ if and only if $P(k, y_1, \ldots, y_m) = 0$ has a solution for integers $y_1, \ldots, y_m$.

Proof. We will first show that every term in $\mathcal{N}$ is equal to a term in the form of a polynomial.
Atomic terms are variables and $\overline{0}$, which are polynomials. If t_1 and t_2 are polynomial terms, then $s(t_1) = t_1 + \overline{1}, t_1 + t_2$, and $t_1 \cdot t_2$ are also polynomial terms.

Therefore, φ is the same as

$$\exists y_1 \cdots \exists y_n P_1(x, y_1, \ldots, y_n) = P_2(x, y_1, \ldots, y_n),$$

for two polynomials P_1 and P_2. Setting $P = P_1 - P_2$ we obtain a polynomial $P(x, y_1, \ldots, y_n)$ with integer coefficients for which $\mathcal{N} \vDash \varphi(\overline{k})$ if and only if $P(k, y_1, \ldots, y_n) = 0$ has a solution for natural numbers $y_1, \ldots, y_n$. This is not quite what we were looking for, since this Diophantine equation may have solutions over integers even if it does not have a solution over naturals! We will fix that by using the Lagrange's Four Square Theorem. Consider the polynomial Q of $4n + 1$ variables that is obtained by replacing each y_i in P by $a_i^2 + b_i^2 + c_i^2 + d_i^2$. In other words, we consider the following polynomial:

$$Q(x, a_1, b_1, c_1, d_1, \ldots, d_n) = P(x, a_1^2 + b_1^2 + c_1^2 + d_1^2, \ldots, a_n^2 + b_n^2 + c_n^2 + d_n^2).$$

Note that given a natural number k, the Diophantine equation $Q = 0$ has integer solutions for integers a_i, b_i, c_i, d_i if and only if the equation $P(k, y_1, \ldots, y_n) = 0$ has a solution for natural numbers $y_1, \ldots, y_n$. This completes the proof of the lemma. □

Theorem 15.5. *There is no effective procedure to solve all Diophantine equations.*

Proof. Suppose there is a procedure to solve Diophatine equations. Let R be a unary r.e. relation which is not recursive. (See Example 15.1.) By Theorem 15.4, there is a formula $\varphi(x)$ of the form

$$\exists y_1 \cdots \exists y_m (t_1(x, y_1, \ldots, y_m) = t_2(x, y_1, \ldots, y_m)),$$

where t_1, t_2 are terms, and that φ defines R in $\mathcal{N}$. By the previous theorem, there is polynomial $P(x, y_1, \ldots, y_n)$ for which for every natural number k we have $\mathcal{N} \vDash \varphi(\overline{k})$ if and only if $P(k, y_1, \ldots, y_n) = 0$ has a solution over integers $y_1, \ldots, y_n$. Since there is an effective procedure that determine if $P(k, y_1, \ldots, y_n) = 0$ has a solution over integers $y_1, \ldots, y_n$, the set of all natural numbers k for which $\mathcal{N} \vDash \varphi(\overline{k})$ must be recursive. However this defined the relation R, which means R must be recursive, a contradiction! □

To make things simpler, let us make the following notation and definition:

Notation. We will abbreviate an n-tuple $(x_1, \ldots, x_n)$ by $\vec{x}$. We also abbreviate $\exists x_1 \cdots \exists x_n$ by $\exists \vec{x}$.

Definition 15.2. A formula $\varphi(\vec{x})$ is called an **equational $\exists$-formula** if it is of the following form

$$\exists \vec{y}(t_1(\vec{x}, \vec{y}) = t_2(\vec{x}, \vec{y})),$$

where t_1 and t_2 are terms. Note that the formula $t_1(\vec{x}) = t_2(\vec{x})$ is also considered an equational $\exists$-formula.

Sketch of proof of Theorem 15.4. First, note that every r.e. relation is of form $\exists x\ S$, where S is a recursive relation. Furthermore, if S is defined by a formula φ in $\mathcal{N}$, then R is defined by the formula $\exists x\ \varphi$ (why?). Therefore, it is enough to prove the theorem for recursive relations.

Next, notice that a relation $R(\vec{x})$ can be written as $\exists x_{n+1}(K_R(\vec{x}) = x_{n+1} \wedge x_{n+1} = 1)$, which means if we show $K_R(\vec{x}) = x_{n+1}$ and $x_{n+1} = 1$ can be defined by equational $\exists$-formulas and $\vee$ and $\exists$ preserve equational $\exists$-formulas, then R is defined by an equational $\exists$-formula. We will thus show $F(\vec{x}) = x_{n+1}$ can be defined by an equational $\exists$-formula. We will prove this when F is one of the "starting functions", and we will also show that the three rules of composition, μ-search, and Primitive Recursion turn relations defined by equational $\exists$-formulas into relations of the same type. We will break up the steps into the following:

Step 1. *If R and S are relations defined by equational $\exists$-formulas, then $R \vee S$ and $R \wedge S$ are also defined by equational $\exists$-formulas.* Suppose R and S are defined by formulas

$$\exists \vec{y}\ t_1(\vec{x}, \vec{y}) = t_2(\vec{x}, \vec{y}), \text{ and } \exists \vec{z}\ t_3(\vec{x}, \vec{z}) = t_4(\vec{x}, \vec{z}).$$

Note that $(R \vee S)(\vec{x})$ holds if and only if $R(\vec{x})$ or $S(\vec{x})$ holds. This is equivalent to saying $t_1(\vec{x}, \vec{y}) = t_2(\vec{x}, \vec{y})$ or $t_3(\vec{x}, \vec{z}) = t_4(\vec{x}, \vec{z})$ for some $\vec{y}$ and $\vec{z}$. This is equivalent to $(t_1 - t_2)(t_3 - t_4) = 0$ or $t_1t_3 + t_2t_4 = t_2t_3 + t_1t_4$. Setting

$$t_5(\vec{x}, \vec{y}, \vec{z}) = t_1(\vec{x}, \vec{y})t_3(\vec{x}, \vec{z}) + t_2(\vec{x}, \vec{y})t_4(\vec{x}, \vec{z}), \text{ and } t_6(\vec{x}, \vec{y}, \vec{z}) = t_2(\vec{x}, \vec{y})t_3(\vec{x}, \vec{z}) + t_1(\vec{x}, \vec{y})t_4(\vec{x}, \vec{z})$$

we conclude that $(R \vee S)(\vec{x})$ holds if and only if

$$\exists \vec{y}\ \exists \vec{z}\ t_5(\vec{x}, \vec{y}, \vec{z}) = t_6(\vec{x}, \vec{y}, \vec{z})$$

Therefore, $R \vee S$ is defined in $\mathcal{N}$ by an equational $\exists$-formula.

Similarly $t_1 = t_2$ and $t_3 = t_4$ is equivalent to $(t_1 - t_2)^2 + (t_3 - t_4)^2 = 0$, which is equivalent to $t_1^2 + t_2^2 + t_3^2 + t_4^2 = 2t_1t_2 + 2t_3t_4$. Therefore, $R \wedge S$ is defined by the formula

$$\exists \vec{y}\ \exists \vec{z}\ t_1^2 + t_2^2 + t_3^2 + t_4^2 = \overline{2}t_1t_2 + \overline{2}t_3t_4.$$

Step 2. *The relations $t_1 < t_2$ and $t_1 \neq t_2$, where t_1 and t_2 are terms, are defined by equational $\exists$-formulas.* The former relation can be written as $\exists y\ t_1 + y + \overline{1} = t_2$. The latter relation $t_1 \neq t_2$ is the same as $(t_1 < t_2) \vee (t_2 < t_1)$ and thus it is defined by an equational $\exists$-formula by an application of Step 1 and what we just proved.

Now, we will focus on relations of the type $F(x_1, \ldots, x_n) = x_{n+1}$, where F is recursive. We will show these are defined in $\mathcal{N}$ by equational $\exists$-formulas.

Step 3. *F is a starting functions.* If $F = \pi_i$ is a projection function given by $\pi_i(\vec{x}) = x_i$, then $\pi_i(\vec{x}) = x_{n+1}$ is defined by $x_i = x_{n+1}$.

If F is the constant function $\overline{0}$, then the relation $F(\vec{x}) = x_{n+1}$ is defined by $\overline{0} = x_{n+1}$.

If $F = K_<$, then $K_<(x_1, x_2) = x_3$ is equivalent to $((x_2 < x_1) \wedge x_3 = \overline{0}) \vee ((x_1 < x_2) \wedge x_3 = \overline{1})$. By Steps 1 and 2 we are done.

If $F = s$ is the successor function, then the relation $s(x) = y$ is defined by the formula $x + \overline{1} = y$.

The function $F = +$ is defined by the formula $x + y = z$.

The function $F = \cdot$ is defined by $x \cdot y = z$.

Step 4. *$\exists$ preserves equational $\exists$-formulas.* If R is defined in $\mathcal{N}$ by an equational $\exists$-formula φ, then $\exists x\ R$ is defined by $\exists x\ \varphi$ (why?).

Step 5. *Composition preserves equational $\exists$-formulas.* Suppose $F(\vec{x}) = G(H_1(\vec{x}), \ldots, H_m(\vec{x}))$ is a composition of functions $G, H_1, \ldots, H_m$ which are defined by equational $\exists$-formulas. We note that $F(\vec{x}) = x_{n+1}$ if and only if the following holds:

$$\exists \vec{y} \left(\bigwedge_{i=1}^{m} H_i(\vec{x}) = y_i \wedge G(\vec{y}) = x_{n+1} \right),$$

where $\vec{y} = (y_1, \ldots, y_m)$. By assumption each of the relations $H_i(\vec{x}) = y_i$ and $G(\vec{y}) = x_{n+1}$ are defined by equational $\exists$-formulas. By Steps 1 and 4 the above relation can also be defined by an equational $\exists$-formula.

Step 6. *μ-recursion preserves equational $\exists$-formulas.* Suppose for every $\vec{x}$ there is $b \in \mathbb{N}$ such that $G(\vec{x}, b) = 0$. Let $F(\vec{x}) = (\mu b)[G(\vec{x}, b) = 0]$. Suppose also that G is a function for which $G(\vec{x}, x_{n+1}) = x_{n+2}$ is defined by an equational $\exists$-formula. We will show $F(\vec{x}) = x_{n+1}$ is also defined by an equational $\exists$-formula.

By definition, F is defined by

$$\exists x_{n+1}\ \ (G(\vec{x}, x_{n+1}) = 0 \wedge \forall y\ (y < x_{n+1} \rightarrow G(\vec{x}, y) \neq 0))$$

Note that the relation $G(\vec{x}, x_{n+1}) = 0$ is equivalent to $G(\vec{x}, x_{n+1}) = x_{n+2} \wedge x_{n+2} = 0$. Since both $G(\vec{x}, x_{n+1}) = x_{n+2}$ and $x_{n+2} = 0$ are defined in $\mathcal{N}$ by equational $\exists$-formulas, so is $G(\vec{x}, x_{n+1}) = 0$.

Note also that $G(\vec{x}, y) \neq 0$ is equivalent to $\exists z \; G(\vec{x}, y) = z \wedge z \neq 0$. Therefore, by steps 1, 2, and 4 this relation is also defined by an equational $\exists$-formula.

So, it is enough to prove the following:

Step 7. *Universal bounded quantifiers.* If an n-ary relation $R(\vec{x})$ is defined in $\mathcal{N}$ by an equational $\exists$-formula, then so is the relation defined by $\forall y \; (y < x_{n+1} \rightarrow R(\vec{x}, y))$. We will skip the proof of this.

Step 8. *Primitive recursion preserves equational $\exists$-formulas.* Suppose F is a function defined by

- $F(0, \vec{x}) = G(\vec{x})$, and
- $F(a + 1, \vec{x}) = H(a, F(a, \vec{x}), \vec{x})$,

where both relations $G(\vec{x}) = x_{n+1}$ and $H(a, b, \vec{x}) = x_{n+1}$ are defined by equational $\exists$-formulas. We will prove F is also defined by an equational $\exists$-formula.
The relation $F(a, \vec{x}) = x_{n+1}$ is then equivalent to the following:

$$\exists c \;\; (\beta(c, 0) = G(\vec{x}) \wedge \beta(c, a) = x_{n+1} \wedge \forall i \; (i < a \rightarrow \beta(c, i + 1) = H(a, \beta(a, i), \vec{x})))$$

The relation $\beta(c, 0) = G(\vec{x})$ can be written as $\beta(c, 0) = z \wedge G(\vec{x}) = z$. Similar for the relation $\beta(c, i + 1) = H(a, \beta(a, i), \vec{x})$. Based of steps 1, 4, 5, and 7 it is enough to prove β is definable in $\mathcal{N}$ by an equational $\exists$-formula.

Step 9. *$\beta(x, y)$ is definable in $\mathcal{N}$ by an equational $\exists$-formula.* This was done previously when we discussed the β-function. □