Math 475 Summary and Homework

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Notations

- $\in$ belongs to.
- $\forall$ for all.
- $\exists$ there exists or for some.
- $|A|$ the size of set A.
- $\operatorname{Im} f$ the image of function f.
- $[n]$ the set $\{1, 2, \ldots, n\}$.
- $\mathbb{N}$ the set of non-negative integers.
- $\mathbb{Z}^+$ the set of positive integers.
- $P(n, k) = (n)_k = \dfrac{n!}{k!(n-k)!}$.
- $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.
- c_n the $n-$th Catalan number.
- $S(n, k)$ the Stirling numbers of the second kind.
- $p(n)$ the number of partitions of integer n.
- $p_k(n)$ the number of partitions of n into at most k parts.
- $p(n, k)$ the number of partitions of n into precisely k parts.
- $p_d(n)$ the number of partitions of n into distinct parts.
- $p_{d,e}(n)$ the number of partitions of n into even number of distinct parts.
- $p_{d,o}(n)$ the number of partitions of n into odd number of distinct parts.
- $\phi(n)$ Euler's totient function. The number of positive integers not exceeding n that are relatively prime to n.

- D_n the number of derangements of $[n]$.
- $[x^n]f(x)$ the coefficient of x^n in the power series $f(x)$.
- $\exp(x)$ the exponential function e^x.
- $V(G)$ and $E(G)$ the vertex set and edge set of G.
- $G - e$, removing edge e from G.
- $G + e$, adding edge e to G.
- C_n the n-cycle, or the cycle of order n.
- P_n the path of order n.
- K_n the complete graph of order n.
- $G[S]$ the subgraph of G induced by S.
- $d(u, v)$ or $d_G(u, v)$ the distance between vertices u and v.
- $k(G)$ the number of connected components of G.
- $G \cong H$, the graph G is isomorphic to the graph H.
- $G \cup H$, the union of graphs G and H.
- $G \sqcup H$, the disjoint union of graphs G and H.
- $K_{n,m}$, the complete bipartite graph with partite sets $[n]$ and $[m]$.
- $H_{r,n}$, the r-regular Harary graph of order n.
- $A \subsetneq B$, the set A is a proper subset of the set B.
- A_{ij}, the (i, j) entry of a matrix A.
- $\kappa(G)$, the vertex-connectivity of a graph G.
- $\lambda(G)$, the edge-connectivity of a graph G.

Contents

This note could contain some typos. Feel free to message me if you see any typos.

1 Week 1

1.1 Preliminaries

You are supposed to be comfortable with the methods of proof by contradiction and proof by induction. Here are a few of examples:

Example 1.1. Prove that if x is a rational number and y is irrational, then $x + y$ is irrational.

Example 1.2. Prove that for every positive integer n,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Example 1.3. Prove that the n-th term of the Fibonacci sequence F_n is less than 2^n, where the Fibonacci sequence is defined as $F_0 = 0, F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$.

Definition 1.1. A *set* is a collection of unordered elements. The number of elements of a set A is denoted by $|A|$.

Definition 1.2. The set $\{1, 2, \ldots, n\}$ is denoted by $[n]$.

Definition 1.3. The *union* of two sets A and B is the set of all elements that are in A or in B (or both). The union of A and B is denoted by $A \cup B$. The *intersection* of A and B is the set of all elements that are in both A and B. The intersection of A and B is denoted by $A \cap B$.

Remark. Unlike the daily use of the word "or", in mathematics "or" is not exclusive. In other words, the definition of $A \cup B$ could be correctly stated as follows:

The union of two sets A and B is the set of all elements that are in A or in B.

In other words, The phrase "or both" in the above definition is redundant.

Definition 1.4. Two sets are called *disjoint* whenever they have no element in common. n sets $A_1, A_2, \ldots, A_n$ are called *pairwise disjoint* if for every $i \neq j$, the two sets A_i and A_j are disjoint.

Addition Principle. Let A and B be two disjoint sets. Then $|A \cup B| = |A| + |B|$. In general if $A_1, A_2, \ldots, A_n$ are pairwise mutually disjoint sets, then $|\bigcup_{k=1}^{n} A_k| = \sum_{k=1}^{n} |A_k|$.

Theorem 1.1. *Let $n \geq 2$ be an integer and $A_1, A_2, \ldots, A_n$ be pairwise disjoint finite sets. Then* $|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i|$.

Example 1.4. Find the number of integers n with $1 < n \leq 1000$ that are either perfect fourth powers or perfect cubes.

Subtraction Principle. Let B be a subset of a set A, then $|A - B| = |A| - |B|$.

Example 1.5. Find the number of integers between 1 and 100, inclusive, that are not multiples of 3.

Multiplication Principle. Let X and Y be two finite sets with $|X| = n$ and $|Y| = m$. Then

- $|X \times Y| = mn$.
- Assume A is a subset of $X \times Y$ for which for every $x \in X$ there are precisely k values of y for which $(x, y) \in A$. Then $|A| = nk$.

Remark. The above theorem can be stated for any number of finite sets.

Example 1.6. How many three digit positive integers are there for which all adjacent digits are distinct?

Permutations. Let $k \leq n$ be two positive integers and S be a set with n element. A *k-permutation* of S is an ordered list of k elements of S. When $k = n$, we call each n-permutation a *permutation.* The only 0-permutation of n elements is the empty permutation.

Theorem 1.2. *Let $0 \leq k \leq n$ be two integers. Then, the number of k-permutations of n distinct objects is* $\dfrac{n!}{(n-k)!}$.

Notation: The number in the above theorem is denoted by $(n)_k$ or $P(n, k)$. Thus, $P(n, k) = (n)_k = \dfrac{n!}{(n-k)!}$.

Remark. The above formula shows why we define 0! to be 1.

Definition 1.5. Let S and T be two sets and d be a positive integer. A function $f : S \to T$ is said to be *d-to-one* iff for every $t \in \text{Im } f$, there are precisely d distinct elements $s \in S$ for which $f(s) = t$.

Example 1.7. Let A be the set of non-zero integers. The function $f : A \to A$ defined by $f(x) = x^2$ is 2-to-one.

Division Principle. Suppose $f : S \to T$ is a d-to-one function. Then $|\text{Im } f| = \dfrac{|S|}{d}$.

Example 1.8. Three people are sitting around a round table. The chairs are unmarked. In how many ways can this be done? Can you generalize it to n people?

Definition 1.6. A *circular permutation* of n objects is a way of arranging them on a circle, where two arrangements are considered the same if one can be obtained by a rotation of the other.

Theorem 1.3 (Circular Permutations)**.** *The number of circular permutations of n distinct objects is* $(n-1)!$.

Example 1.9. How many permutations of the letters a, b, b are there?

Theorem 1.4 (Permutations with repetition)**.** *Suppose we have n objects of k different type. Furthermore, assume there are a_j objects of type j, where $a_1 + \cdots + a_k = n$. Then, the number of permutations of these n objects is*

$$\frac{n!}{a_1!a_2!\cdots a_k!}$$

Definition 1.7. Given non-negative integers $a_1, a_2, \ldots, a_k$ with $a_1 + a_2 + \cdots + a_k = n$, the number $\dfrac{n!}{a_1!a_2!\cdots a_k!}$ is denoted by $\dbinom{n}{a_1 a_2 \cdots a_k}$ and is called a *multinomial coefficient.*

Definition 1.8. Given two positive integers $k \leq n$ and a set A of size n, we say a subset B of A is a *k-subset* if $|B| = k$.

Theorem 1.5 (Subsets or Combinations)**.** *The number of k-subsets of a set of size n is* $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

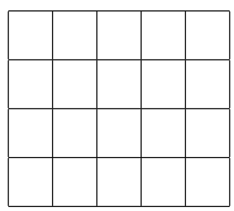
Theorem 1.6 (The Binomial Theorem)**.** *For every positive integer n,*

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + y^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.$$

1.2 One-to-One Correspondence or Bijections

One way to show two sets have the same number of elements is to define a bijection (aka One-to-One Correspondence) between them.

Example 1.10. The roads of a town are all either parallel or perpendicular. In other words all roads are from south to north or from west to east. A taxi driver is to move four block north and five block east. The driver will take the shortest path for the entire trip. At every intersection he decides to make a turn or continue straight. In how many ways can this be done?



Example 1.11. For every positive integer n, the number of divisors of n larger than $\sqrt{n}$ is the same as the number of divisors of n less than $\sqrt{n}$.

Theorem 1.7. *The number of subsets of $[n]$ is 2^n.*

Example 1.12. Prove that for any two integers $0 \leq k \leq n$, we have $\binom{n}{k} = \binom{n}{n-k}$.

1.3 Two-Way Counting

Example 1.13. Prove that $\sum\limits_{k=0}^{n} \binom{n}{k} = 2^n$

Example 1.14. Prove that for any two integers $0 < k \leq n$, we have $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Example 1.15. Prove that for every positive integer n, we have

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}.$$

Example 1.16. Prove that for any positive integer n,

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

1.4 Recursions

Sometimes direct counting is difficult, but one could do the counting by referring to the smaller cases.

Definition 1.9. A sequence a_n, with $n \geq 0$, is said to be *defined recursively* if

- a_0 is defined.
- For every positive integer n, a_n is defined in terms of $a_0, a_1, \ldots, a_{n-1}$.

Such a sequence is called a *recurrence sequence* or a *recursive sequence*. The relation that defines a_n in terms of $a_0, a_1, \ldots, a_{n-1}$ is called the *recurrence relation*.

Remark. If in the recurrence relation for a_n we have multiple terms prior to a_n, then we need to define several initial terms of the sequence. For example the Fibonacci sequence is defined as:

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}, \quad \text{for all} \quad n \geq 2$$

Example 1.17. $n!$ can be defined recursively as $0! = 1$, and $n! = n \cdot (n-1)!$.

Example 1.18. Find the number of binary sequences of length 7 without an odd number of consecutive 1's. For example the sequences 0000000 and 1100011 are counted, but 1000000 is not.

1.5 Pigeonhole Principle

Suppose we place $rn + 1$ pigeons are placed in n pigeonholes, then one hole contains at least $r + 1$ pigeons, otherwise if all holes contain at most r pigeons, then there would be at most rn pigeons placed in the holes. In mathematical terms:

Theorem 1.8 (Pigeonhole Principle)**.** *Let $A_1, A_2, \ldots, A_n$ be n sets and r be a positive integer such that*

$$|A_1 \cup A_2 \cup \cdots \cup A_n| > rn.$$

Then, there exists j for which $|A_j| \geq r + 1$.

Example 1.19. Suppose 51 distinct numbers from the set [100] are selected. Prove that there are two of them that add up to 101.

Example 1.20. Prove that if a, b, and c are three integers, then $(a-b)(a-c)(b-c)$ is even.

Example 1.21. Prove that if a, b, c and d are four integers, then the integer

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

is divisible by 3.

Example 1.22. Let q be an irrational number. Prove that there is a positive integer n and an integer m for which $|nq - m| < 0.01$.

1.6 Catalan Numbers

Definition 1.10. Let A and B be two lattice points in the xy-plane. A *northeastern lattice path* from A to B is a list of lattice points $A = A_0, A_1, A_2, \ldots, A_n = B$ for which for each i, $A_{i+1} = A_i + (1, 0)$ or $A_{i+1} = A_i + (0, 1)$.

Definition 1.11. Let n be a non-negative integer. The number of northeastern lattice paths from $(0, 0)$ to (n, n), for which no lattice point in the path is above the line $y = x$ is the n-th Catalan number and is denoted by c_n.

Example 1.23. Evaluate c_n for all $0 \leq n \leq 6$.

Theorem 1.9. *The sequence of Catalan numbers satisfies the recursion:*

$$C_0 = 1, \quad \textit{and} \quad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \textit{ for all } n \geq 0.$$

1.7 More Examples

Example 1.24 (10 pts)**.** Using induction on n, prove the Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution. For $n = 1$, the left hand side is $x + y$ and the right hand side is $\binom{1}{0}x + \binom{1}{1}y = x + y$.

Suppose for some positive integer n, we have $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. Then

$$(x+y)^{n+1} = (x+y)(x+y)^n = (x+y)\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} = \sum_{k=0}^{n}\binom{n}{k}x^{k+1}y^{n-k} + \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k+1}.$$

The first sum can be written as $\sum_{k=1}^{n+1} \binom{n}{k-1} x^k y^{n+1-k}$. Therefore, the coefficient of $x^k y^{n+1-k}$ in $(x+y)^{n+1}$ for all k, with $0 < k < n+1$, is $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$, by the Pascal's identity. Since $\binom{n}{0} = \binom{n+1}{0}$ and $\binom{n}{n} = \binom{n+1}{n+1}$, we obtain the result for $n + 1$. □

Example 1.25. Let $a_1, a_2, \ldots, a_{33}$ be 33 positive integers for which none of them has a prime divisor more than 11. Prove that there are $i \neq j$ for which the product $a_i a_j$ is a perfect square.

Solution. Each a_i can be written as $2^{x_i}3^{y_i}5^{z_i}7^{t_i}11^{u_i}$. Since each integer has two possible remainders when divided by 2, the number of possibilities of the 5-tuple $(x_i, y_i, z_i, t_i, u_i)$ modulo 2 is $2^5 = 32$. Since we have 33 integers, by pigeonhole principle, there must be two a_i and a_j whose corresponding exponents are the same modulo 2. In other words, there are $i \neq j$ for which $x_i \equiv x_j, y_i \equiv y_j, z_i \equiv z_j, t_i \equiv t_j, u_i \equiv u_j$ modulo 2. Therefore the sums $x_i + x_j, y_i + y_j, z_i + z_j, t_i + t_j$, and $u_i + u_j$ are all even, which means $a_i a_j$ is a perfect square. □

Example 1.26. Prove that for every two integers $0 < k < n$, we have $\frac{1}{n}\binom{n}{k} = \frac{1}{n-k}\binom{n-1}{k}$, once using algebra and once using two-way counting.

Solution. Method #1. $\frac{1}{n}\binom{n}{k} = \frac{n!}{n \cdot k! \cdot (n-k)!} = \frac{(n-1)!}{k!(n-k)!} = \frac{(n-1)!}{k!(n-1-k)!(n-k)} = \frac{1}{n-k}\binom{n-1}{k}.$

Method #2. Clearing the denominators we need to prove $(n-k)\binom{n}{k} = n\binom{n-1}{k}$. Consider the set

$$A = \{(a, S) \mid S \subseteq [n],\ a \in [n] - S,\ \text{and } |S| = k\}.$$

We evaluate $|A|$ in two ways.

Now, we select a first and then we choose S. There are n way to select a, and from the remaining $n-1$ elements, there are $\binom{n-1}{k}$ ways to select S. Thus, $|A| = n\binom{n-1}{k}$.

Now, we select S first and then we choose a. There are $\binom{n}{k}$ ways to select S, and there are $n-k$ ways to select a from elements of $[n] - S$. Thus, $|A| = \binom{n}{k}(n-k)$. This yields the desired equality. □

Example 1.27. Let n and m be two positive integers. How many strictly increasing sequences of length m are there for which all elements of the sequence are from $[n]$?

Solution. Every strictly increasing sequence will uniquely be determined by its m values, and any m distinct values from the set $[n]$ determine a unique strictly increasing sequence. Thus, the answer is $\boxed{\binom{n}{m}}$.

1.8 Exercises

All students are expected to do all of the exercises listed in the following two sections.

1.8.1 Problems for grading

The following problems must be submitted on Friday, February 7, 2020 at the beginning of the class. **Late submission will not be accepted.**

All proofs must be complete and solutions must be fully justified.

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

Exercise 1.1 (5 pts)**.** *Prove that if* $A_1, A_2, \ldots, A_n, A_{n+1}$ *are pairwise disjoint sets, then* $\bigcup_{i=1}^{n} A_i$ *and* A_{n+1} *are disjoint. (Hint: Use proof by contradiction.)*

Exercise 1.2 (10 pts)**.** *Prove the Theorem using induction on* n*: Let* $n \geq 2$ *be an integer and* $A_1, A_2, \ldots, A_n$ *be pairwise disjoint finite sets. Then* $|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i|$.

Exercise 1.3 (10 pts)**.** *Prove the following generalization of the Binomial Theorem, called the Multinomial Theorem:*

$$(x_1 + x_2 + \cdots + x_n)^m = \sum_{r_1+r_2+\cdots+r_n=m} \binom{m}{r_1, r_2, \cdots, r_n} x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}.$$

(Hint: Use a similar method to the proof of the Binomial Theorem that was done in class.)

Exercise 1.4 (10 pts). *Let n be a positive integer. Prove that for every positive integer $k \leq n$, the number of subsets of $[n]$ whose maximum element is k is 2^{k-1}. Use this to prove $\sum_{k=1}^{n} 2^{k-1} = 2^n - 1$.*

Exercise 1.5 (15 pts). *Let $n > 1$ be an integer.*

(a) Prove that the number of pairs of integers (a, b) with $1 \leq a < b \leq n$ is $\binom{n}{2}$.

(b) By taking cases for a (i.e. $a = 1, a = 2, \ldots, a = n$) and using the Addition Principle, prove that the number of pairs of integers (a, b) with $1 \leq a < b \leq n$ is also equal to $\sum\limits_{k=1}^{n}(k-1)$.

(c) Deduce the equality $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.

Exercise 1.6 (10 pts). *Let A be a subset of $\mathbb{Z}$ consisting of n distinct integers. Prove that for some integer k, with $1 \leq k \leq n$, there are k distinct elements $a_1, a_2, \cdots, a_k \in A$, for which $a_1 + a_2 + \cdots + a_k$ is divisible by n. (Hint: Let $A = \{b_1, b_2, \ldots, b_n\}$ and consider the partial sums $b_1, b_1 + b_2, b_1 + b_2 + b_3, \ldots, b_1 + b_2 + \cdots + b_n$. Then use the pigeonhole principle.)*

Exercise 1.7 (15 pts). *Let $m < r$ and $n < s$ be positive integers.*

(a) How many northeastern lattice paths from $(0, 0)$ to (r, s) are there that pass through (m, n)? (As usual, you must fully justify your answer.)

(b) How many northeastern lattice paths from $(0, 0)$ to (r, s) are there that do not pass through (m, n)?

(c) How many northeastern lattice paths from $(0, 0)$ to (r, r) are there that lie below or on the line $y = x$ and pass through (m, m)? (Your answer may be in terms of Catalan numbers.)

Exercise 1.8 (5 pts). *Prove that the number of subsets of $[n]$ with even number of elements is the same as the number of subsets of $[n]$ with odd number of elements. Deduce, there are 2^{n-1} subsets of $[n]$ with an odd number of elements.*

(Hint: Use the Binomial Theorem with $x = 1$ and $y = -1$ or use 1-1 correspondence.)

Exercise 1.9 (10 pts). *Prove that for every positive integer n, there is a positive integer whose digits consist of only 7's and 0's and is divisible by n. (You must use a combinatorial argument.)*

(Hint: Consider the sequence $7, 77, 777, \ldots$ and use the pigeonhole principle.)

Exercise 1.10 (10 pts). *Let n be a positive integer. Find a closed formula (i.e. a simple explicit formula without summation) for*

$$\sum_{k=1}^{n} \binom{2n+1}{k}.$$

1.8.2 Problems for Practice

Exercise 1.11. *Using two-way counting prove that*

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

(Hint: Count the number of all triples (a, b, c) for which $1 \le a < b \le n+1$ and $1 \le a < c \le n+1$ in two ways.)

Exercise 1.12. *Using two-way counting prove that*

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

The following exercises are from *Introduction to Enumerative and Analytic Combinatorics, Second Edition, by Miklos Bona.*

Pages 38-42: 7, 9, 14, 16, 17, 22, 23, 30, 34.

Pages 49-53: 3, 22, 29, 44.

1.8.3 Challenge Problems

Challenge problems are for those who want to get more out of this class. Feel free to work on the problems from the book indicated by a + sign.

Pages 39-42: 15, 25, 26, 28, 33.

Pages 49-53: 37, 45, 47.

Exercise 1.13. *Show that each positive integer n can be uniquely written as $n = \binom{a}{1} + \binom{b}{2} + \binom{c}{3}$ where $0 \le a < b < c$.*

Exercise 1.14. *Find a closed formula for $\sum_{k=0}^{n} \binom{n}{k} \min(k, n-k)$.*

Exercise 1.15. *Let p be a prime and a, b, c be integers such that p does not divide ab. Prove that there are integers x, y such that $ax^2 + by^2 - c$ is divisible by p.*

Exercise 1.16. *Let n be a positive integer. Find a closed formula for $\sum_{k=1}^{\lfloor n/3 \rfloor} \binom{n}{3k}$.*

2 Week 2

2.1 Weak Compositions and Compositions

Definition 2.1. Let n be a non-negative integers and let $a_1, \ldots, a_k$ be non-negative integers for which

$$a_1 + a_2 + \cdots + a_k = n.$$

Then the ordered k-tuple $(a_1, a_2, \ldots, a_k)$ is called a *weak composition* of n into k parts. When $a_1, a_2, \ldots, a_k$ are all positive, the k-tuple $(a_1, a_2, \ldots, a_k)$ is called a *composition* of n into k parts.

Theorem 2.1. *Let n be a non-negative integer and k be a positive integer. Then, the number of weak compositions of n into k parts is*

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

Furthermore, the number of compositions of n into k parts is $\binom{n-1}{k-1}$.

Sketch of proof. We will use one-to-one correspondence. To every weak partition of n into k parts we assign a permutation of n stars and $k-1$ bars as follows:

$$\overbrace{\star\star\cdots\star}^{a_1}|\overbrace{\star\star\cdots\star}^{a_2}|\cdots|\overbrace{\star\star\cdots\star}^{a_k}$$

For the second part, note that $(a_1, a_2, \ldots, a_k)$ is a composition of n into k parts iff $(a_1-1, a_2-1, \ldots, a_k-1)$ is a weak composition of $n-k$ into k parts. $\square$

Example 2.1. How many triples of integers (a, b, c) are there that satisfy all of the following?

- $a+b+c = 97$, and
- $a, b, c \geq 3$.

Example 2.2. Let n and m be two positive integers. How many increasing sequences of length m are there for which all elements of the sequence are from $[n]$?

2.2 Stirling Numbers of the Second Kind

Definition 2.2. Let $k \leq n$ be two positive integers. A set $\{B_1, B_2, \ldots, B_k\}$ consisting of nonempty, pairwise disjoint subsets of $[n]$ is called a partition of $[n]$ into k blocks whenever

$$\bigcup_{j=1}^{k} B_j = [n].$$

The set $[2]$ has two partitions listed below:

- $\{\{1,2\}\}$ is a partition of $[2]$ into 1 block.
- $\{\{1\},\{2\}\}$ is a partition of $[2]$ into 2 blocks.

The following shows all partitions of $[4]$ into 2 blocks.

- $\{\{1\},\{2,3,4\}\}$.
- $\{\{2\},\{1,3,4\}\}$.
- $\{\{3\},\{1,2,4\}\}$.
- $\{\{4\},\{1,2,3\}\}$.
- $\{\{1,2\},\{3,4\}\}$.
- $\{\{1,3\},\{2,4\}\}$.
- $\{\{1,4\},\{2,3\}\}$.

Example 2.3. Find the number of partitions of $[3]$, and $[4]$.

Example 2.4. How many partitions of $[n]$ into one block are there? How about two blocks?

Definition 2.3. The number of partitions of $[n]$ into k blocks is denoted by $S(n,k)$ and is called a *Stirling number of second kind.*

Remark. Note that when $k > n$, there are no partitions of $[n]$ into k blocks. Therefore $S(n,k) = 0$ whenever $k > n$. We also set $S(n,0) = 0$, whenever $n > 0$ and $S(0,0) = 1$.

Theorem 2.2. *For all positive integers $n \geq k$, we have*

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

Example 2.5. Find $S(4,1), S(4,2), S(4,3)$, and $S(4,4)$.

Stirling numbers of the second kind appear in different places. In what follows we will see one other place that they appear.

Definition 2.4. Let $k \leq n$ be positive integers. We consider all increasing sequences of elements of $[k]$ of length $n-k$. Then we evaluate the product of the elements of each sequence and add all of the resulting products. The result is denoted by $h(n,k)$. We also define $h(0,0) = 1$, $h(n,0) = 0$, when $n > 0$, and $h(n,k) = 0$, when $n < k$. We also define $h(n,n) = 1$.

Example 2.6. Evaluate $h(4,1)$, $h(4,2)$, and $h(4,3)$, and compare them with Stirling numbers of the second kind.

Solution. To evaluate $h(4,1)$, we need to find all increasing sequences of length $4-1=3$ whose elements are in $[1]$. There is one such sequence.

$$1, 1, 1$$

Thus $h(4,1) = 1 \cdot 1 \cdot 1 = 1$.

To evaluate $h(4,2)$, we need to find all increasing sequences of length $4-2=2$, whose terms are in $[2]$. They are

$$1, 1;\ 1, 2;\ 2, 2$$

Thus, $h(4,2) = 1 \cdot 1 + 1 \cdot 2 + 2 \cdot 2 = 7$.

For $h(4,3)$, we need to list all increasing sequences of length $4-3=1$, whose elements are in $[1]$:

$$1;\ 2;\ 3$$

Thus, $h(4,3) = 1 + 2 + 3 = 6$.

We see that $h(4,k) = S(4,k)$ for $k = 1, 2, 3$.

Note that by Example 2.2 the number of sequences of length $n-k$ whose terms are from $[k]$ is $\binom{n-1}{k-1}$.

Theorem 2.3. *For all integers $0 \leq k \leq n$, we have*

$$S(n,k) = h(n,k).$$

Sketch of proof. To prove this, we will show these two sequences satisfy the same recurrence relation. Then we proceed by induction on $n+k$. □

Theorem 2.4. *For all positive integers n, k satisfying $n \geq k$,*

$$S(n+1,k) = \sum_{i=0}^{n} \binom{n}{i} S(n-i,k-1).$$

Definition 2.5. The number of all partitions of $[n]$ is called the $n-$th Bell number and is denoted by $B(n)$. Note that $B(n) = \sum_{k=0}^{n} S(n,k)$. We also define $B(0) = 1$.

Theorem 2.5. *Bell numbers satisfy the following recursion:*

$$B(0) = 1,\ B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k), \quad \textit{for all } n \geq 0.$$

2.3 Integer Partitions

Definition 2.6. Given a positive integer n, we call a sequence of positive integers $(a_1, a_2, \ldots, a_k)$ a **partition of n into k parts**, whenever $a_1 \geq a_2 \geq \cdots \geq a_k$, and $a_1 + a_2 + \cdots + a_k = n$. The number of partitions of n into at most k parts is denoted by $p_k(n)$. The number of partitions of n is denoted by $p(n)$. The number of partitions of n into distinct parts is denoted by $p_d(n)$.

Example 2.7. Evaluate $p(n)$ and $p_d(n)$ for $n = 1, 2, 3, 4$.

Note that every partition of n is also a composition of n, however because of the additional restriction that the sequence must be decreasing, not every composition is a partition. For example $(4,1,2)$, and $(4,2,1)$ are both compositions of 7 but only $(4,2,1)$ is a partition of 7.

Definition 2.7. The Ferrers diagram of a partition $(a_1, a_2, \ldots, a_k)$ for n is a diagram consisting of k rows, in which the $i-$th row consists of a_i dots, for every i with $1 \leq i \leq k$.

Example 2.8. The Ferrers diagram of the partition $(4,3,2)$ of 9 is

• • • •
• • •
• •

Definition 2.8. The *conjugate* of a Ferrers diagram is the Ferrers diagram whose i-th row is the i-th column of the original Ferrers diagram. The partition associated to this conjugate Ferrers diagram is called the conjugate of the original partition.

The conjugate of the Ferrers diagram above is seen below:

• • •
• • •

• •

•

This is the Ferrers diagram for the partition $(3, 3, 2, 1)$. This idea leads to the following theorems:

Theorem 2.6. *For all positive integers $k \leq n$, the number of partitions of n that have k parts is equal to the number of partitions of n in which the largest part is equal to k.*

Theorem 2.7. *For every positive integer n, the number of partitions of n in which the first two parts are equal is equal to the number of partitions of n in which each part is at least 2.*

For both theorems above the conjugate defines a one-to-one correspondence and thus completes the proof of the theorem.
The following theorem is more difficult to prove and we omit the proof here.

Theorem 2.8 (Euler's Pentagonal Number Theorem)**.** *Let n be a positive integer and let $p_{d,e}(n)$ denote the number of partitions of n into even number of distinct parts. Similarly let $p_{d,o}(n)$ be the number of partitions of n into odd number of distinct parts. Then*

$$p_{d,e}(n) - p_{d,o}(n) = \begin{cases} (-1)^m & \textit{if } n = \dfrac{3m^2 \pm m}{2} \textit{ for some } m \in \mathbb{Z}^+ \\ 0 & \textit{otherwise} \end{cases}$$

Note: Every integer of form $\dfrac{3m^2 \pm m}{2}$ is called a pentagonal number.

Example 2.9. Manually check the previous theorem for all $n \leq 8$.

Solution. The first few pentagonal numbers are listed below

m	$\frac{3m^2 \pm m}{2}$
1	$1, 2$
2	$5, 7$
3	$12, 15$

Partitions into distinct parts are listed in the following table:

n	Pentagonal?	partitions into distinct parts	$p_{d,e}(n)$	$p_{d,o}(n)$	$p_{d,e}(n) - p_{d,o}(n)$
1	Yes, $m = 1$	1	0	1	-1
2	Yes, $m = 1$	2	0	1	-1
3	No	$3; 2+1$	1	1	0
4	No	$4; 3+1$	1	1	0
5	Yes, $m = 2$	$5; 4+1; 3+2$	2	1	1
6	No	$6; 5+1; 4+2; 3+2+1$	2	2	0
7	Yes, $m = 2$	$7; 6+1; 5+2; 4+3; 4+2+1$	3	2	1
8	No	$8; 7+1; 6+2; 5+3; 5+2+1; 4+3+1$	3	3	0

These all match the previous theorem. □

2.4 More Examples

Example 2.10. A fruit basket contains 25 pieces of fruit. Assume we have a large supply of apples, oranges and bananas. How many different kinds of fruit baskets can we create?

Solution. Suppose x, y, z represent the number of apples, oranges and bananas. We must have $x+y+z = 25$ and x, y, z are non-negative integers. Therefore we are counting the number of weak compositions of 25 into 3 parts. The answer is thus $\boxed{\binom{27}{2}}$. □

Example 2.11. Find the number of three digit positive integers whose digit sum is 10.

Solution. Suppose the three digit integer is abc. We must have $a + b + c = 10$ and that $a \geq 1, b, c \geq 0$. Subtracting one from a we obtain $(a - 1) + b + c = 9$ and that $(a - 1), b, c \geq 0$. Thus, we get a weak composition of 9 into three parts. Note that if (x, y, z) is a weak composition of 9, then the three digit number $(x + 1)yz$ has digit sum 10, unless $x + 1 = 10$. This means there is a one to one correspondence between the desired set and all weak compositions of 9, except for $(9, 0, 0)$. The number of weak compositions of 9 into three parts is $\binom{11}{2}$. The answer is $\boxed{\binom{11}{2} - 1}$. □

Example 2.12. Let n be a positive integer. Prove that the number of partitions of $2n$ into n parts is equal to $p(n)$.

Solution. Consider the Ferrers diagram of a partition of $2n$ into n parts. Removing the first column of this diagram we obtain a Ferrers diagram for n, which gives us a partition of n. Conversely to every Ferrers diagram for a partition of n we can add a first column with n dots and turn that into a partition of $2n$ into n parts. This shows there is a bijection between partitions of $2n$ into n parts and partitions of n. This completes the proof. □

Example 2.13. Let n and k be two positive integers. Find the number of sequences $x_1, x_2, \ldots, x_k$ of non-negative integers for which $\sum_{j=1}^{k} x_j \leq n$.

Solution. Note that $\sum_{j=1}^{k} x_j \leq n$ if and only if $\sum_{j=1}^{k+1} x_j = n$ for some non-negative integer x_{k+1}. Thus, there is a one-to-one correspondence between the given sequences and weak compositions of n into $k+1$ parts. The answer is $\boxed{\binom{n+k}{k}}$. □

Example 2.14. For every positive integer n evaluate $p(n, n)$, and $p(n, n - 1)$

Solution. Suppose $a_1 \geq a_2 \geq \cdots \geq a_n \geq 1$ and $a_1 + a_2 + \cdots + a_n = n$. Since all of these integers are at least 1, their sum is n precisely when the are all 1. Therefore, $p(n, n) = 1$. Similarly if $a_1 + a_2 + \cdots + a_{n-1} = n$, then $a_1 = 2$ and the rest are 1. Therefore, $p(n, n - 1) = 1$. □

2.5 Exercises

All students are expected to do all of the exercises listed in the following two sections.

2.5.1 Problems for grading

The following problems must be submitted on February 14, 2020 at the beginning of the class. **Late submission will not be accepted.**

All proofs must be complete and all solutions must be fully justified.

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

Exercise 2.1 (10 pts)**.** *Find the number of triples of integers (a, b, c) that satisfy both of the following:*

- $a \geq 1, b \geq 2, c \geq 3$, *and*
- $a + b + c = 70$.

Exercise 2.2 (10 pts)**.** *How many compositions of 75 into four odd parts are there? How about five odd parts?*

Exercise 2.3 (10 pts)**.** *Let $n \geq 3$ be an integer. In class we proved $S(n, 2) = \dfrac{2^n - 2}{2}$. Find a simple formula (without any summations) for $S(n, 3)$.*

Exercise 2.4 (10 pts)**.** *Let $k \leq r$, and n be positive integers. How many solutions does the equation*

$$x_1 + x_2 + \cdots + x_r = n$$

have over non-negative integers for which precisely k of the x_i's are equal to 0?

Exercise 2.5 (10 pts)**.** *Let a, m, and n be positive integers. Show that the number of solutions to*

$$x_1 + \cdots + x_n = m$$

over integers between $-a$ and a, inclusive, is the same as the number of solutions to

$$x_1 + \cdots + x_n = -m$$

over integers between $-a$ and a, inclusive.

(Hint: Use the one-to-one correspondence $\mathbf{x} \mapsto -\mathbf{x}$.)

Exercise 2.6 (10 pts)**.** *Prove Theorem 2.5: Bell numbers satisfy the following recursion:*

$$B(0) = 1, \ B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k), \quad \text{for all } n \geq 0.$$

(Hint: Use a similar proof to that of Theorem 2.4.)

Exercise 2.7 (10 pts)**.** *Let $k \leq n$ be two positive integers. Prove that the number of partitions of n into k parts for which each part is at least two is equal to the number of partitions of $n - k$ into k parts.*

Exercise 2.8 (10 pts)**.** *Let $k \leq n$ be two positive integers. Prove that the number of partitions of n into k distinct parts is the same as the number of partitions of $n - \frac{k(k-1)}{2}$ into k parts.*

Exercise 2.9 (10 pts)**.** *Let n be a positive integer, and recall that $(x)_j = x(x-1)\cdots(x-j+1)$ for any positive integer j and any $x \in \mathbb{R}$.*

(a) Prove that for every positive integer a, we have $a^n = \sum_{j=1}^{n} S(n,j)\ (a)_j$.

(b) Prove that for every real number x we have $x^n = \sum_{j=1}^{n} S(n,j)\ (x)_j$.

(Hint: For the first part, use two-way counting. For the second part, note that a polynomial of degree n does not have more than n roots.)

2.5.2 Problems for Practice

pages 97-98: 3, 6, 8, 9

page 112-113: 8, 22, 28

Exercise 2.10. *For every positive integer n evaluate $p(n, n-2)$, and $p(n, n-3)$.*

2.5.3 Challenge Problems

Challenge problems are for those who want to get more out of this class. Feel free to work on the problems from the book indicated by a + sign.

Exercise 2.11. *Let, n and m be two positive integers and $a < b$ be two integers. Show that the number of solutions of $x_1 + x_2 + \cdots + x_n = m$ over integers between a and b, inclusive is the same as the number of solutions of $x_1 + x_2 + \cdots + x_n = (a+b)n - m$ over integers between a and b, inclusive.*

Exercise 2.12. *Let S be a subset of real numbers. A subset A of S is said to have k-gap if for every two distinct $x, y \in A$, (i.e. x and y belong to A) we have $|x - y| \geq k$.*

(a) How many 3-element subsets of $\{1, 2, 3, \ldots, n\}$ have 1-gap?

(b) How many 3-element subsets of $\{1, 2, 3, \ldots, n\}$ have 2-gap?

(c) Let k be a positive integer. How many 3-element subsets of $\{1, 2, 3, \ldots, n\}$ have k-gap?

(d) Let k, r be two positive integers. How many r-element subsets of $\{1, 2, 3, \ldots, n\}$ have k-gap?

Exercise 2.13. *Let $A_1A_2\cdots A_{40}$ be a regular 40-sided polygon. How many triangles can be formed whose vertices are the vertices of this 40-gon and whose angles are more than 10°?*

3 Week 3

3.1 Principle of Inclusion-Exclusion (PIE)

Example 3.1. Consider the set $S = [500]$.

(a) How many elements of S are divisible by 2 or 3?

(b) How many elements of S are divisible by 2, 3 or 5?

Example 3.2. Let $n \geq 3$ be an integer. How many surjective functions $f : [n] \to [3]$ are there?

Theorem 3.1. *Let $A_1, A_2, \ldots, A_n$ be finite sets. Then*

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}|.$$

3.1.1 Euler's Totient Function

Definition 3.1. Euler's totient function is the function $\phi : \mathbb{Z}^+ \to \mathbb{Z}^+$ for which $\phi(n)$ is the number of positive integers not exceeding n that are relatively prime to n.

Example 3.3. Let p and q be two distinct primes, and n be a positive integer. Evaluate $\phi(p^n)$ and $\phi(pq)$.

Theorem 3.2. *Let $n = p_1^{k_1} \cdots p_m^{k_m}$ be the standard prime factorization of a positive integer n. Then*

$$\phi(n) = n \prod_{j=1}^{m} \left(1 - \frac{1}{p_j}\right).$$

3.1.2 Derangements

Definition 3.2. A permutation $\sigma : [n] \to [n]$ is called a **derangement** whenever $\sigma(i) \neq i$ for all i. The number of derangements of $[n]$ is denoted by D_n.

Example 3.4. Evaluate D_1, D_2, D_3, and D_4.

Theorem 3.3. *The number of derangements, D_n, satisfies the following recursion:*

$$D_1 = 0, D_2 = 1, D_n = (n-1)[D_{n-2} + D_{n-1}], \text{ for all } n \geq 2.$$

Theorem 3.4. *The number of derangements is given by the following formula:*

$$D_n = n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.$$

3.1.3 Surjections and Stirling Numbers of the Second Kind

Theorem 3.5. *Let n and k be two positive integers. The number of surjective functions $f : [n] \to [k]$ is*

$$k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - + \cdots = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n.$$

Theorem 3.6. *For every two positive integers n and k we have*

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n.$$

3.2 The Twelvefold Way

Many counting problems can be turned into one of twelve problems. Suppose we are placing n balls into k boxes. Depending on whether the balls and boxes are identical or distinguishable and depending on what restrictions we impose on the number of balls in each box we get twelve different problems. These problems are all listed in the following table.

Balls	Boxes	Number of balls per box	Number of possibilities
identical	identical	any	$p_k(n)$
identical	identical	≥ 1	$p_k(n) - p_{k-1}(n)$
identical	identical	≤ 1	$\begin{cases} 1 & \text{if } n \leq k \\ 0 & \text{otherwise} \end{cases}$
identical	distinguishable	any	$\binom{n+k-1}{k-1}$
identical	distinguishable	≥ 1	$\binom{n-1}{k-1}$
identical	distinguishable	≤ 1	$\binom{k}{n}$
distinguishable	identical	any	$S(n,1) + S(n,2) + \cdots + S(n,k)$
distinguishable	identical	≥ 1	$S(n,k)$
distinguishable	identical	≤ 1	$\begin{cases} 1 & \text{if } n \leq k \\ 0 & \text{otherwise} \end{cases}$
distinguishable	distinguishable	any	k^n
distinguishable	distinguishable	≥ 1	$k!S(n,k)$
distinguishable	distinguishable	≤ 1	$k(k-1)\cdots(k-n+1) = (k)_n$

3.3 More Examples

Example 3.5. Given two positive integers k and n, find the number of increasing sequences of positive integers $a_1 \leq a_2 \leq \ldots \leq a_k$ for which $a_1 = 1$ and $a_k = n$.

Solution. Each sequence is determined by the number of 1's, 2's, etc. in the sequence. Let x_j be the number of j's in the sequence. Since the sequence has k terms, we have $x_1 + \cdots + x_n = k$. We also need to have $x_1, x_n \geq 1$. Letting $y_1 = x_1 - 1, y_n = x_n - 1$, and $y_j = x_j$ for all $2 \leq j \leq n-1$, we get $y_j \geq 0$ for all j. This yields an equation $y_1 + \cdots + y_n = k - 2$. By the formula for weak compositions the answer is $\binom{k-2+n-1}{k-2} = \boxed{\binom{n+k-3}{k-2}}$. ☐

Example 3.6. Prove that for every positive integer n, we have $\sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^n = n!$.

Solution. By a theorem $S(n,n) = \frac{1}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^n$. Note that $S(n,n) = 1$, since there is only one way to partition $[n]$ into n block. This yields the result. ☐

Example 3.7. Find the number of ways 12 people can be broken into three groups, if

(a) each group must have at least one member.

(b) the groups are named A, B, and C, and each group must have at least one member.

(c) the groups can have any number of members.

(d) only the number of group members is important to us, but not who is in which group. The groups may have any number of members.

Solution. (a) This is a partition of [12] into 3 blocks. The answer is $S(12,3) = \dfrac{3^{12} - 3 \cdot 2^{12} + 3}{6}$.
(b) Since the groups are labeled the answer would be $3!S(12,3)$.
(c) The answer is $S(12,1) + S(12,2) + S(12,3)$.
(d) This is a partition since the members are indistinguishable. The answer is $p_3(12)$. □

3.4 Exercises

3.4.1 Problems for grading

The following problems must be submitted on February 21, 2020 at the beginning of the class. **Late submission will not be accepted.**

Exercise 3.1. *(15 pts) Let m and n be two positive integers.*

(a) How many functions $f : [n] \to [m]$ are there?

(b) How many functions $f : [n] \to [m]$ are one-to-one?

(c) How many functions $f : [n] \to [m]$ are bijective?

(You may have to take cases.)

Exercise 3.2. *(10 pts) Let n be a positive integer. Prove*

(a) $\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^{n+1} = \binom{n+1}{2} n!$

(b) $\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^{n+2} = \binom{n+2}{3} n! + 3\binom{n+2}{4} n!$

(Hint: Use two-way counting.)

Exercise 3.3. *(10 pts) A solitaire type of card game is played as follows: The player has two shuffled decks, each with the usual 52 cards. With the decks face down the player turns up a pair of cards, one from each deck. If they are matching cards, he has lost the game. If they are not he continues and turns up another pair of cards , one from each deck. Again he loses if he gets two matching cards. The player wins if he can turn up all 52 pairs, none matching.*

(a) What is the probability of a win?

(b) *Suppose the win is defined differently: The player wins if there is exactly one matching pair in the entire 52 pairs. What is the probability of a win?*

You do not need to simplify your answer.

Exercise 3.4. *(10 pts) Let n and k be two positive integers for which $n \geq 2k$. Find a formula for the number of partitions of $[n]$ into k blocks, none of which is a singleton.*

(Your answer may involve summations.)

Exercise 3.5. *(10 pts) How many permutations of the 26 letters of the English alphabet do not contain any of the strings fish, short or man?*

Exercise 3.6. *(10 pts) Let n be a positive integer. How many triples of sets (A, B, C) satisfy both of the following conditions?*

$$A \cup B \cup C = [n], \quad \textit{and} \quad A \cap B \cap C = \emptyset$$

(Hint: Venn Diagram may help.)

Exercise 3.7. *(10 pts) Find the number of integers between 1 and 10^6, inclusive, that are neither perfect squares, nor perfect cubes, nor perfect fourth powers.*

Exercise 3.8. *(10 pts) Find the number of seven-digit positive integers whose digit sum is 20.*

Exercise 3.9 (10 pts)**.** *Let m and n be two positive integers. Find a formula for the number of n-tuples of non-zero integers $(x_1, \ldots, x_n)$ that satisfy $|x_1| + \cdots + |x_n| = m$. Your answer must be in closed form.*

3.4.2 Problems for practice

Page 98-100: 8, 9, 11, 18, 31, 39.

Page 112-115: 11, 14, 44, 45

Exercise 3.10. *Using the recursion in Theorem 3.3 prove the explicit formula for D_n in Theorem 3.4.*

3.4.3 Challenge Problems

Exercise 3.11. *For every two positive integers m, and n let $f(m, n)$ denote the number of n-tuples $(x_1, \ldots, x_n)$ of integers for which $|x_1| + \cdots + |x_n| \leq m$. Prove that $f(m, n) = f(n, m)$ for every $m, n \in \mathbb{Z}^+$.*

Exercise 3.12. *Let n be a positive integer. We call a permutation $(x_1, \ldots, x_{2n})$ of the numbers $1, 2, \ldots, 2n$ pleasant if $|x_i - x_{i+1}| = n$ for at least one $i \in \{1, 2, \ldots, 2n-1\}$. Prove that the number of pleasant permutations is more than $\frac{(2n)!}{2}$.*

4 Week 4

4.1 Power Series

To deal with a sequence a_n we can use a function associated to this sequence that stores all of the terms of the sequence. This is helpful since we can use the power of calculus to manipulate this function.

Definition 4.1. Given any sequence of real (or complex) numbers a_n, the series $f(x)$ defined by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is called a *power series* (centered at zero). The domain of the function $f(x)$ is the set of all values of x that make the above power series convergent.

Note that all power series that we discuss in this course are centered at zero, so we simply refer to all of them as power series without mentioning the center.
A well-known theorem in Real (or Complex) Analysis states the following:

Theorem 4.1. *For any power series precisely one of the following occurs:*

(a) The power series converges only for $x = 0$.

(b) The power series converges for all $x \in \mathbb{R}$ (or $x \in \mathbb{C}$).

(c) There is a positive real number R for which the power series converges for all x, with $|x| < R$ and diverges for all x with $|x| > R$.

The value R in (c) above is called the **radius of convergence** of the power series. When (a) occurs, the radius of convergence is said to be zero and when (b) occurs the radius of convergence is said to be infinity.

The following theorem allows us to add and multiply power series.

Theorem 4.2. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be two power series each with a positive radius of convergence R_1 and R_2, respectively. Then,*

- *$f(x) = g(x)$ for all x with $|x| < \min(R_1, R_2)$ if and only if for all $n \geq 0$, $a_n = b_n$. Furthermore, in that case $R_1 = R_2$.*
- *The coefficients a_n are obtained by the formula $a_n = \dfrac{f^{(n)}(0)}{n!}$.*
- *$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$, for all x with $|x| < \min(R_1, R_2)$.*
- *$f(x) \cdot g(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0) x + \cdots = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_k b_{n-k}) x^n$, for all x with $|x| < \min(R_1, R_2)$.*
- *$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for all x with $|x| < R_1$.*

- $\int_0^x f(t)\ dt = a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \cdots = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ *for all x with $|x| < R_1$.*

Here are some particularly important power series that you have seen in Calculus II:

Theorem 4.3. *(i)* $\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$*, for all $x \in (-1, 1)$.*

(ii) $e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$*, for all $x \in \mathbb{R}$.*

(iii) $\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$*, for all $x \in \mathbb{R}$.*

(iv) $\cos x = \frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$*, for all $x \in \mathbb{R}$.*

(v) $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = -\sum_{n=1}^{\infty} \frac{x^n}{n}$*, for all x with $-1 \le x < 1$.*

Recall that for every two positive integers $k \le n$, we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

This definition can be extended to when n is *any* real number, as follows.

Definition 4.2. For any real number a and any positive integer k, we define

$$\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!}, \text{ and } \binom{a}{0} = 1.$$

The following theorem which is a generalization of the Binomial Theorem is a standard theorem in Real Analysis.

Theorem 4.4 (Binomial Theorem)**.** *Let a be a real number and let $x \in (-1, 1)$. Then*

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k.$$

Example 4.1. Evaluate $\binom{\frac{1}{2}}{k}$ for all positive integers k. Write your answer in terms of combinations of integers.

Solution.

$$\begin{aligned}
\binom{\frac{1}{2}}{k} &= \frac{\frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdots \frac{-(2k-3)}{2}}{k!} \\
&= \frac{(-1)^{k-1} 1 \cdot 3 \cdot 5 \cdot (2k-3)}{2^k \cdot k!} \\
&= (-1)^{k-1} \frac{(2k-2)!}{2^k \cdot k! \cdot 2 \cdot 4 \cdots (2k-2)} \\
&= (-1)^{k-1} \frac{(2k-2)!}{2^{2k-1} \cdot k! \cdot (k-1)!} \\
&= \frac{(-1)^{k-1}}{2^{2k-1} \cdot k} \binom{2k-2}{k-1}
\end{aligned}$$

□

Example 4.2. Find a power series for $\sqrt{1-4x}$.

Solution. By the Binomial Theorem, and the previous example we obtain

$$\sqrt{1-4x} = (1-4x)^{1/2} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k}(-4x)^k = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^{2k-1}k}\binom{2k-2}{k-1}(-4)^k x^k = 1 - \sum_{k=1}^{\infty} \frac{2}{k}\binom{2k-2}{k-1}x^k. \quad \square$$

4.2 Formal Power Series

If the sequence a_n grows fast, then the series $\sum\limits_{n=0}^{\infty} a_n x^n$ only converges at $x = 0$. For example the power series $\sum\limits_{n=0}^{\infty} n!x^n$ only converges at $x = 0$. This limits our ability to work with these kinds of power series. For our purpose we can often ignore the convergence of a power series for particular values of x. In other words, you could think of $\sum_{n=0}^{\infty} a_n x^n$ as a "polynomial" with infinite degree. So, from now on think of "x" as just a "symbol" or a "placeholder" rather than a real number.

Definition 4.3. A **formal power series** is an infinite sum of the form

$$a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n,$$

where a_n is a sequence.

For two power series $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum\limits_{n=0}^{\infty} b_n x^n$, we say

$$f(x) = g(x) \text{ if and only if } \forall n \geq 0 \ a_n = b_n.$$

We define their sum as

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

Their product is defined as

$$f(x) \cdot g(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_k b_{n-k}) x^n.$$

The derivative and integral of formal power series are also defined similar to before.

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \text{ and } \int_0^x f(t)\ dt = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}.$$

Definition 4.4. For a power series $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$, the coefficient of x^n, i.e. a_n, is denoted by $[x^n]f(x)$.

Division is tricky, though. Not every formal power series has a multiplicative inverse, and thus we cannot always define division. For example if $\frac{1}{x}$ were to be a formal power series of form $a_0 + a_1 x + a_2 x^2 + \cdots$, then we would have to have

$$a_0 x + a_1 x^2 + a_2 x^3 + \cdots = 1,$$

which is impossible since the left side has no constant term (i.e. coefficient of x^0) while the constant term on the right hand side is 1.

This leads to a natural question: When does a formal power series have a multiplicative inverse, and what do the multiplicative inverse and dividing even mean anyway?

Definition 4.5. Given a formal power series $f(x)$, we say the formal power series $g(x)$ is the **multiplicative inverse** of $f(x)$ whenever $f(x)g(x) = 1$, in which case the formal power series $g(x)$ is denoted by $\frac{1}{f(x)}$.

Definition 4.6. Suppose $f(x)$ is a formal power series that has a multiplicative inverse and $g(x)$ is a formal power series. The **quotient** of $g(x)$ by $f(x)$, denoted by $\frac{g(x)}{f(x)}$, is defined as the product $g(x) \cdot \frac{1}{f(x)}$.

Example 4.3. The formal power series $1+x+x^2+\cdots$ is the multiplicative inverse of the power series $1-x$. Since their product is 1.

The following theorem answers the question of when the multiplicative inverse for a formal power series exists.

Theorem 4.5. *The multiplicative inverse of a formal power series* $\sum_{n=0}^{\infty} a_n x^n$ *exists as a formal power series (and is unique) if and only if* $a_0 \neq 0$.

Proof. ($\Rightarrow$) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Suppose $f(x)g(x) = 1$, for some formal power series $g(x) = \sum_{n=0}^{\infty} b_n x^n$. By comparing the constant terms we obtain $a_0 b_0 = 1$, and thus $a_0 \neq 0$.
($\Leftarrow$) Now, suppose $a_0 \neq 0$. We would like to define b_n in such a way that $a_0 b_0 = 1$, and that $\sum_{k=0}^{n} a_k b_{n-k} = 0$ for all $n \geq 1$. So, since $a_0 \neq 0$, we can define a sequence b_n recursively by

$$b_0 = \frac{1}{a_0}, \text{ and } b_n = \frac{-\sum_{k=1}^{n} a_k b_{n-k}}{a_0}, \text{ for all } n \geq 1.$$

Therefore, $b_n a_0 = -\sum_{k=1}^{n} a_k b_{n-k}$, which implies $\sum_{k=0}^{n} a_k b_{n-k} = 0$, for all $n \geq 1$. This shows

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = a_0 b_0 + \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n = 1,$$

which shows $f(x)$ has a multiplicative inverse. □

Remark. Note that the equality, sums, products, derivatives and intergals for power series and formal power series are given using the same formulas (see Theorem 4.2 and Definition 4.3). Therefore, if an identity is valid for power series with positive radius of convergence, it would also be valid for formal power series. This means we can use all of the formulas listed in Theorem 4.3 when dealing with formal power series.

4.3 Solving Recurrence Relations

In order to find an explicit formula for a sequence a_n it often helps to define a formal power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and then find this power series.

Example 4.4. Let a_n be a sequence given by the recurrence relation $a_0 = 1$ and $a_{n+1} = 2a_n + 1$ for all $n \geq 0$. Find a formula for a_n.

4.3.1 Ordinary Generating Functions

Definition 4.7. Let a_n, $n \geq 0$ be a sequence of real numbers. Then the formal power series

$$F(x) = \sum_{n=0}^{\infty} a_n x^n$$

is called the **ordinary generating function** of the sequence a_n.

Example 4.5. Let a_n be a sequence defined recursively by $a_0 = 2, a_1 = 5$ and $a_n = 5a_{n-1} - 6a_{n-2}$ for all $n \geq 2$. Find an explicit formula for a_n.

Solution. Let $F(x)$ be the ordinary generating function of the sequence a_n.
Multiplying the recursive formula by x^n we obtain $a_n x^n = 5a_{n-1}x^n - 6a_{n-2}x^n$. Summing this for $n \geq 2$, we get the following:

$$\begin{aligned}
\sum_{n=2}^{\infty} a_n x^n &= \textstyle\sum_{n=2}^{\infty}(5a_{n-1}x^n - 6a_{n-2}x^n) \\
&= 5x \sum_{n=2}^{\infty} a_{n-1}x^{n-1} - 6x^2 \sum_{n=2}^{\infty} a_{n-2}ax^{n-2} \\
&= 5x(F(x) - a_0) - 6x^2 F(x) \\
&= (5x - 6x^2)F(x) - 10x
\end{aligned}$$

The left hand side is equal to $F(x) - a_0 - a_1 x = F(x) - 2 - 5x$. This implies

$$F(x)-2-5x = (5x-6x^2)F(x)-10x \Rightarrow (6x^2-5x+1)F(x) = 2-5x \Rightarrow F(x) = \frac{2-5x}{6x^2-5x+1} = \frac{2-5x}{(3x-1)(2x-1)}$$

Applying the method of partial fractions we obtain

$$\frac{2-5x}{(3x-1)(2x-1)} = \frac{A}{3x-1} + \frac{B}{2x-1}.$$

Multiplying both sides by $(3x-1)(2x-1)$ we obtain $2-5x = A(2x-1)+B(3x-1)$, which yields $A = B = -1$. Therefore

$$F(x) = \frac{-1}{3x-1} + \frac{-1}{2x-1} = \frac{1}{1-3x} + \frac{1}{1-2x} = \sum_{n=0}^{\infty}(3x)^n + \sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty}(3^n + 2^n)x^n.$$

The above equality is obtained using the geometric series formula. This shows $a_n = 3^n + 2^n$. □

Example 4.6. Find an explicit formula for the Fibonacci sequence defined by

$$f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2.$$

Solution. Similar to the previous example, let $F(x)$ be the ordinary generating function for the Fibonacci sequence. With a similar method to the example above we obtain

$$F(x) - f_0 - f_1 x = \sum_{n=2}^{\infty} f_n x^n = x(F(x) - f_0) + x^2 F(x).$$

Therefore, $F(x) = \frac{-x}{x^2+x-1}$. As before we will use the method of partial fractions. The roots of $x^2+x-1 = 0$ are $\frac{-1\pm\sqrt{5}}{2}$. For simplicity call these two roots r are s. So we can write

$$\frac{-x}{x^2+x-1} = \frac{A}{x-r} + \frac{B}{x-s}.$$

Clearing the denominators we obtain $-x = A(x-s) + B(x-r)$. Substituting $x = r$ once and then $x = s$, we obtain $A = \frac{-r}{r-s}$ and $B = \frac{-s}{s-r}$. This implies

$$\begin{aligned} F(x) &= \frac{1}{r-s}\left(\frac{-r}{x-r} - \frac{-s}{x-s}\right) \\ &= \frac{1}{r-s}\left(\frac{1}{1-(x/r)} - \frac{1}{1-(x/s)}\right) \\ &= \frac{1}{r-s}\left(\sum_{n=0}^{\infty}(x/r)^n - \sum_{n=0}^{\infty}(x/s)^n\right) \\ &= \frac{1}{r-s}\left(\sum_{n=0}^{\infty}\frac{x^n}{r^n} - \frac{x^n}{s^n}\right) \\ &= \frac{1}{r-s}\left(\sum_{n=0}^{\infty}\left(\frac{1}{r^n} - \frac{1}{s^n}\right)x^n\right) \end{aligned}$$

By looking at the coefficient of x^n we obtain $f_n = \frac{1}{r-s}\cdot\left(\frac{1}{r^n} - \frac{1}{s^n}\right)$, which gives the following formula for the terms of the Fibonacci sequence:

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

□

Example 4.7. Find a formula for the n-th Catalan number.

Solution. Recall that Catalan numbers satisfy the recursion $c_0 = 1$, $c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k}$ for all $n \geq 0$. Let $C(x)$ be the ordinary generating function for c_n. Multiplying both sides by x^{n+1} and adding up for $n \geq 0$ yields

$$C(x) - c_0 = x\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} c_k c_{n-k}\right)x^n \qquad (*)$$

Also note that the above equality is equivalent to the recursion for c_n. The expressions $\sum_{k=0}^{n} c_k c_{n-k} x^n$ appear as coefficients of the square of the formal power series $C(x)$. The right hand side is $x(C(x))^2$ and the left hand side is $C(x) - 1$. Therefore, we obtain $x(C(x))^2 - C(x) + 1 = 0$. Furthermore, $C(x)$ is the *only* formal power series satisfying $x(C(x))^2 - C(x) + 1 = 0$, since (*) only holds for $C(x)$. Using the quadratic formula we know that $C(x) = \frac{1\pm\sqrt{1-4x}}{2x}$ both satisfy the equation above and thus, by Example 4.2 we obtain

$$2xC(x) = 1 \pm \left(1 - \sum_{k=1}^{\infty}\frac{2}{k}\binom{2k-2}{k-1}x^k\right)$$

Only the negative sign gives us a formal power series on the right hand side with no constant term. Thus, we obtain the following:

$$2xC(x) = 1 - \left(1 - \sum_{k=1}^{\infty} \frac{2}{k}\binom{2k-2}{k-1}x^k\right) = 2x\sum_{k=1}^{\infty}\frac{1}{k}\binom{2k-2}{k-1}x^{k-1}$$

Therefore,

$$\boxed{c_n = \frac{1}{n+1}\binom{2n}{n}}$$

□

Remark. Note that $\sqrt{1-4x}$ on its face is not a formal power series, as it is not of form $\sum a_n x^n$. What we mean by $\sqrt{1-4x}$ as a formal power series is the formal power series $\sum\limits_{n=0}^{\infty} \binom{1/2}{n}(-4)^n x^n$, which is the power series we get from the Binomial Theorem.

Also, note that we have not shown that quadratic equations only have two solutions in formal power series (and this is not even true!). In other words, even though $\frac{1-\sqrt{1-4x}}{2x}$ does satisfies the quadratic equation $x(C(x))^2 - C(x) + 1 = 0$ in order to show $C(x) = \frac{1-\sqrt{1-4x}}{2x}$, we need to show $C(x)$ is the *only* formal power series that satisfies $x(C(x))^2 - C(x) + 1 = 0$, and that $\frac{1-\sqrt{1-4x}}{2x}$ is in fact a formal power series. This is what we showed above.

4.3.2 Exponential Generating Functions

Example 4.8. Find an explicit formula for the sequence a_n given by

$$a_0 = 1, \text{ and } a_n = na_{n-1} + n \text{ for all } n \geq 1.$$

Solution. Let $F(x)$ be the ordinary generating function of a_n. Multiplying both sides of the recursion by x^n and adding up we obtain $\sum\limits_{n=1}^{\infty} a_n x^n = \sum\limits_{n=1}^{\infty} na_{n-1}x^n + \sum\limits_{n=1}^{\infty} nx^n$. The left hand side is $F(x) - 1$. The right hand side is not very easy to write in terms of $F(x)$, however it *can* be written in terms of $F(x)$ as $x^2F'(x) + x(F(x) - a_0) + x\sum\limits_{n=1}^{\infty} nx^{n-1} = x^2F'(x) + xF(x) - x + x\frac{d}{dx}\left(\frac{1}{1-x}\right)$. Solving this problem requires solving the differential equation $F(x) = x^2F'(x) + xF(x) - x + \frac{x}{(1-x)^2} + 1$. While this is certainly possible to solve using techniques from ODE, it is more complicated than it needs to be.

Instead, we will use a different type of generating function called exponential generating functions. Let $A(x) = \sum\limits_{n=0}^{\infty} \frac{a_n}{n!}x^n$. Dividing the recursion by $n!$ and summing it up for $n \geq 1$, we obtain

$$\sum_{n=1}^{\infty} \frac{a_n}{n!}x^n = \sum_{n=1}^{\infty} \frac{a_{n-1}}{(n-1)!}x^n + \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}.$$

The left hand side is $A(x) - 1$, while the right hand side is $xA(x) + xe^x$. This gives us the equation $A(x) - 1 = xA(x) + xe^x$. Solving this for $A(x)$ we obtain

$$A(x) = \frac{1}{1-x} + xe^x \cdot \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n + x\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} x^n + x\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{k!}x^n.$$

By comparing the coefficient of x^n we obtain $\frac{a_n}{n!} = 1 + \sum_{k=0}^{n-1} \frac{1}{k!}$, for all $n \geq 1$, therefore,

$$\boxed{a_0 = 1, a_n = n!\left(1 + \sum_{k=0}^{n-1} \frac{1}{k!}\right) \text{ for all } n \geq 1.}$$

□

Definition 4.8. Let $a_n, n \geq 0$, be a sequence of complex numbers. The formal power series

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

is called the **exponential generating function** of the sequence a_n.

4.4 More Examples

Example 4.9. Find a formula for the sum $\sum_{n=1}^{\infty} nx^n$.

Solution. This power series looks like the derivative of $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Differentiating we get $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$. Therefore, $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$. □

Example 4.10. Find an explicit formula for the sequence a_n satisfying the recursion $a_1 = 1$, and $a_{n+1} = 4a_n + 4n$ for all $n \geq 1$.

Solution. Let $A(x) = \sum_{n=1}^{\infty} a_n x^n$, and sum up the equality $a_{n+1}x^{n+1} = 4a_n x^{n+1} + 4nx^{n+1}$. We obtain

$$A(x) - x = 4xA(x) + 4x^2 \sum_{n=1}^{\infty} nx^{n-1} = 4xA(x) + 4x^2 \frac{d}{dx}\left(\frac{1}{1-x}\right).$$

Solving we get $A(x) = \frac{x}{1-4x} + \frac{4x^2}{(1-4x)(1-x)^2}$. Using partial fractions we obtain

$$A(x) = \frac{25}{36(1-4x)} + \frac{8}{9(1-x)} - \frac{4}{3(1-x)^2} - \frac{1}{4} = \frac{25}{36}\sum_{n=0}^{\infty}(4x)^n + \frac{8}{9}\sum_{n=0}^{\infty} x^n - \frac{4}{3}\frac{d}{dx}\left(\frac{1}{1-x}\right) - \frac{1}{4}.$$

Differentiating $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ we have $\frac{1}{1-x} = \sum_{n=1}^{\infty} nx^{n-1}$. Therefore,

$$A(x) = \frac{25}{36}\sum_{n=0}^{\infty}(4x)^n + \frac{8}{9}\sum_{n=0}^{\infty} x^n - \frac{4}{3}\sum_{n=1}^{\infty} nx^{n-1} - \frac{1}{4}.$$

Therefore,

$$\boxed{a_n = \frac{25}{36}4^n + \frac{8}{9} - \frac{4}{3}(n+1)}$$

□

Example 4.11. Find the OGF and EGF for the sequence $a_n = n^2 + 3n$.

First Solution. Note that $a_0 = 0$, so we can ignore this term when finding the generating functions. We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Differentiating and then multiplying by x we obtain $\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n$. Differentiating again and multiplying by x, we get $\sum_{n=1}^{\infty} n^2 x^n = \frac{x+x^2}{(1-x)^3}$. Putting these together we get the OGF associated with $n^2 + 3n$ is

$$\sum_{n=1}^{\infty}(n^2+3n)x^n = \frac{x+x^2}{(1-x)^3} + \frac{3x}{(1-x)^2} = \frac{-2x^2+4x}{(1-x)^3}.$$

For the EGF, we need to evaluate

$$E(x) = \sum_{n=1}^{\infty} \frac{(n^2+3n)}{n!}x^n = \sum_{n=1}^{\infty} \frac{n+3}{(n-1)!}x^n = \sum_{n=0}^{\infty} \frac{n+4}{n!}x^{n+1} = \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n-1)!} + \sum_{n=0}^{\infty} \frac{4x^{n+1}}{n!}$$

Note that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Therefore, $E(x) = x^2e^x + 4xe^x$. □

Second Solution. Similar to the previous part, we know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Differentiating this two times we get $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$, and $\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$. Note that we can write n^2+3n as a linear combination of n, and $n(n-1)$. So, let's first do that. $n^2+3n-n(n-1)$ is linear and equals 4n. Therefore, $n^2+3n = n(n-1)+4n$. Therefore,

$$A(x) = \sum_{n=0}^{\infty}(n^2+3n)x^n = \sum_{n=2}^{\infty} n(n-1)x^n + 4\sum_{n=1}^{\infty} nx^n = x^2\frac{2}{(1-x)^3} + 4x\frac{1}{(1-x)^2} = \frac{4x-2x^2}{(1-x)^3}.$$

Similar to the previous part we use $n^2+3n = n(n-1)+4n$. This gives us

$$E(x) = \sum_{n=2}^{\infty} \frac{n(n-1)}{n!}x^n + 4\sum_{n=1}^{\infty} \frac{nx^n}{n!} = x^2e^x + 4xe^x.$$

□

4.5 Exercises

All students are expected to do all of the exercises listed in the following two sections.

4.5.1 Problems for grading

The following problems must be submitted on Friday, March 6, 2020 at the beginning of the class. **Late submission will not be accepted.**

All proofs must be complete and solutions must be fully justified.

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

Exercise 4.1 (20 pts)**.** *Using the method of generating functions, find an explicit formula for each of the following sequences.*

(a) $a_0 = 1$, $a_n = 3a_{n-1} + n$ *for all* $n \geq 1$.

(b) $a_0 = a_1 = 1$, $a_n = 5a_{n-1} - 6a_{n-2}$ *for all* $n \geq 2$.

(c) $a_1 = 1$, $a_2 = 3$, $a_n = 4a_{n-1} - 4a_{n-2}$ *for all* $n \geq 3$.

(d) $a_0 = 1$, $a_n = na_{n-1} + 3n$ *for all* $n \geq 1$.

Exercise 4.2 (10 pts). *Suppose* $P(x) = \sum_{n=0}^{\infty} p_n x^n$ *and* $Q(x) = \sum_{n=0}^{\infty} q_n x^n$ *are two given formal power series for which* $q_0 = 0$, *and* $p_0 \neq 0$. *Prove that there is a unique formal power series* $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *that satisfies*

$$(A(x))^2 - P(x)A(x) + Q(x) = 0, \text{ and } a_0 \neq 0.$$

(Hint: See proof of Theorem 4.5.)

Exercise 4.3 (10 pts). *Write* $\binom{-3}{n}$ *in terms of combinations of positive integers. Use that to find a power series for* $\frac{1}{(2-x)^3}$.

Exercise 4.4 (10 pts). *Let* n *and* k *be two positive integers and* r *be a positive integer less than both* k *and* n. *Use the identity* $(1+x)^n(1+x)^k = (1+x)^{n+k}$ *to prove*

$$\sum_{j=0}^{r} \binom{n}{j}\binom{k}{r-j} = \binom{n+k}{r}.$$

Exercise 4.5 (15 pts). *Let* D_n *be the number of derangements of* $[n]$ *for every* $n > 0$ *and set* $D_0 = 1$.

(a) Using two-way counting prove that $n! = \sum_{k=0}^{n} \binom{n}{k} D_k$.

(b) Let $D(x)$ *be the exponential generating function associated to* D_n. *Prove that* $D(x)e^x = \frac{1}{1-x}$.

(c) Use the previous part to find an explicit formula for D_n.

Exercise 4.6 (10 pts). *Let* n *be a positive integer.* $2n$ *points* $A_1, A_2, \ldots, A_{2n}$ *are equally spaced on a circle. Find the number of ways we can draw* n *non-intersecting chords whose endpoints are all of these* $2n$ *points.*

Exercise 4.7 (15 pts). *The purpose of this problem is to show that for every integer* $n \geq 2$, *the product* $3^{n-1} \prod_{k=2}^{n} (3k-4)$ *is divisible by* $n!$. *Define a sequence* a_n *by* $a_1 = 1$, $a_n = \frac{3^{n-1}}{n!} \prod_{k=2}^{n} (3k-4)$ *for all* $n \geq 2$.

(a) Write down $\binom{\frac{1}{3}}{n}$ *in terms of* a_n.

(b) Let $A(x)$ *be the ordinary generating function associated with* a_n. *Find* $A(x)$.

(c) Use the previous part to find a recurrence relation for a_n. *Use that to show* a_n *is an integer for all* n. *Deduce that* $3^{n-1} \prod_{k=2}^{n} (3k-4)$ *is divisible by* $n!$.

Exercise 4.8 (10 pts). *Let* a_n *be the sequence given by the recursion*

$$a_0 = 1, \text{ and } a_n = -\sum_{k=1}^{n} \frac{a_{n-k}}{k!} \text{ for all } n \geq 1.$$

Using OGF of a_n *prove that* $a_n = \frac{(-1)^n}{n!}$ *for all* $n \geq 1$.

Exercise 4.9 (10 pts)**.** *Find OGF and EGF for the sequence* $n^3 + 3n$.

(Hint: See the second solution to Example 4.11.)

4.5.2 Problems for practice

p. 164-165: 4, 5, 17, 18

p. 176-177: 3, 4, 6

4.5.3 Challenge Problems

Exercise 4.10. *Let* $P(x)$ *be a polynomial. Find a formula for the ordinary and exponential generating functions of the sequence* $\{P(n)\}_{n=0}^{\infty}$.

Exercise 4.11. *Let* a_n *be a sequence for which* $a_0 = 3$, *and* $a_{n+1} = \sum_{k=0}^{n} a_k a_{n-k} - \frac{1}{3}\sum_{k=0}^{n-1}\sum_{\ell=k}^{n-1} a_k a_{\ell-k} a_{n-1-\ell}$, *for all* $n \geq 0$. *Find a formula for* a_n.

5 Weeks 5 and 6

5.1 Applications of Ordinary Generating Functions

Theorem 5.1 (Product Formula for OGF, First Version)**.** *Suppose* f_n, g_n *and* h_n *are three sequences whose ordinary generating functions are* $F(x), G(x)$, *and* $H(x)$, *respectively. Assume*

$$h_n = \sum_{k=0}^{n} f_k g_{n-k}, \text{ for all } n \geq 0.$$

Then $H(x) = F(x)G(x)$.

The proof of this theorem follows from the definition of product of two formal power series. This theorem is often used to solve counting problems in the following manner.

Example 5.1. Let a_n be a sequence defined by

$$a_n = \sum_{j=0}^{n} 2^j (n-j).$$

Find a closed formula for a_n.

Definition 5.1. An **interval** is a (possibly empty) set $\{i+1, \dots, i+j\}$ of consecutive integers. The **length** of this interval is said to be j.

Theorem 5.2 (Product Formula for OGF, Second Version)**.** *Suppose for every* $n \geq 0$, f_n *and* g_n *are the number of ways we can carry out tasks 1 and 2 on a set of size* n, *respectively. Suppose* h_n *is the number of ways one can divide the set* $[n]$ *into two (possibly empty) intervals* $A = \{1, \dots, i\}$ *and* $B = \{i+1, \dots, n\}$ *and then carry out task 1 on* A *and task 2 on* B. *Let* $F(x)$, $G(x)$ *and* $H(x)$ *be the ordinary generating functions associated with* f_n, g_n *and* h_n, *respectively. Then* $H(x) = F(x)G(x)$.

Example 5.2. Alex is taking a multiple choice test with n questions, where $n \geq 3$. He choose an even integer j between 2 and $n-1$, inclusive, and randomly marks the first j questions from the beginning of the test either A or B. He then randomly selects one of the remaining questions and marks it C, and the rest of the questions D. How many different outcomes are possible?

Solution. Let a_n be the number of ways this can be done. Since we are dividing $[n]$ into two intervals and performing two tasks, we can use the Product Formula. The first task is zero when j is odd and 2^j when j is even. Thus the OGF for the first task is $F(x) = \sum_{k=1}^{\infty} 2^{2k}x^{2k} = \frac{4x^2}{1-4x^2}$. The second task can be done in k ways over an interval of length k. Thus, the OGF for the second task is $G(x) = \sum_{k=1}^{\infty} kx^k = x\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2}$. Therefore, the OGF for the sequence a_n, $A(x)$ is

$$\begin{aligned} A(x) &= \frac{4x^3}{(1-4x^2)(1-x)^2} \\ &= \frac{1}{1-2x} - \frac{1}{9(2x+1)} + \frac{4}{9(1-x)} - \frac{4}{3(1-x)^2} \\ &= \sum_{n=0}^{\infty} (2x)^n - \frac{1}{9}(-2x)^n + \frac{4}{9}x^n - \frac{4}{3}(n+1)x^n \end{aligned}$$

Therefore, $\boxed{a_n = 2^n + \frac{(-1)^{n+1}}{9}2^n + \frac{4}{9} - \frac{4}{3}(n+1)}$ □

The following examples show that Ordinary Generating Functions are very helpful when dealing with partitions, compositions and weak compositions.

Example 5.3. Find the number of weak compositions of n into k parts, where k and n are positive integers.

Example 5.4. With an ample supply of bananas, apples, strawberries, and grapes we are to make fruit salads. Each fruit salad must consist of n pieces of fruit, where n is a given positive integer. The number of banana pieces in each fruit salad must be a multiple of 5, the number of apple pieces must be even, and the number of strawberry pieces must be less than 5. In terms of n, how many different types of fruit salad can be made?

Example 5.5. Find the generating functions of $p(n)$.

Solution. Here we are performing n tasks. We divide $[n]$ into n intervals. The k-th interval determines how many k's appear in the partition of n. In other words, the number of k's is the length of the k-th interval divided by k. The OGF for each k is given by $1+x^k+x^{2k}+\cdots$. Therefore, multiplying we get the OGF of $p(n)$.

Here is another way of looking at this: To form a partition for n, we need to write n as

$$n = \underbrace{1+\cdots+1}_{a_1 \text{ times}} + \underbrace{2+\cdots+2}_{a_2 \text{ times}} + \cdots + \underbrace{n+\cdots+n}_{a_n \text{ times}},$$

where $a_j \geq 0$ for all j. In other words, we write $\sum_{j=1}^{n} ja_j = n$. Therefore, every partition of n, yields a product of form

$$x^n = x^{a_1} \cdot (x^2)^{a_2} \cdots (x^n)^{a_n}.$$

Furthermore, every such product results in a partition of n. Thus, $p(n)$ is the coefficient of x^n in the product

$$(1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots) \cdots (1 + x^n + x^{2n} + x^{3n} \cdots).$$

Note that to make this independent of n, we can keep multiplying by $(1 + x^m + x^{2m} + \cdots)$ for all $m > n$, without changing the coefficient of x^n. Thus, the generating function for $p(n)$ is

$$(1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots) \cdots = \prod_{j=1}^{\infty} \frac{1}{1 - x^j} = \sum_{n=0}^{\infty} p(n)x^n,$$

where $p(0) = 1$. □

Remark. The product of infinitely many formal power series is not always well-defined. For example if we multiply $(1 + x + x^2 + x^3 + \cdots)$ by itself infinitely many times, the only term with a finite coefficient is the constant term. Every other term appears infinitely many times after "expanding" the infinite product. This is described in the following definition.

Definition 5.2. Let $F_0(x), F_1(x), F_2(x), \dots$ be a sequence of formal power series. Consider the sequence of formal power series $G_n(x) = \prod_{j=0}^{n} F_j(x)$. Suppose for every positive integer n, there is some positive integer N for which the coefficient of x^n in all power series $G_N(x), G_{N+1}(x), G_{N+2}(x), \dots$ is the same number a_n. Then, we define

$$\prod_{j=0}^{\infty} F_j(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Note that each product $\prod_{j=0}^{n} F_j(x)$ is called a **partial product** of $\prod_{j=0}^{\infty} F_j(x)$.

Example 5.6. Let k be a fixed positive integer. Find the OGF of the sequence $p(n, k)$.

Solution. Note that $p(n, k)$ is equal to the number of partitions of n into parts the largest of which is k (See Theorem 2.6). Therefore the generating function for $p(n, k)$ is

$$\sum_{n=0}^{\infty} p(n, k)x^n = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots) \cdots (x^k + x^{2k} + \cdots) = \boxed{x^k \prod_{j=1}^{k} \frac{1}{1 - x^j}},$$

where $p(0, k) = 0$. □

To find a generating function for $p(n, k)$ where both k and n are changing we need two variables. We will use the same method used in Example 5.5, however we need another variable to be the "placeholder", to count the number of terms that are added. We can deal with that as follows:

$$y^k x^n = (yx)^{a_1} \cdot (yx^2)^{a_2} \cdots (yx^n)^{a_n},$$

where a_j's are the number of j's in the partition of n, and precisely k of the a_j's are non-zero. Therefore, a two-variable formal power series for $p(n, k)$ is as follows:

$$(1+yx+(yx)^2+\cdots)(1+yx^2+(yx^2)^2+\cdots)\cdots = \prod_{j=1}^{\infty}\frac{1}{1-yx^j} = 1+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}p(n,k)y^kx^n.$$

Definition 5.3. Let $F(x) = \sum_{n=0}^{\infty} f_n x^n$ be a formal power series and $A(x) = \sum_{n=1}^{\infty} a_n x^n$ be a formal power series having constant term $a_0 = 0$. Then the composition of the power series is the power series

$$F(A(x)) = \sum_{n=0}^{\infty} f_n(A(x))^n = \sum_{n=0}^{\infty} b_n x^n,$$

where $(A(x))^0 = 1$, and b_n is the coefficient of x^n in the finite sum $\sum_{k=0}^{n} f_n(A(x))^k$.

Note that for every n, the coefficient of x^n in $(A(x))^m$ for all $m \geq n+1$ is zero, which means x^n may only appear in the finite sum $\sum_{k=0}^{n} f_n(A(x))^k$.

Theorem 5.3 (Composition Formula for OGF). *Let a_k be the number of ways we can carry out task 1 on any set of size k, with $a_0 = 0$. Let b_k be the number of ways we can carry out task 2 on any set of size k. Let c_n be the number of ways we can split $[n]$ into non-empty intervals, then carry out task 1 on each interval and then carry out task 2 on the set of intervals. Let $A(x), B(x)$, and $C(x)$ be the ordinary generating functions associated with sequences a_n, b_n and c_n, respectively. Then $C(x) = B(A(x))$.*

Example 5.7. A soccer coach has her n players stand in a line. Then she breaks the line at a few places, to form non-empty units, and then chooses a leader for each unit. Finally she chooses one of the units for a specific task. Find the generating function for the sequence c_n that counts the number of ways she can perform this.

5.2 Applications of Exponential Generating Functions

Theorem 5.4 (Product Formula for EGF, First Version). *Let f_n, g_n and h_n, $n \geq 0$ be sequences such that*

$$h_n = \sum_{k=0}^{n}\binom{n}{k} f_k g_{n-k}, \text{ for all } n \geq 0.$$

Let $F(x), G(x)$, and $H(x)$ be EGF associated to f_n, g_n, and h_n, respectively, then $H(x) = F(x)G(x)$.

When solving counting problems we may need the following version of the Product Formula.

Theorem 5.5 (Product Formula for EGF, Second Version). *Suppose for every $k \geq 0$, f_k and g_k are the number of ways we can carry out tasks 1 and 2 on a set of size k, respectively. Suppose h_n is the number of ways one can select a (possibly empty) subset S of $[n]$; then carry out task 1 on S and task 2 on $[n] - S$. Let $F(x)$, $G(x)$ and $H(x)$ be the exponential generating functions associated with f_n, g_n and h_n, respectively. Then $H(x) = F(x)G(x)$.*

Remark. Note that the two tasks in the Product Formula are ordered. In other words, even if the two tasks are the same tasks, the order in which this task is applied is important. For example if we apply task 1 to $\{1,2\}$ and then apply it to $\{3,4,5\}$, that is different from first applying task 1 to $\{3,4,5\}$ and then to $\{1,2\}$. The first one is counted in a_2a_3 while the second one is counted in a_3a_2.

Example 5.8. Alex is taking a multiple choice test with n questions, where $n \geq 2$. He choose an integer j between 1 and $n-1$, inclusive, and randomly marks j questions either A or B. He then randomly selects one of the remaining questions and marks it C, and he marks the rest of the questions D. How many different outcomes are possible?

Theorem 5.6 (Composition Formula for EGF). *Let a_k be the number of ways we can carry out task 1 on any set of size k, with $a_0 = 0$. Let b_k be the number of ways we can carry out task 2 on any set of size k. Let c_n be the number of ways we can partition $[n]$ into non-empty blocks, then carry out task 1 on each block and then carry out task 2 on the set of blocks. Let $A(x), B(x)$, and $C(x)$ be the exponential generating functions associated with sequences a_n, b_n and c_n, respectively. Then $C(x) = B(A(x))$.*

Example 5.9. There are n people at a dinner party. We divide them into an unspecified number of groups, have each group sit at a different round table and serve one of the three dinner choices to each table. In how many ways can this be done?

Example 5.10. Find the exponential generating function of the number of partitions of $[n]$ into blocks of even size.

Solution. The first task is to decide whether a block is even or odd. If it is even then task one produces 1, otherwise it produces 0. Thus, the EGF for this task is $F(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}$. Note that adding e^x and e^{-x} eliminates all the odd powers of x and doubles the even terms. In other words, $F(x) = (e^x + e^{-x})/2 - 1$ The second task has the generating function $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$, since each such partition is counted once. Thus, the desired generating function is $\exp(F(x)) = \boxed{e^{\frac{e^x+e^{-x}-2}{2}}}$.

Theorem 5.7. *Let $S : s_1, s_2, s_2, \ldots$ be an increasing sequence of positive integers. Let $h_S(n)$ be the number of ways $[n]$ can be partitions into blocks so that all block sizes are in S. Then the exponential generating function of $h_S(n)$ is* $\exp\left(\sum_{j=1}^{\infty} \frac{x^{s_j}}{j!}\right)$.

Proof. The first task assign 1 to every set whose size is in S. The second task assigns 1 to every partition. The EGF for the first task is $\sum_{j=1}^{\infty} \frac{x^{s_j}}{j!}$ and the EGF for the second task is $\sum_{j=1}^{\infty} \frac{x^{s_j}}{j!}$. Thus, the EGF for the $\square$

5.3 Other Generating Functions

Example 5.11. Let a_n be a sequence satisfying $a_0 = 1$, $a_n = n^2 a_{n-1} + n!$. Find an explicit formula for a_n.

5.4 More Examples

Example 5.12. Given a positive integer n, find the number of weak compositions of n into three parts, the second of which is even.

Solution. The OGF for this sequence is $F(x) = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x + x^2 + \cdots) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x} = \frac{1}{(1-x)^3(1+x)}$. Partial fraction decomposition for $F(x)$ is

$$F(x) = \frac{1}{8(1-x)} + \frac{1}{4(1-x)^2} + \frac{1}{2(1-x)^3} + \frac{1}{8(1+x)}$$

Example 5.13. Find the number of weak compositions of n into three parts, the last of which is a multiple of 3.

Solution. The OGF for this sequence is $F(x) = (1 + x + x^2 + \cdots)^2(1 + x^3 + x^6 + \cdots) = \dfrac{1}{(1-x)^2(1-x^3)} = \dfrac{1}{(1-x)^3(1+x+x^2)} = \dfrac{1}{(1-x)^3(r-x)(s-x)}$, where $r, s = \dfrac{-1 \pm \sqrt{3}i}{2}$. Partial decomposition yields the answer. $\square$

Example 5.14. Find the OGF of $p_d(n)$, the number of partitions of n into distinct parts.

Solution. The k-th task can be done in precisely 1 way if the length of the k-th interval is 0 or k and in zero ways otherwise, since we can have at most one k in the partition. Therefore, the OGF for the k-th task is $1 + x^k$. The answer, therefore, is $\boxed{\sum_{n=0}^{\infty} p_d(n)x^n = \prod_{k=1}^{\infty}(1 + x^k)}$ with $p_d(0) = 1$. $\square$

Example 5.15. Prove that for any positive integer n, the number of partitions of n into odd parts is the same as the number of partitions of n into distinct parts.

Solution. Since we could have any number of each odd part, the OGF for each odd part $2k+1$ is $\sum\limits_{n=0}^{\infty} x^{(2k+1)n}$. Multiplying these we get the OGF for the number of partitions of n into odd parts $\prod\limits_{k=0}^{\infty}(\sum\limits_{n=0}^{\infty} x^{(2k+1)n})$ which is equal to $F(x) = \prod\limits_{k=0}^{\infty} \dfrac{1}{1 - x^{2k+1}}$.

As seen in Example 5.14, the OGF for the number of partitions of n into distinct parts is $G(x) = \prod\limits_{n=1}^{\infty}(1+x^n)$. To finish up the proof, we will have to show $F(x) = G(x)$. Note that

$$G(x) = \prod_{n=1}^{\infty} \frac{1 - x^{2n}}{1 - x^n} = \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\frac{1-x^8}{1-x^4}\cdots.$$

We see that all the terms on top cancel with all the even terms at the bottom and that precisely gives us $F(x)$. However this argument is not quite rigorous, since when dealing with infinite series infinite cancellation may not be allowed. So, we will make this argument more rigorous. The partial products of $G(x)$ are $\prod\limits_{n=1}^{N} \dfrac{1-x^{2n}}{1-x^n}$, which after cancellation is the same as $\prod\limits_{0 \le k < N/2} \dfrac{1}{1 - x^{2k+1}} \cdot \prod\limits_{N/2 \le k \le N} (1 - x^{2k})$. Since all exponents of x in the product $\prod\limits_{N/2 \le k \le N}(1 - x^{2k})$ are larger than $N - 1$, the coefficients of $x^0, x^1, \ldots, x^{N-1}$ in $G_N(x) = \prod\limits_{n=1}^{N} \dfrac{1-x^{2n}}{1-x^n}$ and $F_N(x) = \prod\limits_{0 \le k < N/2} \dfrac{1}{1-x^{2k+1}}$ are the same. Note that $G_N(x)$ and $F_N(x)$ are partial products of $G(x)$ and $F(x)$. Therefore, $F(x) = G(x)$, as desired. $\square$

Example 5.16. A book must consist of $n \ge 2$ pages. Each page can be either text or an illustration. The book can have any number of chapters but each chapter must have at least one illustrations and one text page. In how many ways is this possible?

Solution. Let a_n and b_n be the number of ways the first and second tasks can be done, respectively. The first task is choosing weather each page is a text or an illustration. This can be done in $2^n - 2$ ways, because we cannot have all text or all illustration pages. We also have $a_0 = 0$. The second task is essentially doing nothing because every legitimate chapter will be accepted in one way. Since the order is important we need to use OGF. The generating functions are thus, $A(x) = \sum\limits_{n=2}^{\infty}(2^n-2)x^n = \dfrac{4x^2}{1-2x} - \dfrac{2x^2}{1-x}$ and $B(x) = \sum\limits_{n=2}^{\infty} x^n = \dfrac{x^2}{1-x}$. Thus, we need to find $A(x)B(x) = \dfrac{4x^4}{(1-2x)(1-x)} - \dfrac{2x^4}{(1-x)^2}$. The sequence can now be found using partial fraction decomposition. □

Example 5.17. Find the EGF for the sequence of Bell numbers.

Solution. Bell numbers count all partitions of $[n]$. Thus, we can use Theorem 5.7 with $S = \mathbb{Z}^+$. Therefore, the EGF for the Bell numbers is e^{e^x-1}. □

5.5 Exercises

All students are expected to do all of the exercises listed in the following two sections.

5.5.1 Problems for grading

The following problems must be submitted on Friday, March 13, 2020 at the beginning of the class. **Late submission will not be accepted.**

All proofs must be complete and solutions must be fully justified.

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

Exercise 5.1 (10 pts)**.** *Let n, and k be two positive integers. In class we used the method of generating functions to find the number of weak compositions of n into k parts. Using the method of generating functions find the number of compositions of n into k parts.*

Exercise 5.2 (15 pts)**.** *Find a formula in closed form for $p(n,3)$, the number of partitions of a positive integer n into 3 parts. Your formula may involve complex numbers! You may use a computer algebra system to get the partial fraction decomposition.*

Hint: See Example 5.6.

Exercise 5.3 (15 pts)**.** *Let $P(x,y) = \prod\limits_{j=1}^{\infty}(1+yx^j)$.*

(a) Find an interpretation for the coefficient of x^n in the power series expansion of $P(x,1)$.

(b) Find an interpretation for the coefficient of x^n in the power series expansion of $P(x,-1)$.

(c) Interpret the coefficients of x^n in the power series expansions of $\dfrac{P(x,1)+P(x,-1)}{2}$ and $\dfrac{P(x,1)-P(x,-1)}{2}$.

Hint: See the explanation after Example 5.6.

Exercise 5.4 (10 pts)**.** *Let a_n be the number of ways one can pay n cents using pennies, nickels, and dimes. Find the ordinary generating function of a_n.*

Exercise 5.5 (10 pts)**.** *Find the EGF for the sequence a_n that counts the number of partitions of $[n]$ in which all blocks have even sizes and the number of blocks is also even.*

Exercise 5.6 (10 pts)**.** *Let k be a positive integer. Find a closed form for $\sum_{n=0}^{\infty} S(n,k)\frac{x^n}{n!}$. Use that to find a formula for $S(n,k)$.*

Exercise 5.7 (10 pts)**.** *n people are standing in a line at the post office. Two customer service representatives splits the line at an arbitrary point (so, there are $n-1$ places that the lines could be split.) The first representative offers each customer in the first portion of the line two choices: either first class mail or overnight. The second representative only has time to service two of the customers. So, they randomly pick two customers (thus, the second part of the line must have at least two people, otherwise that can be done in zero ways) and offer each customer one of the 3 choices: overnight, flat-rate, or first class mail. The rest of the customers in the second part of the line get one forever stamp each. What is the OGF for the number of possible outcomes?*

Exercise 5.8 (10 pts)**.** *Similar to the previous problem assume a line with n people is formed. Several post office representatives break the line into non-empty pieces and each one helps one group in the line. There are three different options (first class mail, express and flat-rate) and each representative can only offer one of the three options to the entire group. At the end of the process one of the groups is randomly picked for a survey. How many different ways can this be done?*

Exercise 5.9 (10 pts)**.** *Find an explicit formula for a_n if $a_0 = 1$, and $a_n = n^3 a_{n-1} + (n!)^2$ for all $n \geq 1$.*

5.5.2 Problems for Practice

Page 164-165: 12, 21, 25.

Page 176-177: 15.

Exercise 5.10. *Let k be a positive integer. Find the OGF of the sequence $S(n,k)$.*

Solution. Let $F_k(x) = \sum_{n=1}^{\infty} S(n,k)x^n$ be the OGF of the sequence $S(n,k)$ for every k. We know $S(n+1,k) = S(n,k-1) + kS(n,k)$ for all $n \geq 1$. Multiplying by x^{n+1} and summing up we obtain

$$\sum_{n=1}^{\infty} S(n+1,k)x^{n+1} = \sum_{n=1}^{\infty} S(n,k-1)x^{n+1} + k\sum_{n=1}^{\infty} S(n,k)x^{n+1}$$

The left hand side is $F_k(x) - S(1,k)x$. The right hand side is $xF_{k-1}(x) + kxF_k(x)$, for all $k \geq 2$. Note that $S(1,k) = 0$, for all $k \geq 2$. Therefore, $F_k(x)(1-kx) = xF_{k-1}(x)$, for all $k \geq 2$. Using this repeatedly we obtain

$$F_k(x) = \frac{x}{1-kx}F_{k-1}(x) = \frac{x}{1-kx} \cdot \frac{x}{1-(k-1)x}F_{k-2}(x) = \cdots = \frac{x^{k-1}}{(1-kx)\cdots(1-2x)}F_1(x)$$

$F_1(x) = \sum_{n=1}^{\infty} x^n = \dfrac{x}{1-x}$. Thus, $\boxed{F_k(x) = \prod_{j=1}^{k}\left(\frac{x}{1-jx}\right)}$. □

5.5.3 Challenge Problems

Exercise 5.11. *Let p be an odd prime. Find the number of non-empty subsets of $\{1, 2, \ldots, p-1\}$ that have a sum that is divisible by p.*

Exercise 5.12. *Let n be a positive integer. Show that the number of partitions of n into parts which have at most one of each distinct even part (e.g. 1 + 1 + 1 + 2 + 3 + 4) equals the number of partitions of n in which each part can appear at most three times (e.g. 1 + 1 + 1 + 2 + 2 + 4 + 4 + 4).*

Exercise 5.13. *Let n be a positive integer. Show that the number of partitions of n, where each part appears at least twice, is equal to the number of partitions of n into parts all of which are divisible by 2 or 3.*

Exercise 5.14. *Let $\alpha(n)$ be the number of representations of a positive integer n as sum of 1's and 2's, taking order into account. For example, since*

$$4 = 1 + 1 + 2 = 1 + 2 + 1 = 2 + 1 + 1 = 2 + 2 = 1 + 1 + 1 + 1,$$

we have $\alpha(4) = 5$. Let $\beta(n)$ be the number of representations of n that are sums of integers greater than 1, again taking order into account. For example, since

$$6 = 4 + 2 = 3 + 3 = 2 + 4 = 2 + 2 + 2,$$

we have $\beta(6) = 5$. Show that $\alpha(n) = \beta(n + 2)$.

Exercise 5.15. *Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of all non-negative integers. For every subset $S \subseteq \mathbb{N}$ and every $n \in \mathbb{N}$ let $r_S(n)$ be the number of pairs of integers (s_1, s_2) for which $s_1, s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Can $\mathbb{N}$ be partitioned into two subsets A and B for which $r_A(n) = r_B(n)$ for all $n \in \mathbb{N}$? If so, find all such partitions, and if not, prove no such partition exists.*

Exercise 5.16. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two sequences of integers for which neither is a permutation of the other. Suppose in addition that the two sequences $a_i + a_j$, with $1 \leq i < j \leq n$ and $b_i + b_j$, with $1 \leq i < j \leq n$ are permutations of one another. Prove that n must be a power of 2.*

6 Week 7

6.1 Introduction to Graphs

Definition 6.1. A **(simple) graph** G is an ordered pair (V, E), where V is a finite non-empty set, called the set of **vertices** or **nodes**, and E is a set of 2-element subsets of V, called **edges**. The set $V = V(G)$ is called the **vertex set** and the set $E = E(G)$ is called the **edge set** of G. Two graphs G and H are called **equal** if $V(G) = V(H)$ and $E(G) = E(H)$. An edge $\{u, v\}$ is sometimes denoted by uv or vu.

Definition 6.2. Two vertices u and v of a graph G are called **adjacent**, **neighbors** or **connected** if $uv \in E(G)$. An edge $e = uv$ is said to be **incident** to vertices u and v. The vertices u and v are called the **endpoints** of the edge uv. Two distinct edges are called **incident** if they share an endpoint.

Definition 6.3. Let G be a graph, u be a vertex of G and e be an edge of G.

- The graph $G - u$ is the graph obtained from G by removing u along with all edges that have u as an endpoint. In other words, $G - u$ is the graph whose vertex set is the set $V(G) - \{u\}$ and whose edge set is $\{e \in E(G) \mid u \notin e\}$. Note that for $G - u$ to be a graph we need the order of G to be at least 2.
- The graph $G - e$ is the graph whose vertex set is $V(G)$ and whose edge set is $E(G) - \{e\}$.

Remark. Note that since E consist of 2-element subsets of V, each edge must have two *distinct* endpoints. In other words, no "loops" are allowed. Also, since E is a set, no element of E is repeated, which means no "multiple edge" is allowed. In some textbooks, multiple edges and loops in the definition of a graph are allowed, and thus graphs without loops and multiple edges are called *simple* graphs. In our class we only discuss graphs with no loops or multiple edges.

Definition 6.4. Let u and v be two vertices of a graph G, and $e = uv$ be an edge in the complete graph on $V(G)$. The graph $G + e$ is a graph whose vertex set is $V(G)$ and whose edge set is $E(G) \cup \{e\}$.

Definition 6.5. The number of vertices of a graph G is called the **order** of G, and the number of edges of G is called the **size** of G.

Definition 6.6. A graph H is called a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then we say H is a **spanning subgraph** of G. We say H is a **vertex induced subgraph** of G if whenever $u, v \in V(H)$ and $uv \in E(G)$, then $uv \in E(H)$. In other words G is obtained by selecting a subset S of $V(G)$ and including all edges of G that are between vertices of S.

Definition 6.7. For a nonempty set of vertices S of a graph G, the **subgraph induced by** S is the vertex induced subgraph of G with vertex set S. This induced subgraph is denoted by $G[S]$. For a nonempty set X of edges of a graph G, the graph whose edge set is X and whose vertex set is the set of all vertices of G that are incident with at least one edge in X is called the **edge-induced subgraph of** G and is denoted by $G[X]$.

Definition 6.8. Let u, v be two vertices of a graph G.

- A uv**-walk** is a sequence $u = u_0, u_1, \ldots, u_m = v$ of vertices of G for which $u_j u_{j+1}$ is an edge of G for every j, $0 \leq j \leq m - 1$. We say this uv-walk **traverses** each edge $u_j u_{j+1}$. A walk is called **closed**, if $u = v$. The number m, which is the number of edges traversed by the walk, is called the **length** of this walk.
- A uv**-trail** is a walk with no edge traversed more than once.
- A uv**-path** is a trail with no vertex traversed more than once.
- A closed trail of positive length is called a **circuit**.
- A **cycle** is a circuit with the first and last vertices as the only repeated vertices.

Definition 6.9. A graph G is called **connected** if for any two distinct vertices u and v in G, there is a uv-path. Otherwise it is called **disconnected**. The **distance** of two vertices u, v, denoted by $d_G(u, v)$ or simply $d(u, v)$, is the minimum length of a uv-path of G. If there is no uv-path, then we set $d(u, v) = \infty$. Any uv-path of length $d(u, v)$ is called a **geodesic** from u to v. The **diameter** of a connected graph G, denoted by diam G, is the maximum distance between any two vertices of G.

Definition 6.10. A vertex in a graph is called **isolated** if it is not adjacent to any vertices.

Theorem 6.1. *Suppose u, v are two vertices of a graph G. If G has a uv-walk of length at most ℓ, then it has a uv-path of length at most ℓ.*

Theorem 6.2. *Let u, v and w be three vertices of a graph. Then the distance satisfies the following properties:*

- $d(u, v) = d(v, u)$.
- $d(u, v) = 0$ *iff* $u = v$.
- $d(u, v) + d(v, w) \geq d(u, w)$.

Definition 6.11 (Special Graphs)**.** Let n be a positive integer.

- The **trivial graph** is the graph with 1 vertex and no edge. Every other graph is called **nontrivial**.
- The **path graph** on n vertices, denoted by P_n, is a path with n vertices.
- The **cycle graph** on n vertices, where $n \geq 3$, (or the n**-cycle**), denoted by C_n, is a cycle with n vertices. If n is even C_n is called an **even cycle**, and if n is odd C_n is called an **odd cycle**.
- The **complete graph** on n vertices, denoted by K_n, is the graph with a vertex set of size n, $V = \{v_1, \ldots, v_n\}$ and the edge set $E = \{v_i v_j \mid 1 \leq i < j \leq n\}$.

Definition 6.12. The **complement** of a graph G, denoted by $\overline{G}$ has vertex set $V(\overline{G}) = V(G)$, and edge set $E(\overline{G}) = \{uv \mid u, v \in V(G), u \neq v, \text{ and } uv \notin E(G)\}$.

Definition 6.13. Two graphs G and H are called **isomorphic**, if there is a bijection $f : V(G) \to V(H)$, for which $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. If G and H are isomorphic then we write $G \cong H$. Sometimes we say G is a copy of H.

Theorem 6.3. *Let R be a relation on the vertices of a graph G defined by uRv iff there is a uv-path in G. Then R is an equivalence relation.*

Definition 6.14. The subgraphs of a graph G induced on the equivalence classes of the relation R in the previous theorem are called the **connected components of G.** The number of connected components of G is denoted by $k(G)$.

Definition 6.15. Let G be a graph. Then,

- If $U \subsetneq G$, then $G - U$ is the subgraph of G induced on the vertex set $V(G) - U$.

- If $X \subseteq E(G)$, then $G-X$ is the subgraph of G whose vertex set is $V(G)$ and whose edge set is $E(G)-X$.

Theorem 6.4. *Let G be a graph with at least 3 vertices. G is connected if and only if there are two distinct vertices u and v for which $G-u$ and $G-v$ are connected.*

Theorem 6.5. *If a graph G is disconnected then* $\operatorname{diam} \overline{G} \leq 2$ *and thus $\overline{G}$ is connected.*

Theorem 6.6. *Isomorphism has the following properties:*

(a) $\cong$ is an equivalence relation.

(b) If $G \cong H$, then $\overline{G} \cong \overline{H}$.

(c) If $G \cong H$, then G and H have the same size and order.

Proof. Exercise! □

Definition 6.16. The **union** of graphs $G_1, G_2, \ldots, G_n$, denoted by $G_1 \cup G_2 \cup \cdots \cup G_n$ or $\bigcup_{j=1}^{n} G_j$, is the graph whose vertex set is $\bigcup_{j=1}^{n} V(G_j)$ and whose edge set is $\bigcup_{j=1}^{n} E(G_j)$. When the vertex sets $V(G_1), \ldots, V(G_n)$ are pairwise disjoint, the union of these graphs is denoted by $\bigsqcup_{j=1}^{n} G_j$ and is called the **disjoint union** of $G_1, \ldots, G_n$. We say a graph G is **decomposed** into graphs $G_1, G_2, \ldots, G_n$ if $\bigcup_{j=1}^{n} G_j$ and no two G_j's share an edge.

Example 6.1. Prove that K_4 can be decomposed into copies of P_4.

Solution. Let K_4 be the complete graph on [4]. Then K_n is the edge disjoint union of two paths $1, 2, 3, 4$ and $2, 4, 1, 3$. □

6.2 More Examples

Example 6.2. Let $n \geq 3$ be a positive integer.

(a) Find the number of subgraphs of K_n that are $n-$cycles.

(b) Find the number of subgraphs of K_n that are paths of order n.

Solution. (a) To form an n-cycle, we start from vertex v_1, then choose its neighbors. This can be done in $\binom{n-1}{2}$ ways. The remaining $(n-3)$ vertices can be placed between the neighbors of v_1 in $(n-3)!$ ways. So, the answer is $(n-3)!\binom{n-1}{2} = \boxed{\frac{(n-1)!}{2}}$.
Another way of counting that would be to place the vertices on a circle. This can be done in $(n-1)!$ ways. Accounting for the reflection, the answer is $(n-1)!/2$.

(b) Each path is created by placing the n vertices in a row, however each path is created twice (once forward and once backwards). Thus, the answer is $\boxed{\frac{n!}{2}}$. □

Example 6.3. For any integer $n \geq 2$, let G_n be the graph whose vertex set is $[n]$, and $jk \in E(G_n)$ if and only if $\gcd(k, \ell) \neq 1$. Describe all isolated vertices of G_n.

Solution. Note that 1 is isolated, since $\gcd(1, k) = 1$ for every k. If i is composite, then $i = jk$ for some $j, k \in \mathbb{Z}$ with $2 \leq j < i$. Thus, ij is an edge. Therefore, i is not isolated. If $i \leq \frac{n}{2}$, then i is connected to $2i$ and thus not isolated. So far we have shown the only possible isolated vertices are primes more than $n/2$ along with 1. If $p > n/2$ is prime and p is connected to a, then $\gcd(a, p) \neq 1$. Since p is prime, $\gcd(a, p) = p$, which means p must divide a. This means $a \geq 2p > n$, which is a contradiction. Therefore, the set of isolated vertices of G_n is $\{1\} \cup \{p \mid p \text{ is prime and } \frac{n}{2} < p \leq n\}$. □

Example 6.4. For every positive integer n let G_n be the graph whose vertex set is the set of all polynomials $a_0 + a_1t + a_2t^2 + \cdots + a_nt^n$, where $a_j \in \{0, 1\}$ for all j. Two distinct vertices are connected if they have the same degree.

(a) Find the order and size of G_n. (Note that the degree of the zero polynomial is defined to be $-\infty$.)

(b) Find the number of connected components of G_n.

Solution. (a) Each a_j has two options and there are $n+1$ coefficients a_j. Thus, the order of G_n is $\boxed{2^{n+1}}$. For every positive integer $m \leq n$ a polynomial has degree m if its leading term is t^m. Thus, there are 2^m polynomials of degree m. 1 and 0 are the only constant polynomials and are not connected. Therefore, the size of G_n is $\sum_{j=1}^{n} \binom{2^j}{2} = \frac{1}{2}\sum_{j=1}^{n}(2^{2j} - 2^j) = \frac{1}{2}\left(\frac{4^{n+1}-4}{3} - (2^{n+1}-2)\right) = \boxed{\frac{4^{n+1} - 3 \cdot 2^{n+1} + 2}{6}}$.

(b) The equivalence relation that creates the connected components has two polynomials in relation if and only if they have the same degree. Note that possible degrees of vertices of G_n are $-\infty, 0, 1, \ldots, n$. Thus, G_n has $\boxed{n+2}$ connected components. □

Example 6.5. Prove that K_n, with $n \geq 2$, can be decomposed into copies of P_3 if and only if n or $n-1$ is a multiple of 4.

Solution. Suppose K_n can be decomposed into copies of P_3. Since the size of K_n is $\binom{n}{2}$ and the size of P_3 is 2, $\binom{n}{2}$ must be even. Thus, $n(n-1)$ must be a multiple of 4. Note that either n or $n-1$ is odd. If n is odd, then $n-1$ must be a multiple of 4, and if $n-1$ is odd, then n must be a multiple of 4.

Now, we will prove that if K_n can be decomposed into copies of P_3, then K_{n+4} can also be decomposed into copies of P_3. Let G be the complete graph on vertices $v_1, v_2, \ldots, v_{n+4}$. This graph can be decomposed into the complete graph on $v_1, \ldots, v_n$, the complete graph on $v_{n+1}, v_{n+2}, v_{n+3}, v_{n+4}$, the paths v_{n+1}, v_j, v_{n+2}, and v_{n+3}, v_j, v_{n+4}, for all $j \leq n$. Note that K_4 can be decomposed into three copies of P_3: $1, 2, 3$; $1, 4, 2$, and $1, 3, 4$. Also, we know the complete graphs on n vertices can be decomposed into copies of P_3. Thus, the complete graph on $n+4$ vertices can be decomposed into copies of P_3.

Now, by induction on m, we will prove K_{4m} and K_{4m+1} can both be decomposed into copies of P_3.

Basis step: We will show K_4 and K_5 can be decomposed into copies of P_3. Above, we showed that for K_4.

For K_5, we can decompose the complete graph on [5] into five copies of P_3, two of which are $1, 5, 2$, and $3, 5, 4$ and the other three are obtained from decomposing K_4 onto copies of P_3.

Inductive Step: Assume K_{4m} and K_{4m+1} can both be decomposed into copies of P_3. By what we showed above K_{4m+4} and K_{4m+1+4} can both be decomposed into copies of P_3. This proves the claim for $m + 1$. By induction K_{4m} and K_{4m+1} can both be decomposed into copies of P_3. Therefore, if n or $n-1$ is divisible by 4, then K_n can be decomposed into copies of P_3. □

Example 6.6. For an integer $n > 1$ let G_n be the graph whose vertex set is $[n]$ and that $E(G) = \{\{m, k\} \mid m \neq k, \text{ and } \gcd(m, k) = 1\}$. Prove that G_n is connected, and find the diameter of G_n.

Solution. Note that 1 is connected to all vertices. Thus, for every $1 < j < k$, the path $j, 1, k$ shows $d(j, k) \leq 2$. Thus, diam $(G_n) \leq 2$. The graphs G_2 and G_3 are complete. Therefore, diam $(G_2) =$ diam $(G_3) = 1$. If $n \geq 4$, then 2 and 4 are not adjacent and thus $d(2, 4) > 1$. Therefore

$$\text{diam } G_n = \begin{cases} 2 & \text{if } n \geq 4 \\ 1 & \text{if } n = 2, 3 \end{cases}$$

□

Example 6.7. Let n be a positive integer. What is the maximum number of edges that a disconnected graph of order n can have?

Solution. If G is a disconnected graph of order n, it must have at least two connected components. Let G_1 be a connected component of G with order k and let G_2 be the union of the other connected components. The size of G is the size of G_1 plus the size of G_2. Therefore, the size of G is at most $\binom{k}{2} + \binom{n-k}{2} = k^2 - kn + \frac{n^2-n}{2}$. This is a quadratic in terms of k with vertex at $k = n/2$ which is between 1 and n. Thus the maximum is obtained at $k = 1$ or $k = n - 1$. Both of these values give us $\boxed{\dfrac{n^2 - 3n + 2}{2}}$. □

6.3 Exercises

All students are expected to do all of the exercises listed in the following two sections.

6.3.1 Problems for grading

The following problems must be submitted on Friday, April 3, 2020 before 1 PM EST. The submission will be on Gradescope via Elms. **Late submission will not be accepted.**

Instructions for submission: To submit your solutions please note the following:

- Each problem must go on a separate page.
- It is highly recommended (but not required at the moment) that you LaTeX your homework.
- To submit your homework go to Elms. Hit "GradeScope" on the left panel. That should allow you to upload a PDF file of your homework.

- You could use the (free) DocScan app to scan and upload your homework.
- Sometime in the next week do a test and make sure this all works out so you do not face any issues right before the deadline.
- Homework must be submitted before 1 PM EST on the due date. GradeScope will not allow late submissions.

All proofs must be complete and solutions must be fully justified.

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

Exercise 6.1 (15 pts). *For an integer $n > 1$, let G_n be the graph whose vertices are all positive divisors of n. Two distinct vertices k and ℓ are connected if and only if k divides ℓ or ℓ divides k.*

(a) Draw G_{30} and find its order and size.

(b) Show that for every n there are at least two vertices that are connected to all other vertices. Which vertices are those two vertices?

(c) Prove that G_n is the complete graph if and only if n is a prime power.

Hint: For the third part, use proof by contradiction.

Exercise 6.2 (10 pts). *Let G be a graph of order n and size m. How many vertex induced subgraphs does G have? How many edge-induced subgraphs does G have?*

Exercise 6.3 (10 pts). *Suppose u and v are two vertices of a graph and $u = u_0, u_1, \ldots, u_m = v$ is a uv-geodesic. Prove that $d(u, u_j) = j$.*

Exercise 6.4 (10 pts). *Determine if the following statements are true or false. Fully justify your answers.*

(a) If the order of a connected graph G is at least four, then there are three distinct vertices u, v and w for which $G - u$, $G - v$, and $G - w$ are all connected.

(b) There is a connected graph G that has three u, v and w vertices for which $d(u, v) = d(u, w) = d(v, w) = diam(G) = 3$.

Exercise 6.5 (10 pts). *Let G be a graph, and let u and v be two distinct vertices of G. Prove that $d_G(u, v) > 1$ if and only if $d_{\overline{G}}(u, v) = 1$.*

Exercise 6.6 (10 pts). *Let n be a positive integer. Consider the following statement:*

$P(n)$: There is a connected graph G whose complement $\overline{G}$ is also connected and has four vertices x, y, u, v for which $d_G(u, v) = d_{\overline{G}}(x, y) = n$.

(a) Prove $P(1), P(2)$, and $P(3)$.

(b) Prove that $P(n)$ is false for all $n \geq 4$.

Exercise 6.7 (15 pts). *Prove the following properties of isomorphism.*

(a) $\cong$ is an equivalence relation.

(b) If $G \cong H$, then $\overline{G} \cong \overline{H}$.

(c) If $G \cong H$, then G and H have the same size and the same order.

Exercise 6.8 (10 pts). *Let n be a positive integer. Prove that there is a graph G of order n for which $G \cong \overline{G}$ if and only if n or $n-1$ is divisible by 4.*

Hint: For one direction use the size of G. For the other direction show if there is such a graph of order n, then there is such a graph of order $n+4$, and then use induction.

Exercise 6.9 (10 pts). *For a positive integer n, define a graph G whose vertices are all subsets of $[n]$ and two distinct vertices are adjacent if and only if their intersection has precisely one element.*

(a) Find the order and size of G.

(b) Evaluate $k(G)$.

6.3.2 Problems for Practice

The following problems are from *A First Course in Graph Theory, Gary Chartrand, and Ping Zhang.*

p. 7-8: 3, 5

p. 17-18: 12, 15, 17

Exercise 6.10. *Prove that K_n can be decomposed into three pairwise isomorphic graphs if and only if n or $n-1$ is divisible by 3.*

6.3.3 Challenge Problems

Exercise 6.11. *Let k be a positive integer. $12k$ people have participated in a party in which everyone shakes hands with $6k+3$ other people. We know that the number of people who shake hands with every two people is a fixed number. Find k.*

Exercise 6.12. *Let $1 \leq m < n$ be integers. n vertices numbered $1, 2, \ldots, n$ are placed on the circumference of a circle in that order. Two vertices j and k are connected if and only if j and k are m arcs apart. For example vertex 1 is connected to vertices $m+1$ and $n-m+1$. Find the necessary and sufficient condition for this graph to be a cycle.*

7 Week 8

7.1 Bipartite Graphs

Definition 7.1. A graph G is called **bipartite** if the vertex set $V(G)$ can be partitioned into two subsets X and Y, called **partite sets** for which every edge of G has one endpoint in X and one endpoint in Y.

Remark. Note that since blocks in every partition are non-empty, X and Y in the above definition must be non-empty. This means every bipartite graph must have at least two vertices.

Remark. In the above definition, it may be possible to partition the vertices of a bipartite graph G into partite sets in multiple ways. For example let H_1 and H_2 be two 1-paths on vertex sets $\{v_1, v_2\}$ and $\{w_1, w_2\}$, respectively, and let $G = H_1 \cup H_2$. The sets $X = \{v_1, w_1\}$, and $Y = \{v_2, w_2\}$ are two partite sets of G, and so are the sets $X_1 = \{v_1, w_2\}$, and $Y_1 = \{v_2, w_1\}$.

Example 7.1. Every even cycle is bipartite.

Solution. Let $v_1, v_2, \ldots, v_{2k}$ be an even cycle, where $v_j v_{j+1}$ is an edge for every j, with $v_{2k+1} = v_1$. Then $X = \{v_1, v_3, \ldots, v_{2k-1}\}$ and $Y = \{v_2, v_4, \ldots, v_{2k}\}$ is a partition for the vertices of this cycle and all edges are between a vertex of X and a vertex of Y. □

Example 7.2. Prove that the complete graph K_n is bipartite iff $n = 2$.

Solution. Note that by definition K_2 is bipartite and K_1 is not. Suppose $n \geq 3$ and K_n is bipartite. Then by pigeonhole principle one of the partite sets X or Y has at least 2 distinct elements, say u and v, however uv is an edge, which is a contradiction. □

The following theorem gives a complete classification of all bipartite graphs.

Theorem 7.1. *A nontrivial graph is bipartite iff it does not contain an odd cycle.*

Proof. Suppose G is bipartite with partite sets X and Y. Assume G has an odd cycle of length n. Let the vertex set of this cycle be $\{v_1, v_2, \ldots, v_n\}$ with v_j adjacent to v_{j+1} with $v_{n+1} = v_1$. Suppose $v_1 \in X$, since vertices of X are not adjacent, $v_2 \in Y$, similarly $v_3 \in X$, etc. If n is odd this shows $v_n \in X$, however v_n and v_1 are adjacent, which is a contradiction.

Now, suppose G has no odd cycles. We will prove that G is bipartite. The idea is to start with one vertex and place it in a partite set X, then place its neighbors in a set Y, and then their neighbors in X, and so on. However this process may not reach all vertices if G is not connected, so we will prove this first for connected graphs.

Suppose H is a connected nontrivial graph and no odd cycles. Let u be a vertex of H and let X consist of all vertices v of H for which $d(u, v)$ is even and Y consist of all vertices w of H for which $d(w, u)$ is odd. First, note that X and Y partition $V(H)$. ($u \in X$ and all neighbors of u are in Y, thus X and Y are non-empty. Furthermore, X and Y are disjoint and $X \cup Y = V(H)$.) We will have to show no two vertices in X can be adjacent. Suppose v and w are adjacent vertices in X. We know there are uv- and uw-paths of even length. This along with the edge vw gives a circuit with an odd number of edges. If there are any repeated vertices, we could remove the cycle that is formed to get a new smaller circuit. Since all cycles have an even number of edges, each time we still get a circuit with an odd number of edges. Repeating this we get a cycle with an odd number of edges, which is a contradiction since G has no odd cycles.

Now, suppose G is a nontrivial graph with no odd cycles. Let $G_1, \dots, G_k$ be all connected components of G. If a connected component has order more than 1 it is bipartite by what we showed above. Suppose $G_1, \dots, G_j$ are all connected components with order at least 2, and $G_{j+1}, \dots, G_k$ are the isolated vertices.
If $j \geq 1$, and the partite sets of G_i are X_i and Y_i, then the partite sets of G are $X = X_1 \cup \dots \cup X_j$ and $Y = Y_1 \cup \dots \cup Y_j \cup G_{j+1} \cup \dots \cup G_k$.
If $j = 0$, i.e. G has no connected components with more than one vertex, then G has no edges and thus it is bipartite, with partite sets $X = \{u\}$ and $Y = V(G) - \{u\}$, where u is a vertex of G. □

Definition 7.2. Let r, s be two positive integers. The **complete bipartite graph** $K_{r,s}$ is the bipartite graph whose partite sets X and Y have size r and s, respectively and every vertex in X is adjacent to every vertex in Y. We call the graph $K_{1,s}$ a **star**.

Example 7.3. Find the order and size of $K_{r,s}$.

Solution. The order of $K_{r,s}$ is $r + s$, since it has $r + s$ vertices. Every edge has an endpoint in X and an endpoint in Y, where $|X| = r$ and $|Y| = s$. Thus, there are rs edges, which means the size of $K_{r,s}$ is rs. □

Similar to what we saw above we may define the following.

Definition 7.3. Let $k \geq 2$ be an integer. A graph G is called k**-partite** if the vertex set $V(G)$ can be partitioned into k subsets $X_1, X_2, \dots, X_k$, called **partite sets** for which every edge of G has one endpoint in some X_i and the other endpoint in another X_j where $i \neq j$.

Example 7.4. Any graph of order n is n-partite, by selecting all the partite sets to be singletons.

Definition 7.4. Given positive integers $r_1, r_2, \dots, r_k$, with $k \geq 2$, the **complete** k**-partite graph** $K_{r_1,r_2,\dots,r_k}$ is the k-partite graph with partite sets $X_1, X_2, \dots, X_k$, where $|X_j| = r_j$ for all j and every two vertices in X_j and X_i, where $i \neq j$, are adjacent. A graph that is a complete k-partite graph for some k is called a **complete multipartite graph**.

Example 7.5. K_n is a complete n-partite graph.

The question of classifying all k-partite graphs is a difficult one.

Definition 7.5. Let G and H be two graphs. The **Cartesian product** $G \times H$ is a graph whose vertex set is $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if one of the following occurs:

- $u_1 = u_2$ and $v_1v_2 \in E(H)$, or

- $v_1 = v_2$ and $u_1u_2 \in E(G)$.

Example 7.6. $K_2 \times K_2 \cong C_4$.

Solution. Let the vertex set of K_2 be $\{0,1\}$. The vertex set of $K_2 \times K_2$ will then be $v_1 = (0,0), v_2 = (0,1), v_3 = (1,1)$, and $v_4 = (1,0)$. By definition the edges of $K_2 \times K_2$ are v_jv_{j+1} for all j with $v_5 = v_1$. This is precisely what P_4 is. □

Example 7.7. $K_2 \times K_2 \times K_2$ is isomorphic to the three dimensional cube.

Solution. Similar to above the vertex set of this graph is (a,b,c) with $a,b,c \in \{0,1\}$. Two vertices are connected if and only if they are different at precisely one entry. Drawing the diagram for this we see it is a three dimensional cube. □

Definition 7.6. The graph $K_2^n = \underbrace{K_2 \times \cdots \times K_2}_{n \text{ times}}$, which is the Cartesian product of n copies of K_2 is called the n-dimensional **hypercube**.

Example 7.8. The n-dimensional hypercube is isomorphic to the graph G whose vertex set is the set of all sequences of length n whose terms are 0 and 1, and two vertices are adjacent if and only if they differ at a single term.

Theorem 7.2. *If G and H are bipartite, then $G \times H$ is also bipartite.*

Proof. Let X, X' and Y, Y' be partite sets of G and H, respectively. We know the vertex set of $G \times H$ is

$$(X \cup X') \times (Y \cup Y') = (X \times Y) \cup (X \times Y') \cup (X' \times Y) \cup (X' \times Y').$$

We will show that $U = (X \times Y) \cup (X' \times Y')$ and $W = (X \times Y') \cup (X' \times Y)$ are partite sets of $G \times H$. By symmetry, we only need to prove vertices of U are not adjacent to one another. On the contrary suppose $(u_1,u_2), (v_1,v_2) \in U$ are adjacent. By symmetry, there are two cases.
Case I: $(u_1,u_2), (v_1,v_2) \in X \times Y$, which implies $u_1, v_1 \in X$. Since X is a partite set of G, the vertices u_1 and u_2 are not adjacent in G. Therefore, $u_1 = v_1$. Similarly $u_2 = v_2$, which is a contradiction.
Case II: $(u_1,u_2) \in X \times Y, (v_1,v_2) \in X' \times Y'$. Note that $u_1 \in X$ and $v_1 \in X'$, which implies $u_1 \neq v_1$. Similarly, $u_2 \neq v_2$, which is a contradiction since $(u_1,u_2), (v_1,v_2) \in U$ are assumed to be adjacent in $G \times H$.

Therefore, $G \times H$ is bipartite. □

7.2 Degrees

Definition 7.7. Let v be a vertex of a graph G, the **degree** of v, denoted by $\deg(v)$, is the number of vertices that are adjacent to v. The smallest degree and the largest degree in a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

Theorem 7.3 (Handshaking Lemma, or the First Theorem of Graph Theory)**.** *In every graph the sum of degrees is equal to twice the size of the graph.*

Proof. Each vertex v is incident to $\deg(v)$ edges. Therefore, in total all vertices are incident to $\sum\limits_{v\in V(G)} \deg(v)$ edges. However, each edge is counted twice, because each edge has two endpoints. Therefore, $\sum\limits_{v\in V(G)} \deg(v) = 2m$, where m is the size of G. □

Definition 7.8. A vertex is called an **odd vertex** if its degree is odd. It is called an **even vertex** if its degree is even.

Corollary 7.1. Every graph has an even number of odd vertices.

Proof. Note that the sum of degrees is even. Thus, there must be an even number of vertices whose degrees are odd. Thus, every graph has an even number of odd vertices. □

Theorem 7.4. *If G is a graph of order n for which*

$$\deg u + \deg v \geq n - 1$$

for every two non-adjacent vertices, then G is connected and $\operatorname{diam} G \leq 2$. *Consequently, if $\delta(G) \geq (n-1)/2$, then G is connected and* $\operatorname{diam} G \leq 2$.

Proof. Let u and v be two distinct vertices of G. If u and v are adjacent, then $d(u, v) = 1$. Otherwise, let A and B be the sets of neighbors of u and v, respectively. By assumption $|A| + |B| \geq n - 1$. Since neither u nor v is in A or B, $A \cup B$ has at most $n - 2$ elements. Thus, by pigeonhole principle there is a vertex w that belongs to both A and B. Therefore, u, w, v is a uv-path. Thus, $d(u, v) = 2$. Therefore $\dim G \leq 2$. The second part follows from the fact that $\deg u + \deg v \geq 2\delta(G) \geq n - 1$. □

7.3 More Examples

Example 7.9. How many $4-$cycle subgraphs does the graph $K_{4,5}$ have?

Solution. We need to choose 2 vertices from each of the partite sets. This can be done in $\binom{4}{2}\binom{5}{2} = 60$ ways. These four points give us only one 4-cycle. So, the answer is $\boxed{60}$. □

Example 7.10. Let $n \geq 4$ be an integer. How many subgraphs isomorphic to $K_{1,3}$ does the complete graph K_n have?

Solution. To obtain a subgraph isomorphic to $K_{1,3}$ we need to first select 1 vertex for one partite set and 3 of the remaining for the other partite set. Once the partite sets are selected the subgraph $K_{1,3}$ is uniquely determined. This can be done in $\binom{n}{1}\binom{n-1}{3}$ ways. □

Example 7.11. Find the necessary and sufficient condition for the two graphs G and H for which the graph $G \times H$ is a complete graph.

Solution. Suppose $G \times H$ is a complete graph. Let u, v be two vertices of G and x, y be two vertices of H. We know (u, x) and (u, y) are vertices of $G \times H$. If $x \neq y$, then $(u, x) \neq (u, y)$. Since $G \times H$ is a complete graph, xy must be an edge in H. Therefore, H must be a complete graph. Similarly G must be a complete graph. Furthermore, if $u \neq v$ and $x \neq y$, then the vertices (u, x) and (v, y) are not adjacent. Thus, for $G \times H$ to be complete we need to have $|V(G)| = 1$ or $|V(H)| = 1$. So far we showed that if $G \times H$ is complete, then both G and H are complete and one of them must have order 1. We will prove this is a sufficient condition as well. Suppose G and H are complete graphs and G has order 1. Let $V(G) = \{u\}$. Every vertex of $G \times H$ is of form (u, x) with $x \in V(H)$. Since H is a complete graph, all disticnt vertices of form (u, x) are adjacent, and thus $G \times H$ is a complete graph. □

Example 7.12. For every positive integer n let $P(n)$ be the following statement:

If a graph G satisfies $\deg u + \deg v \geq n - 2$ for all vertices $u \neq v$, then G is connected.

Find all values of n for which $P(n)$ is true.

Solution. For $n = 1$, the only graph is K_1 which is connected. For $n \geq 2$, take $G = K_1 \sqcup K_{n-1}$. $\deg u = n-2$ for every vertex u of K_{n-1}. Thus, for every two distinct vertices u and v, we have $\deg u + \deg v \geq n - 2$. However G is disconnected. Thus, $P(n)$ is true if and only if $n = 1$. □

Example 7.13. Assume a graph of order n and size $2n$ has the property that all vertices have degree either 3 or 4. Prove that G is regular.

Solution. Let x and y be the number of vertices of degree 3 and 4, respectively. By assumption $x + y = n$. By Handshaking Lemma we have $3x + 4y = 2(2n) = 4n$. Substituting we obtain $4n = 3(x + y) + y = 3n + y$, and hence $y = n$, which implies $x = 0$. Therefore, G is 4-regular. □

Example 7.14. Find $\deg_{G\times H}(u, v)$ in terms of $\deg_G u$ and $\deg_H v$.

Solution. (u, v) is connected to all vertices of form (x, v) and (u, y), where x is adjacent to u in G and y is adjacent to v in H. Therefore, $\deg_{G\times H}(u, v) = \deg_G u + \deg_H v$. □

Example 7.15. Let $n \geq 2$ be an integer. What is the maximum size of a bipartite graph of order n?

Solution. Suppose G be a bipartite graph of order n whose partite sets have sizes a and $n-a$. The maximum size of G is $a(n - a) = an - a^2$. This is a quadratic of a whose vertex is at $a = n/2$. So, if n is even it is maximized at $n/2$. Otherwise it is maximized at $a = (n \pm 1)/2$. Therefore, if n is even the answer is $n^2/4$ and if n is odd the answer is $(n^2 - 1)/4$. □

7.4 Exercises

All students are expected to do all of the exercises listed in the following two sections.

7.4.1 Problems for grading

The following problems must be submitted on Saturday, April 11, 2020 before 1 PM EST. The submission will be via Gradescope on Elms. **GradeScope will not accept late submissions.**

Instructions for submission: To submit your solutions please note the following:

- Each problem must go on a separate page.
- It is highly recommended (but not required at the moment) that you LaTeX your homework.
- To submit your homework go to Elms. Hit "GradeScope" on the left panel. That should allow you to upload a PDF file of your homework.
- You could use the (free) DocScan app to scan and upload your homework.
- Sometime in the next week do a test and make sure this all works out so you do not face any issues right before the deadline.
- Homework must be submitted before 1 PM EST on the due date. GradeScope will not allow late submissions.

All proofs must be complete and solutions must be fully justified.

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

Exercise 7.1 (10 pts)**.** *Let G and H be two nontrivial graphs, for which $G \times H$ is bipartite. Prove that G and H are both bipartite. (Compare this to Theorem 7.2.)*

Exercise 7.2 (10 pts)**.** *Given positive integers $r_1, r_2, \ldots, r_k$, with $k \geq 2$, find the order and size of $K_{r_1, r_2, \ldots, r_k}$.*

Exercise 7.3 (10 pts)**.** *In a certain graph of size 10 we know each vertex degree is either 4 or 5. Prove that there is only one such graph and find this graph.*

Exercise 7.4 (15 pts)**.** *Let n be a positive integer.*

(a) Prove that if G is a graph of order n such that $\delta(G)+\Delta(G) \geq n-1$, then G is connected and $\operatorname{diam} G \leq 4$.

(b) For every $n \geq 4$, give an example of a disconnected graph G of order n for which $\delta(G) + \Delta(G) = n - 2$. (This shows the bound $n - 1$ cannot be improved.)

(c) For every $n \geq 7$, give an example of a graph with $\delta(G) + \Delta(G) \geq n - 1$ and $\operatorname{diam} G = 4$. *(This shows the inequality* $\operatorname{diam} G \leq 4$ *cannot be improved.)*

Exercise 7.5 (10 pts)**.** *A nontrivial graph G has the property that every edge of G connects an even vertex to an odd vertex. Prove that G is bipartite and has even size.*

Exercise 7.6 (10 pts)**.** *Let $2 \leq k \leq n$ be integers. How many subgraphs of $K_{n,n}$ are $2k$-cycles?*

Exercise 7.7 (10 pts)**.** *Suppose $n \geq 5$ is an integer. Prove that if G is a graph of order n, then either G or $\overline{G}$ is not bipartite. By an example show this statement is not true for $n = 4$.*

7.4.2 Problems for Practice

The following problems are from *A First Course in Graph Theory, Gary Chartrand, and Ping Zhang.*

p. 36-38: 2, 3, 6, 8, 16

p. 42-43: 21, 27, 29, 30

p. 47: 32, 34

7.4.3 Challenge Problems

Exercise 7.8. *Let* $0 \le b < a$*, and* $0 < k < n$ *be four integers. Find the necessary and sufficient condition on* a, b, k, n *for which the sequence* $\underbrace{a, a \ldots, a}_{k\ times}, \underbrace{b, b, \ldots, b}_{n-k\ times}$ *is graphical.*

8 Week 9

8.1 Regular Graphs

Definition 8.1. A graph G is called **regular** if $\delta(G) = \Delta(G)$. In other words, a regular graph is a graph whose vertices all have the same degree. A graph G is called r**-regular** if $\delta(G) = \Delta(G) = r$.

Example 8.1. Let n be a positive integer.

- C_n is a 2-regular graph for all $n \ge 3$.
- K_n is an $(n-1)$-regular graph.

Make sure you check the **Petersen graph** on page 39.
Another way of looking at the Petersen graph is the following: Let P be a graph whose vertex set is the set of all $2-$subsets of [5]. Two vertices are connected if and only if they are disjoint. This graph is the Petersen graph.

Theorem 8.1. *Let* r, n *be two integers satisfying* $0 \le r \le n-1$*. There exists an* r*-regular graph of order* n *if and only if* rn *is even.*

Proof. First assume there is an r-regular graph of order n. The degree sum of this graph is rn, since each vertex has degree r and there are n vertices. Thus, by the Handshaking Lemma rn must be even.

Now, assume rn is even. This means r or n is even.

Suppose $r = 2k$ is even. Place all the vertices $v_1, v_2, \ldots, v_n$ around a circle and connect every vertex v_j to $2k$ vertices $v_{j\pm1}, \ldots, v_{j\pm k}$ before and after v_j, taking each index mod n, if necessary. This yields an r-regular graph of order n.

Suppose $r = 2k+1$ is odd and $n = 2\ell$ is even. Then similar to the previous case, place $v_1, v_2, \ldots, v_{2\ell}$ around a circle. Connect every v_j to $2k$ vertices $v_{j\pm 1}, \ldots, v_{j\pm k}$ before and after v_j, again taking each index mod n if necessary. Also, connect v_j to $v_{j+\ell}$, for every j. Note that since $r < n$, $k < \ell$, and thus $v_{j+\ell}$ is a new neighbor of v_j. Furthermore, with this method $v_{j+\ell}$ will be connected back to $v_{j+\ell+\ell} = v_{j+2\ell} = v_j$. This yields an r-regular graph of order n. □

Definition 8.2. The r-regular graphs of order n defined in the proof of the previous theorem are called **Harary graphs** and are denoted by $H_{r,n}$.

The following theorem shows that every graph can be viewed as an induced subgraph of a regular graph.

Theorem 8.2. *Let G be a graph and r be an integer satisfying $r \geq \Delta(G)$. Then, there is an r-regular graph H for which G is an induced subgraph of H.*

The idea is to place a copy of G' of G next to itself and connect each vertex v to its corresponding vertex v' if $\deg v < r$. Repeat this process and get a regular graph. Each time we are reducing the difference between r and the degrees of the vertices. Thus, we can write down the proof as follows:

Proof. We will prove this by induction on $r - \delta(G)$. If $r - \delta(G) = 0$, then $\Delta(G) \leq r = \delta(G) \leq \Delta(G)$, and thus G itself is r-regular. Therefore, $H = G$ works.

Now suppose G is a graph with $r - \delta(G) = n$ a positive integer. Consider the graph $G' \cong G$ with vertex set $\{v' \mid v \in G\}$ for which $u'v'$ is an edge in G' if and only if uv is an edge of G. To the graph $G \sqcup G'$ add all edges uu' for all u with $\deg u < r$. This gives us a new graph H for which $\deg_H u = \deg_H u' = r$ or $\deg_H u = \deg_H u' = \deg_G u + 1$. Thus, $\delta(H) = \delta(G) + 1$. Therefore, $r - \delta(H) = n - 1$. By inductive hypothesis H is a subgraph of an r-regular graph. Since G is an induced subgraph of H, we are done. □

8.2 Degree Sequence

In this section we will answer the following question:

Question 1. Given a list of non-negative integers, under what conditions does there exist a graph whose vertex degrees are the given list?

Definition 8.3. A list of all vertex degrees of a graph G is called its **degree sequence**. A sequence s of integers is called **graphical** if there is a graph whose degree sequence is s.

Example 8.2. Determine if each of the following sequences is graphical:

(a) 3, 2, 2, 1, 1.

(b) 4, 3, 1, 1, 1.

The above example shows that the answer to Question 1 is not simple. This question can be answered recursively as follows:

Theorem 8.3. *(a) A decreasing sequence of non-negative integers $s : d_1 \geq d_2 \geq \cdots \geq d_n$, with $n \geq 2$ and $d_1 \geq 1$ is graphical if and only if the sequence*

$$s_1 : d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$$

is graphical.

(b) The sequence $\underbrace{1, 1, \ldots, 1}_{k \text{ times}}, \underbrace{0, 0 \ldots, 0}_{\ell \text{ times}}$ is graphical if and only if k is even.

Proof. (a) See page 45.
(b) If $k = 2r$ then the graph obtained by taking the union of r copies of P_2 and ℓ copies of P_1 has the given degree sequence. If k is odd, then this sequence cannot be graphical since by Corollary 7.1 in every graph the number of odd vertices must be even. □

Example 8.3. Check if each sequence is graphical. If it is create a graph whose degree sequence is the given sequence.

(a) $4, 3, 3, 1, 1, 0$

(b) $4, 2, 2, 2, 1, 1$.

For more examples see pages 46-47.

Example 8.4. Prove that the sequence $s : d_1, d_2, \ldots, d_n$ is graphical if and only if

$$s_1 : n - 1 - d_1, n - 1 - d_2, \ldots, n - 1 - d_n$$

is graphical.

Solution. Hint: Show that if G has degree sequence s, then $\overline{G}$ has degree sequence s_1, and vice-versa.

8.3 Matrices and Graphs

Definition 8.4. Let G be a graph of order n and size m with $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. An **adjacency matrix** of G is an $n \times n$ matrix whose (i, j) entry is 1 if $v_i v_j \in E(G)$ and zero otherwise. An **incidence matrix** of G is an $n \times m$ matrix whose (i, j) entry is 1 if v_i is incident with e_j, and zero otherwise.

Example 8.5. The adjacency matrix A of K_n is an $n \times n$ matrix with zero on its diagonal entries and 1 elsewhere.

$$A = \begin{pmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ & & \ddots & & \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}$$

An incidence matrix of K_4 is a 4×6 matrix M shown below:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

In writing this matrix we have set $e_1 = v_1v_2, e_2 = v_1v_3, e_3 = v_1v_4, e_4 = v_2v_3, e_5 = v_2v_4$, and $e_6 = v_3v_4$. Note that changing the order of the edges could yield different incidence and adjacency matrices.

Theorem 8.4. *Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$ and A as an adjacency matrix for G as described in the definition above. Then, for every positive integer k, the (i, j) entry of A^k is the number of v_iv_j walks of length k in G.*

Proof. We prove this by induction on k. For $k = 1$, the (i, j) entry of A shows if there is an edge between v_i and v_j or not. Since the only walk of length 1 is an edge, we are done.

Assume the (i, j) entry of A^k is the number of v_iv_j-walks of length k. Let the (i, j) entries of A and A^k be a_{ij} and b_{ij}, respectively. The (i, j) entry of $A^{k+1} = AA^k$ is $\sum_{\ell=1}^{n} a_{i\ell}b_{\ell j}$. This is the sum of $b_{\ell j}$'s for those ℓ's that v_ℓ is a neighbor of v_i. Note that each v_iv_j-walk of length $k + 1$ is an edge v_iv_ℓ followed by a $v_\ell v_j$-walk of length k. Thus, the number of v_iv_j-walks is the sum of $b_{\ell j}$'s for those ℓ's for which v_ℓ is a neighbor of v_i. This proves the claim by induction. □

8.4 Bridges

Think about a bridge as a road that *must* be crossed when going from one part of a graph to another part of the graph. To be more precise, we define a bridge as follows:

Definition 8.5. Let e be an edge in a graph G. We say e is a **bridge** if $G - e$ has more connected components than G.

Theorem 8.5. *Let G be a graph whose connected components are $G_1, G_2, \ldots, G_n$. Suppose $e = uv$ is a bridge. Assume e is an edge of G_1. Then $G_1 - e$ has two connected components H and K where u is in H and v is in K and $G_1 = (H \sqcup K) + e$. Furthermore, $k(G - e) = k(G) + 1$.*

Proof. Removing e leaves $G_2, \ldots, G_n$ intact. If $G_1 - e$ has more than two connected components, there must be at least two edges connecting these components, which is a contradiction. Thus, $G_1 - e$ must have two connected components. Therefore, $k(G - e) = k(G) + 1$. Also, $G_1 - e$ has two connected components H and K and since G_1 is connected, the edge e must be between H and K, as desired. □

Example 8.6. In any path graph P_n, every edge is a bridge. In any cycle graph C_n, no edge is a bridge.

Theorem 8.6. *An edge e in a graph G is a bridge if and only if e does not belong to any cycles of G.*

Proof. Suppose an edge $e = uv$ does not belong to any cycles of G. We will show that e is a bridge. Assume to the contrary $k(G) = k(G - e)$. This means u and v are in the same connected component in $G - e$. Thus, there is a uv-path, say P, in $G - e$. This path along with the edge $e = uv$ gives us a cycle, which means e belongs to a cycle, which is a contradiction.

Suppose $e = uv$ belongs to a cycle C. We will show connected components of G are also connected in $G - e$. Suppose x and y are two vertices in G for which there is a xy-path. If this path does not use e, then it is also a path in $G - e$. If it does use e, then replacing e by $C - e$ we get an xy-walk in $G - e$, which means x and y are in the same connected component of $G - e$. Therefore, e is not a bridge. □

8.5 Trees

Definition 8.6. A graph G is called **acyclic** or a **forest** if it has no cycles. A **tree** is an acyclic connected graph.

Remark. Note that a graph is a forest if and only if all of its connected components are trees.

Example 8.7. For positive integers m and n,

- P_n is a tree.
- $P_n \sqcup P_m$ is acyclic but not a tree.

Definition 8.7. A **leaf** or an **end-vertex** in a graph is a vertex of degree 1.

Definition 8.8. A **caterpillar** is a tree of order 3 or more, the removal of whose leaves produces a path called the **spine** of the caterpillar.

Example 8.8. All paths and stars of order at least 3 are caterpillars.

The following theorem suggests a categorization of all trees.

Theorem 8.7. *A graph T is a tree if and only if every two vertices of T are connected by a unique path.*

Proof. Suppose every two vertices of T are connected by a unique path. Then, T is connected. If T has a cycle, then between every two distinct vertices of that cycle there are at least two paths. This is a contradiction. Thus, T is connected and acyclic, and hence it is a tree.

Suppose T is a tree, but there are two distinct paths P and Q connecting vertices u and v. Since $P \neq Q$, one of them has an edge that the other one does not. Let $e = xy$ be an edge in P that is not in Q. Assume in the uv-path P, u comes before x, which comes before y, and that comes before v. We will show that e is not a bridge, and hence by Theorem 8.6, e must belong to a cycle, which yields to a contradiction since T is a tree. Assume e is a bridge. By Theorem 8.5, u and v must belong to different connected components of $T - e$. By assumption Q is a uv-path in $T - e$; part of P from u to x is a ux-path in $T - e$; and part of P from y to v in $T - e$. Thus, x and y are in the same connected components of $T - e$. This contradiction completes the proof. □

Theorem 8.8. *Every nontrivial tree has at least two leaves.*

Proof. Let P be a longest path in the tree. Suppose u and v are the endpoints of P. We claim that u and v are leaves. Note that u has no neighbor that is not a vertex of P, otherwise that neighbor along with P gives a longer path. Also, since the tree has no cycles, u can not be adjacent to any vertex of P other except for one. Thus, the degree of u is 1 and hence u is a leaf. Similarly v is also a leaf. This completes the proof. □

Theorem 8.9. *If u is a leaf of a tree T with at least two vertices, then $T - u$ is a tree.*

Proof. Exercise! □

Theorem 8.10. *Every tree of order n has size $n - 1$.*

Proof. Let T be a tree of order n. We will prove by induction on n that the size of T is $n - 1$. For $n = 1$, T is the trivial graph and thus its size is 0, as desired. Suppose T is a tree of order $n + 1$. By Theorem 8.8, T has a leaf u. By Theorem 8.9, $T - u$ is a tree of order n. By inductive hypothesis $T - u$ has $n - 1$ edges, and thus T has n edges, as desired. □

8.6 More Examples

Example 8.9. Prove that every nontrivial graph that is a forest is bipartite.

Solution. Note that every forest has no cycles. Thus, it doesn't have any odd cycles. By Theorem 7.1 it is bipartite. □

Example 8.10. For an integer n, let G_n be the graph whose vertices are all subsets of $[n]$. Two vertices are adjacent if one is a subset of the other. Find the degree sequence of G_n.

Example 8.11. Let u, v be two distinct vertices of a graph G. Suppose P is a uv-path in G. Prove that P is the unique uv-path in G if and only if every edge of P is a bridge.

Solution. Suppose P is the unique uv-path and $e = xy$ is an edge of P for which u, x, y, v appear in P in that order (or $x = u$ or $y = v$). Assume to the contrary that e is not a bridge. This means e lies in a cycle C. Note that part of P from u to x along with $C - e$ and part of P from y to v gives a uv-walk in the graph $G - e$. Thus, by a theorem there is a uv-path Q in $G - e$. Since e is an edge in P but not in Q, we have $P \neq Q$, which contradicts the uniqueness of the uv-path P.

Suppose P and Q are two distinct uv-paths. Suppose to the contrary every edge of P is a bridge. Let P be $u = u_0, u_1, \ldots, u_n = v$ and let $e_j = u_j u_{j-1}$. Since e_0 is a bridge, $G - e_0$ is disconnected, which means Q must contain the edge e_0, but since the first vertex in Q is u_0, the second vertex in Q must be u_1. Similarly, since $G - e_1$ must be disconnected, Q must contain e_1, and thus $u_1 = v_1$. This proves $u_j = v_j$ and thus $P = Q$, a contradiction. □

Example 8.12. Let A be an adjacency matrix of a graph G relative to the sequence of vertices $v_1, v_2, \ldots, v_n$. Prove the following:

(a) A vertex v_i is isolated if and only if $(A^k)_{ii} = 0$ for all $k \geq 1$.

(b) The diagonal entries of A^2 are $\deg v_1, \deg v_2, \ldots, \deg v_n$.

(c) $(A^{2k-1})_{ii} = 0$ for all integers $i, k \geq 1$ if and only if G is bipartite.

Solution. (a) Suppose a v_i is isolated. This means there are no walks of positive length from v_i to any other vertices. Thus, the i-th row of A^k is zero for all $k \geq 1$. In particular $(A^k)_{ii} = 0$.
Now, note that if v_i is not isolated, then $v_i v_j$ is an edge for some j. Thus, $v_i v_j v_i$ is a $v_i v_i$-walk of length 2. Thus, $(A^2)_{ii} \geq 2$, as desired.
(b) For every vertex u, a uu-walk of length 2 must be of form u, v, u, where uv is an edge. Thus, there are precisely $\deg u$ of these walks. Which establishes the result.
(c) Suppose $(A^{2k-1})_{ii} = 0$ for all $i, k \geq 1$. This means there are no $v_i v_i$-walks of odd length. In particular there are no cycles of odd length that have v_i as a vertex. Since this is true for all i, G has no odd cycles and thus G is bipartite.
Now, suppose $(A^{2k-1})_{ii} \neq 0$ for some integers i and k. Assume on the contrary G is bipartite. Since $(A^{2k-1})_{ii} \neq 0$, there is a $v_i v_i$-walk of odd length. Suppose C is the shortest $v_i v_i$-walk of odd length. If this walk has a repeated vertex other than v_i, then it must contain a cycle D. Since G is bipartite, D must be an even cycle. Removing D from C we get a shorter $v_i v_i$-walk with odd edged, which is a contradiction. Therefore, G is not bipartite. □

Example 8.13. Prove that every tree has more leaves than vertices of degree more than 2.

Solution. Let n be the order of a tree T, a be the number of leaves, b be the number of vertices of degree 2, and c the number of vertices of degree at least 3. We know $a + b + c = n$. By Handshaking Lemma and Theorem 8.10 we have $2(n-1) \geq a + 2b + 3c$. Therefore, $2n - 2 \geq a + 2(n - a - c) + 3c = 2n - a + c$. This implies $a \geq c + 2$ and thus $a > c$, as desired. □

Example 8.14. Find all positive integers a and b for which there is a tree whose degree sequence is

$$s : \underbrace{2, \ldots, 2}_{a \text{ times}}, \underbrace{1, \ldots, 1}_{b \text{ times}}$$

Solution. Let G be a graph whose degree sequence is s. By Theorem 8.10, and the Handshaking Lemma we must have $2(a + b - 1) = 2a + b$. This implies $b = 2$. We claim $b = 2$ is also a sufficient condition. Note that the graph P_{a+2} has degree sequence s, where $b = 2$. □

8.7 Exercises

All students are expected to do all of the exercises listed in the following two sections.

8.7.1 Problems for grading

The following problems must be submitted on Friday, April 17, 2020 at the beginning of the class. **Late submission will not be accepted.**

All proofs must be complete and solutions must be fully justified.

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

Exercise 8.1 (10 pts)**.** *Determine if each sequence is graphical. If it is, create a graph whose degree sequence is the given sequence.*

(a) 5, 4, 4, 3, 2, 2

(b) 6, 3, 3, 3, 2, 2, 1, 0

Exercise 8.2. *(10 pts) Let M be an incidence matrix of a graph G. Relate the matrix MM^T with an adjacency matrix of G.*

(Remember that trying out some examples always helps.)

Exercise 8.3. *(10 pts) let G be the complete bipartite graph $K_{r,s}$ with partite sets $U = \{u_1, u_2, \ldots, u_r\}$, and $W = \{w_1, w_2, \ldots, w_s\}$ and let A be the adjacency matrix of G relative to the ordered vertex set*

$$\{u_1, u_2, \ldots, u_r, w_1, w_2, \ldots, w_s\}.$$

Using Theorem 8.4 find a formula for A^k for every positive integer k.

Exercise 8.4. *(10 pts) Prove that if v is a leaf of a tree T of order ≥ 2, then $T - v$ is a tree.*

Exercise 8.5. *(10 pts) Suppose T is a tree with precisely two leaves. Prove that T is a path graph.*

Exercise 8.6. *(20 pts) For every tree T with n vertices let*

$$D(T) = \sum_{u,v \in V(T)} d(u,v),$$

where the distance between every pair of vertices appears in the above sum exactly once. (i.e. there are precisely $\binom{n}{2}$ terms in the sum for $D(T)$.) Prove that

(a) $D(T) \geq (n-1)^2$, and that the equality occurs only when $T = K_{1,n-1}$ if $n \geq 2$ or $T = K_1$ if $n = 1$.

(b) $D(T) \leq \binom{n+1}{3}$, and that the equality occurs only when $T = P_n$.

(Hint: Use induction.)

Exercise 8.7. *(20 pts) Let $2 \leq d < n$ be integers.*

(a) Suppose there is a tree of order n all of whose vertex degrees are either 1 or d. Prove that $d - 1$ must divide $n - 2$.

(b) Prove that if $d - 1$ divides $n - 2$, then there is a tree of order n all of whose vertices are of degree 1 or d.

Exercise 8.8 (10 pts)**.** *Suppose G is a connected graph that is not regular. Prove that G has two adjacent vertices u and v for which $\deg u \neq \deg v$.*

8.7.2 Practice Problems

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Exercise 8.9. *Let $s : d_1, d_2, \ldots, d_n$ be a sequence of positive integers with $n \geq 2$. Prove that there is a tree whose degree sequence is s if and only if $\sum_{k=1}^{n} d_k = 2n - 2$.*

Solution. Suppose T is a tree with degree sequence s. By Handshaking Lemma, $\sum d_j$ is twice the size of T. Since T is a tree of order n, its size must be $n - 1$. Thus, $\sum d_j = 2(n - 1)$, as desired.

Suppose $\sum d_j = 2n - 2$. We will prove by induction on n that there is a tree whose degree sequence is s. For $n = 2$, we have $d_1 + d_2 = 2$, which implies $d_1 = d_2 = 1$. The path P_2 is the desired tree. Now, suppose $\sum d_j = 2n - 2$ for some $n > 2$. Note that at least one d_j must be 1, because if all of them are more than 2, then their sum is at least $2n$, which is a contradiction. Suppose $d_1 = 1$. Also, note that at least one of d_j's is more than 1. Otherwise, $\sum d_j = n$ which is less than $2n - 2$, since $n > 2$. Suppose $d_2 > 1$. Now, consider the sequence $s_1 : d_2 - 1, d_3, d_4, \ldots, d_n$. The sum of the terms is $2n - 2 - 2 = 2(n - 1) - 2$. All terms are positive integers. Thus, by inductive hypothesis there is a tree S for which the degree sequence of S is s_1. Add a new vertex w to S and connect it to the vertex whose degree is $d_2 - 1$. This yields a tree whose degree sequence is s. □

8.7.3 Challenge Problems

Exercise 8.10. *Let G be a bipartite k-regular graph for some $k \geq 2$. Prove that G does not have any bridge.*

9 Week 10

Theorem 9.1. *Every forest of order n with k connected components has size $n - k$.*

Proof. Suppose F is a forest of order n with connected components $T_1, \ldots, T_k$. Let the order of T_j be n_j. By Theorem 8.10 the size of T_j is $n_j - 1$. Therefore, the size of F is $\sum_{j=1}^{k} (n_j - 1)$, which is equal to $n - k$, since $\sum_{j=1}^{k} n_j = n$. □

Theorem 9.2. *Every graph with k connected components has a spanning subgraph that is a forest with k connected components. Therefore, every graph of order n with k connected components has at least $n - k$ edges.*

The idea of the proof is that if the graph is not a forest, we remove edges that belong to cycles until we get a forest. This can be presented in two different ways. The first proof is constructive, meaning that it gives you an algorithm for finding the desired forest. The second proof is non-constructive but such proofs are often shorter and more rigorous.

Constructive Proof. Let G be a graph with k connected components. If G is acyclic, then it is a forest and G is the desired subgraph of G. Otherwise, G has a cycle. Let e_1 be an edge of G that belongs to a cycle. By Theorem 8.6, e_1 is not a bridge. Thus, $k(G - e_1) = k$. If $G - e_1$ is a forest, then $G - e_1$ is the desired subgraph of G, otherwise let e_2 be an edge of $G - e_1$ that belongs to a cycle of $G - e_1$. By Theorem 8.6, $k(G - e_1 - e_2) = k$. Repeating the same argument, we obtain the desired forest. □

Non-constructive Proof. Among all spanning subgraphs of G with k connected components, let H be one with the smallest size. We will prove that H is a forest. On the contrary assume H contains a cycle, and let e be an edge of H that belongs to a cycle. By Theorem 8.6, e is not a bridge, which implies $k(H - e) = k$. This contradicts the choice of H, since the size of $H - e$ is smaller than the size of H. □

Corollary 9.1. The size of every connected graph of order n is at least $n - 1$.

Theorem 9.3. *Let G be a connected graph and H be an acyclic subgraph of G. Then, there is a spanning tree for G containing H.*

Proof. The proof is similar to the proof of the previous theorem. The only difference is that at every step we would need to avoid all edges of H. □

Definition 9.1. Let G be a graph. A spanning subgraph H of G is called a **spanning tree** if H is a tree.

Theorem 9.4. *Let G be a graph of order n and size m. If G satisfies any two of the following properties, then G satisfies the third property as well, and thus G is a tree.*

(a) G is connected.

(b) G is acyclic.

(c) $m = n - 1$.

Proof. See page 91. □

Theorem 9.5. *Let T be a tree of order k. If G is a graph with $\delta(G) \geq k - 1$, then T is isomorphic to some subgraph of G.*

Proof. We will prove this by induction on order of T. If T has order 1, then it is the trivial graph which is a subgraph of every graph.

Suppose T is a tree of order $k + 1$ and G is a graph with $\delta(G) \geq k$. Let u be a leaf of T. By Theorem 8.9, $T - u$ is a tree of order k. By inductive hypothesis $T - u$ is isomorphic to a subgraph H of G. Suppose uv is an edge of T and v' is the vertex in H corresponding to v under an isomorphism from T to H. Note that $\deg_G v' \geq \delta(G) \geq (k + 1) - 1 = k$. Since the order of H is k, the degree of v' is at least k, and v' is not a neighbor of itself, there is a neighbor of v', say w, that is not a vertex of H. Thus, $H + v'w$ is a subgraph of G that is isomorphic to T. □

Example 9.1. Show that $k-1$ in the above theorem is sharp.

Solution. Consider the tree $T = K_{1,k-1}$ where $k \geq 2$ is an integer. If $\delta(G) < k-1$, then G has no vertices of degree at least $k-1$, which implies T is not isomorphic to a subgraph of G. □

Definition 9.2. Let G be a graph. A **weight** for G is a function $w : E(G) \to \mathbb{R}$. A graph equipped with a weight is called a **weighted graph**. The weight of a graph G, denoted by $w(G)$ is evaluated by taking the sum of all weights of edges of G.

$$w(G) = \sum_{e \in E(G)} w(e).$$

Definition 9.3. Let G be a weighted connected graph. A **minimum spanning tree** of G is a spanning tree of G whose weight is the smallest among all spanning trees of G.

The following theorem provides an algorithm for finding a minimum spanning tree.

Theorem 9.6 (Kruskal's Algorithm)**.** *Let G be a connected weighted graph of order $n \geq 2$. Let the sequence of edges e_k be defined recursively as follows:*

- *e_1 is one of the edges of G with minimum weight.*
- *For every $k \leq n-1$, let e_k be an edge of G other than $e_1, e_2, \ldots, e_{k-1}$ for which the subgraph of G induced on edges $e_1, e_2, \ldots, e_k$ is acyclic and $w(e_k)$ is minimum among all such edges.*

Then the subgraph of G induced on edges $e_1, e_2, \ldots, e_{n-1}$ is a minimum spanning tree of G.

Proof. First note that by Theorem 9.3, as long as $k \leq n-1$, there is such an edge e_k satisfying the second condition above. Let T be the graph induced on edges $e_1, e_2, \ldots, e_{n-1}$. Since T is acyclic, its size is $n-1$,by Theorem 9.4 the graph whose vertices is $V(G)$ and whose edge set is $\{e_1, e_2, \ldots, e_{n-1}\}$ is a tree. Therefore, T is a spanning tree of G. We will now show T is a minimum spanning tree.

Suppose on the contrary T is not a minimum spanning tree of G. Among all minimum spanning trees of G, let H be a minimum spanning tree of G that has the largest number of edges in common with T. Suppose e_k is the first edge of T that is not an edge of H. Thus, $e_1, \ldots, e_{k-1}$ are edges of H (Note: k could be 1). Since $H + e_k - e_k = H$ is connected, e_k belongs to a cycle cycle C. Let e_0 be an edge in this cycle that is not in T. The graph $T_0 = H + e_k - e_0$ is thus a spanning tree of G, since it has $n-1$ edges and it is connected. Thus, by minimality of H, we must have $w(H) \geq w(T)$. Therefore, $w(e_k) \geq w(e_0)$. By the choice of e_k in Kruskal's algorithm, $w(e_0) \geq w(e_k)$. Therefore, $w(T_0) = w(H)$. However, T_0 has more edges in common with T, which contradicts the choice of H. □

Theorem 9.7 (Prim's Algorithm)**.** *Let G be a connected weighted graph. Construct a sequence of edges of G as follows:*

- *Start with an arbitrary vertex v of G and select an edge e_1 incident to v with minimum weight.*
- *For every $k \leq n-1$, select the edge e_k in such a way that e_k has the minimum weight among all the edges that have precisely one vertex in common with an edge from the list $e_1, \ldots, e_{k-1}$.*

Then, the subgraph of G induced on the edges $e_1, \ldots, e_{n-1}$ is a minimum spanning tree of G.

Proof. See page 98. □

Example 9.2. Find the number of spanning trees of K_n, for $n = 2, 3, 4$.

See page 101 for more examples.

Theorem 9.8 (Matrix Tree Theorem)**.** *Let G be a nontrivial graph of order n whose vertices are $v_1, v_2, \ldots, v_n$, and let A be the adjacency matrix of G relative to $v_1, v_2, \ldots, v_n$. Let D be the $n \times n$ diagonal matrix whose i-th diagonal entry is $\deg v_i$ for all i. Then, the number of spanning tree of G is the same as any co-factor of the matrix $C = D - A$.*

Definition 9.4. A matrix C in the previous theorem is called a **Laplacian matrix** for the graph G. Note that changing the order of vertices changes a Laplacian matrix of a graph.

The above theorem will give us the following.

Theorem 9.9. *The number of spanning trees of K_n is n^{n-2}.*

Proof (Optional). K_1 has only one spanning trees, so the result for $n = 1$ holds. Suppose $n \geq 2$. We will use the Matrix Tree Theorem along with some facts from linear algebra. The adjacency matrix of K_n has zeros on its diagonal and 1's everywhere else. D in the Matrix Tree Theorem is the $n \times n$ diagonal matrix with $(n-1)$'s on its diagonal. Removing the first row and the first column of $D - A$ we get an $(n-1) \times (n-1)$ matrix E that has $n-1$ on its diagonal entries and -1 everywhere else. By Matrix Tree Theorem, the number of spanning trees of K_n is the determinant of this $(n-1) \times (n-1)$ matrix E. In what follows we will use some linear algebra to evaluate $\det E$.

$$E = \begin{pmatrix} n-1 & -1 & \ldots & -1 & -1 \\ -1 & n-1 & \ldots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \ldots & n-1 & -1 \\ -1 & -1 & \ldots & -1 & n-1 \end{pmatrix}_{(n-1)\times(n-1)}$$

Adding all the rows to the first row does not change the determinant and we obtain the following determinant:

$$\det E = \det \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ -1 & n-1 & \dots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \dots & n-1 & -1 \\ -1 & -1 & \dots & -1 & n-1 \end{pmatrix}$$

Adding the first row to all other rows does not change the determinant. So we obtain the following determinant:

$$\det E = \det \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 0 & n & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & n & 0 \\ 0 & 0 & \dots & 0 & n \end{pmatrix}$$

This is an upper triangular $(n-1) \times (n-1)$ matrix with with 1 in its first diagonal entry and n in all other $n-2$ diagonal entries. Thus, its determinant is n^{n-2}. Therefore, the number of spanning trees of K_n is n^{n-2}, as desired. □

9.1 More Examples

Example 9.3. Find the number of spanning trees of C_n.

Solution. Note that C_n has n vertices and n edges. Since every tree of order n has size $n-1$, to obtain a spanning tree for C_n one edge must be removed. Note that removing an edge keeps the graph connected, because every edge of C_n belongs to a cycle. Thus, C_n has precisely n spanning trees. □

9.2 Exercises

9.2.1 Problems for Grading

The following problems must be submitted on Friday, April 24, 2020 at the beginning of the class. **GradeScope will not accept late submissions.**

All proofs must be complete and solutions must be fully justified.

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

Exercise 9.1 (10 pts)**.** *Find all forests G for which $\overline{G}$ is also a forest.*

Exercise 9.2 (10 pts)**.** *Using the Matrix Tree Theorem find the number of spanning trees of $K_{3,3}$. (Recall that the vertices are labeled.)*

Exercise 9.3 (10 pts)**.** *Let T and T' be two distinct spanning trees of a connected graph G of order n. Show that there exists a sequence $T = T_0, T_1, \ldots, T_k = T'$ of spanning trees of G such that T_i and T_{i+1} have precisely $n-2$ edges in common for every i, $0 \leq i \leq k-1$.*

Exercise 9.4 (10 pts)**.** *Let G be a connected weighted graph whose edges have distinct weights. Prove that G has a unique minimum spanning tree.*

(Hint: Use the idea of the proof of Theorem 9.6.)

Exercise 9.5 (10 pts)**.** *Prove that every tree of order n is isomorphic to a subgraph of $\overline{C}_{n+2}$.*

Exercise 9.6 (10 pts)**.** *Prove that in every nontrivial weighted connected graph every minimum spanning tree contains an edge of minimum weight.*

Exercise 9.7 (10 pts)**.** *Let G be a weighted connected graph. Consider the following algorithm.*

(i) Set $G_0 = G$.

(ii) For every $i \geq 0$, if G_i is a tree, then let $T = G_i$ and stop. Otherwise, let e_i be a non-bridge edge of G_i with the largest weight, then let $G_{i+1} = G_i - e_i$, and repeat step (ii).

Prove that this algorithm produces a minimum spanning tree T.

Exercise 9.8 (10 pts)**.** *Let G be the weighted graph of order $n+1$ with $V(G) = \{v_0, v_1, \ldots, v_n\}$, and $E(G) = \{v_i v_{i+1} \mid i = 0, 1, \ldots, n\} \cup \{v_0 v_i \mid i = 1, 2, \ldots, n\}$. Define a weight on G by $w(v_0 v_i) = n$ for all $i > 0$, and $w(v_i v_{i+1}) = i$ for all i with $1 \leq i \leq n-1$. How many minimum spanning trees does G have?*

Exercise 9.9 (10 pts)**.** *Prove that an edge e of a connected graph G is a bridge if and only if it belongs to every spanning tree of G.*

9.2.2 Problems for Practice

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10 Week 11

10.1 Connectivity

Definition 10.1. A vertex v of a connected graph G is called a **cut-vertex** if $G - v$ is a disconnected graph.

Example 10.1. The trivial graph has no cut-vertex, since if we remove its only vertex we don't get a graph.

Example 10.2. For every $n \geq 2$, the graph $K_{1,n}$ has precisely one cut-vertex.

Example 10.3. K_n has no cut-vertices, since for $n = 1$, it is the trivial graph whose only vertex is not a cut-vertex and for $n \geq 2$, removing every vertex of K_n we obtain a graph that is isomorphic to K_{n-1}.

Theorem 10.1. *Let v be a vertex incident to a bridge in a connected graph G. Then, v is a cut-vertex if and only if* $\deg v \geq 2$.

Corollary 10.1. If a connected graph of order at least 3 contains a bridge, then it contains a cut-vertex.

Theorem 10.2. *Let v be a cut-vertex in a connected graph G and u, w be vertices in distinct components of $G - v$. Then v lies on every uw-path in G.*

Corollary 10.2. A vertex v of a connected graph G is a cut-vertex if and only if there are vertices u, w distinct from v for which v lies on every uw-path in G.

Theorem 10.3. *Let G be a connected nontrivial graph and let $u \in V(G)$. If v is a vertex that is farthest from u, then v is not a cut-vertex.*

Corollary 10.3. Every connected nontrivial graph contains at least two vertices that are not cut-vertices.

10.2 Blocks

Definition 10.2. A connected graph is called **nonseparable** if it has no cut-vertices.

Theorem 10.4. *A graph of order at least 3 is nonseparable if and only if every two vertices lie on a common cycle.*

Theorem 10.5. *Let G be a connected nontrivial graph, and let R be a relation defined on $E(G)$ by eRf, where $e, f \in E(G)$, if and only if $e = f$ or e and f lie on a common cycle. Then R is an equivalence relation.*

Definition 10.3. Let G be a connected nontrivial graph and R be the relation defined in the above theorem. Then the subgraphs induced by the edges of each equivalence class of R are called **blocks** of G.

Remark. Note that since blocks are induced subgraphs on a nonempty set of edges, they have no isolated vertices and they are nontrivial.

Definition 10.4. Let S be a set and $\mathcal{P}$ be a property that at least one subset of S satisfies. Let A be a subset of S.

- We say A is a **maximal** $\mathcal{P}$-subset of S, if A satisfies $\mathcal{P}$ and no subset of S properly containing A satisfies $\mathcal{P}$.
- We say A is a **maximum** $\mathcal{P}$-subset of S, if A satisfies $\mathcal{P}$ and no subset of S whose size is larger than $|A|$ satisfies $\mathcal{P}$.

Similarly, the notations of **minimal** and **minimum** are defined. The set S can also be replaced with a graph and the notions of maximal or minimal $\mathcal{P}$-subgraph of G, and maximum or minimum $\mathcal{P}$-subgraph of G are defined similarly. For maximum and minimum in graphs we use the order of the subgraphs.

Example 10.4. Every connected component of a graph G is a maximal connected subgraph of G. A connected component with the largest number of vertices is a maximum connected subgraph of G.

Theorem 10.6 (Properties of Blocks)**.** *Let G be a connected nontrivial graph. Then,*

(a) Every block of G is a nonseparable graph.

(b) Every two distinct blocks share no edges.

(c) Every two distinct blocks share at most one vertex.

(d) If two distinct blocks share a vertex v, then v is a cut-vertex of G.

Proof. (a) Let B be a block of G. If B is P_2, then by definition B is nonseparable and connected. Suppose B has at least three vertices, and let u and v be two distinct vertices of B and let e_1 and e_2 be two edges of B incident to u and v, respectively. If $e_1 \neq e_2$, then e_1 and e_2 lie on a common cycle and thus u and v lie on a common cycle. If $e_1 = e_2$, then either u or v has another neighbor. Say $vw \neq e_1$ is another edge. The same argument shows that vw and uv lie on a common cycle. Therefore, every two vertices lie on a common cycle. Thus, by Theorem 10.4, B is nonseparable.
(b) This follows from the fact that equivalence classes of an equivalence relation are disjoint.
(c) Suppose two distinct blocks B_1 and B_2 share vertices $u \neq v$. Since B_1 and B_2 are connected, there are uv-paths P_1 and P_2 in B_1 and B_2, respectively. Let x be the neighbor of u in P_1 and y be the neighbor of u in P_2. If z is the first vertex of P_1 after u that is in P_2, then we have a cycle by following P_1 from u to z and then by following P_2 from z back to u. Note that ux and uy are in this cycle, which means uy must be an edge of B_1. This contradicts the fact that B_1 and B_2 are edge-disjoint.
(d) Suppose v is a common vertex of two distinct blocks B_1 and B_2. Let u and w be vertices of B_1 and B_2 that are neighbors of v. (Note that blocks have no isolated vertices.) If v were not a cut-vertex, then there would be a uw-path in G for which v does not belong to. This path along with the path w, v, u gives a cycle. This means uv and vw lie on a common cycle, whcih means vw must be an edge of B_1, which is a contradiction. □

Theorem 10.7. *Every block of a connected nontrivial graph is a maximal nonseparable subgraph.*

Proof. Suppose B is a block of G. By Theorem 10.6 B is nonseparable. Suppose on the contrary that H is a nonseparable subgraph of G properly containing B as a subgraph. Let u be a vertex of B and $v \in V(H)$ be a neighbor of u. We will show that v must belong to B. This along with the fact that H is connected, we get a contradiction. Suppose on the contrary v does not belong to B. Let w be a neighbor of u in B. Since H is nonseparable, there is a vw-path P in H that does not contain v. This path along with w, v, u gives a cycle that contains both edges vw and vu. Therefore, vu must be in B which means v must be in B. This completes the proof. □

10.3 Vertex-Cut and Edge-Cut Sets

Definition 10.5. A **vertex-cut** in a graph G is a set $U \subsetneq V(G)$ such that $G - U$ is disconnected.

Example 10.5. A vertex v is a cut-vertex in a connected graph G if and only if the set $\{v\}$ is a vertex-cut.

Example 10.6. Let G be a graph of order n. Prove that if $G \cong K_n$, then G has no vertex-cuts, and if G is not a complete graph, then it has a vertex-cut of size $n-2$.

Solution. Removing any proper subset of $V(K_n)$ leaves a complete graph, which is always connected, and thus K_n has no vertex-cuts. Now, suppose $G \ncong K_n$. Assume u and v are two non-adjacent vertices of G, then $U = V(G) - \{u, v\}$ is a vertex-cut of size $n-2$, since $G - U$ has two vertices u and v and no edge. □

Definition 10.6. Let G be a graph of order n. If $G \ncong K_n$, then the **vertex-connectivity** $\kappa(G)$ is defined to be the size of a minimum vertex-cut of G; if $G \cong K_n$, then $\kappa(G)$ is defined to be $n-1$.

Similar notions may be defined for edges.

Definition 10.7. An **edge-cut** in a graph G is a set X of edges of G for which $G - X$ is disconnected. The **edge-connectivity** $\lambda(G)$ of a graph G is the cardinality of a minimum edge-cut of G, if $G \ncong K_1$; while $\lambda(K_1) = 0$.

Example 10.7. Let G be a graph.

- If G is nontrivial, then it has an edge-cut. For example $E(G)$ is an edge-cut.
- If G is disconnected, then the empty set is both a vertex-cut and an edge-cut.
- If X is a minimum edge-cut for a connected graph G, then $G - X$ has precisely two connected components.

Theorem 10.8. *For every positive integer n, $\lambda(K_n) = n - 1$.*

Proof. For $n = 1$, the result follows from the definition. Suppose $n \geq 2$ and let X be a minimum edge-cut. Suppose G_1 and G_2 are the connected component of $G - X$. All edges between vertices of G_1 and G_2 must be in X. Therefore, if G_1 has k vertices, we must have $|X| \geq k(n-k)$. We need to prove $k(n-k) \geq n-1$. This is equivalent to $kn - k^2 - n + 1 \geq 0$ which is equivalent to $(k-1)(n-k-1) \geq 0$. Since $1 \leq k \leq n-1$, the inequality holds. □

Theorem 10.9. *For every graph G,*

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

Proof. Let n be the order of G. If $G \cong K_n$, then $\delta(G) = n-1$, $\kappa(G) = n-1$ by definition, and $\lambda(G) = n-1$, by Theorem 10.8. This proves the theorem for when G is a complete graph.

Now, assume G is not a complete graph. Let u be a vertex of G with degree $\delta(G)$. All edges incident to u form an edge-cut. Therefore, $\lambda(G) \leq \delta(G)$. This proves one of the inequalities and also shows that $\lambda(G) \leq n-2$.

Suppose X is a minimum edge-cut of G. Let G_1 and G_2 be the connected components of $G - X$. Suppose G_1 has k vertices. If there is a vertex x in G_1 and y in G_2 that are not adjacent in G, then for every edge e in X we remove one vertex that is incident to e and is neither x nor y. This gives us a vertex-cut whose size is at most $|X|$. If all vertices of G_1 and G_2 are adjacent in G, then $|X| \geq k(n-k)$. This quantity as seen in Theorem 10.8 is at least $n-1$, which is a contradiction since $\lambda(G) \leq n-2$. Therefore, there is always a vertex-cut of size at most $|X|$. This implies $\kappa(G) \leq \lambda(G)$. □

Theorem 10.10. *If G is a 3-regular graph, then $\kappa(G) = \lambda(G)$.*

Proof. Note that since $\kappa(K_n) = \lambda(K_n)$, we may assume G is not a complete graph.

Since G is 3-regular, we have $\delta(G) = 3$. Thus, $\kappa(G) \leq \lambda(G) \leq 3$. Note that if $\kappa(G) = 3$, then $\lambda(G) = 3$, and we are done. If $\kappa(G) = 0$, then G is disconnected, and hence $\lambda(G) = 0$, and we are done. So we are left with two cases: $\kappa(G) = 1$ or $\kappa(G) = 2$. Note that we only need to show $\lambda(G) \leq \kappa(G)$.

Assume $\kappa(G) = 1$, then G has a cut-vertex u. The graph $G - u$ is disconnected. Since the degree of u is 3 and $G - u$ is disconnected, $G - u$ has a component G_1 for which u has precisely one neighbor v in G_1. This means uv is a bridge, which implies $\lambda(G) \leq 1 = \kappa(G)$.

Suppose $\kappa(G) = 2$. Let $U = \{u, v\}$ be a minimum vertex-cut for G. Let G_1, G_2 be two components of $G - U$. If there are at most two edges with one endpoint in G_1 and one endpoint in U, then by removing these edges we get a disconnected graph. Same argument works for G_2. So, assume there are at least three edges between U and G_1 and at least three edges between U and G_2. Since both u and v have degree 3, there are precisely three edges between U and G_1 and three edges between U and G_2. If all the edges between G_1 and U are incident to u, then the graph G would be disconnected which is a contradiction. Suppose u has precisely one neighbor u_1 in G_1 and v has precisely one neighbor v_1 in G_2. Then, the set $\{uu_1, vv_1\}$ is an edge-cut. Therefore, $\lambda(G) \leq 2 = \kappa(G)$. This completes the proof. □

10.4 More Examples

Example 10.8. Let G be a connected graph. Define a relation R on the vertices of G by uRv if and only if $u = v$ or u and v belong to a common cycle. Show that in general R is not an equivalence relation. Find a necessary and sufficient condition for G so that R is an equivalence relation.

Solution. Let H and K be 3-cycles on the vertex sets [3] and $\{3, 4, 5\}$, respectively, and let $G = H \cup K$. Note that in graph G we have $2R3$ and $3R4$, but $2 \not R 4$ because every 24-path must pass through 3 since 3 is a cut-vertex. Thus, R is not necessarily transitive.

Clearly R is always reflexive and symmetric. We will have to check if it is transitive. Suppose uRv and vRw. If $u = v$, $v = w$ or $u = w$, then uRw, and we are done. Suppose u, v, w are distinct vertices. Let C_1 be a common cycle of u and v and let C_2 be a common cycle of v and w. By definition of blocks C_1 and C_2 must belong to two (possibly identical) blocks. If C_1 and C_2 belong to the same block, then since blocks are nonseparable u and w belong to a common cycle and thus uRw. If C_1 belongs to a block B_1 and C_2 belongs to a different block B_2, then v as a common vertex of B_1 and B_2 must be a cut-vertex. Thus, every uw-path must pass through v and thus, u and v do not belong to a common cycle, which means R is not transitive. This happened because two blocks of order at least three had a common vertex. Therefore, the necessary and sufficient condition can be stated as follows:

R is an equivalence relation if and only if no two distinct blocks of order at least 3 share a vertex.

The proof of why this is a necessary and sufficient condition should be written based on the arguments above. □

Example 10.9. Find all minimum edge-cuts of K_n for every $n \geq 2$.

Solution. Suppose X is a minimum edge-cut of G and let G_1 and G_2 be components of $G - X$. Suppose G_1 has k vertices. We know that X must contain all edges between G_1 and G_2. There are precisely $k(n-k)$ edges. Since these edges are an edge-cut for K_n, we must have $k(n-k) = n-1$. Therefore, $(k-1)(n-k-1) = 0$. This means $k = 1$ or $k = n-1$. Thus, X must be all edges incident to one vertex of K_n. Note that this set is indeed an edge-cut. Therefore, every minimum edge-cut of K_n is obtained by taking all edges incident to a fixed vertex of K_n.

Example 10.10. Prove or disprove each of the following statements:

(a) In a graph, every edge-cut contains a minimum edge-cut.

(b) In a graph, every vertex-cut contains a minimum vertex cut.

(c) If X is an edge-cut of a graph G and U is a set of vertices of G for which each edge in X is incident to at least one vertex in U, then $U = V(G)$ or U is a vertex-cut.

Solution. (a) This statement is false. Consider the graph $G = H \cup K$ where H is the complete graph on [3] and K is the complete graph on $\{3, 4\}$. $X = \{23, 13\}$ is a minimal edge-cut, since neither 23 not 13 is an edge and $G - U$ is disconnected. However $\lambda(G) = 1$ since 34 is a bridge.
(b) This statement is false. Consider $G = H \cup K$, where H is the 4-cycle v_1, v_2, v_3, v_4, v_1, and K be the path v_1, v_5. Note that $\kappa(G) = 1$, since G is connected and v_1 is a cut-vertex. The set $\{v_2, v_3\}$ is a minimal vertex-cut of size 2, but neither v_2 nor v_3 are cut-vertices.
(c) This is false. For example in K_2 let $X = E(K_2)$ and let U be the set containing one of the vertices of K_2. □

10.5 Exercises

10.5.1 Problems for Grading

The following problems must be submitted on Friday, May 1, 2020 at the beginning of the class. **Late submission will not be accepted.**

All proofs must be complete and solutions must be fully justified.

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

Exercise 10.1 (10 pts)**.** *Prove that if v is a cut-vertex of a connected graph G, then v is not a cut-vertex of $\overline{G}$.*

Exercise 10.2 (10 pts)**.** *Prove that for every positive integer n, we have $\kappa(K_{n,n}) = \lambda(K_{n,n}) = n$.*

Exercise 10.3 (10 pts)**.** *Prove that every cut-vertex of a connected graph must belong to at least two blocks.*

Exercise 10.4 (10 pts)**.** *Find $\kappa(T)$ and $\lambda(T)$ for every tree T.*

Exercise 10.5 (10 pts)**.** *Let n be a positive integer. Give an example of a graph G with $\delta(G) = n$ and $\kappa(G) = \lambda(G) = 1$.*

Exercise 10.6 (10 pts)**.** *Let e be an edge of a connected graph G. Prove that $\lambda(G) - 1 \leq \lambda(G - e) \leq \lambda(G)$.*

Exercise 10.7 (20 pts)**.** *(a) Suppose in a graph G of order n, we have $\delta(G) \geq (n-1)/2$. Prove that $\lambda(G) = \delta(G)$.*

(b) Suppose G is a graph for which $\Delta(G) \leq (n-1)/2$. Prove that $\lambda(\overline{G}) = \delta(\overline{G})$

(Hint: Use a method similar to the Proof of Theorem 10.8.)

Exercise 10.8 (10 pts)**.** *Prove that a connected graph G of order at least 3 is nonseparable if and only if any two adjacent edges of G lie on a common cycle.*

10.5.2 Problems for Practice

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Definition 11.1. Let G be a connected graph and k be a positive integer. Then, G^k is defined to be the graph with $V(G^k) = V(G)$, and two distinct vertices u and v are adjacent in G^k if and only if $d_G(u, v) \leq k$.

Remark. Note that if $k \geq \text{diam}\,(G)$, then G^k is a complete graph since the distance of no two vertices of G exceeds k.

Definition 11.2. We say a graph G is k-connected for a positive integer k, if $\kappa(G) \geq k$.

Remark. Note that $\kappa(G) \geq 1$ if and only if G is connected. Thus a graph is 1-connected if and only if it is connected.

Theorem 11.1. *If G is a connected graph of order at least 3, then G^2 is 2-connected.*

Proof. We need to show G^2 has no cut-vertices. Suppose v is a cut-vertex of G^2 and let u and w be vertices in different components of $G^2 - v$. Since u and w are not adjacent in G^2, we must have $k = d_G(u,w) \geq 3$. Since G is connected, there is a uw-path of length k. Let $u_0 = u, u_1, \ldots, u_k = w$ be a uw-path in G. Note that by definition of G^2, each u_j is adjacent to u_{j+2} in G^2. Let P_1 be the path in G^2 starting from u and ending at w that contains all u_j's, where j is even, and let P_2 be the path in G^2 starting from u and ending at w that contains all u_j's, where j is odd. Note that since u and w are in different connected components of $G^2 - v$, the vertex v must be on both P_1 and P_2, however these two paths do not share any common vertices, except u and w, but u and w are in $G - v$ and thus neither of them is v. This contradiction completes the proof. □

Theorem 11.2. *$\kappa(H_{r,n}) = r$, where $H_{r,n}$ is a Harary graph.*

11.1 Menger's Theorem

Definition 11.3. Let G be a connected graph and U be a vertex-cut. If $u \neq v$ are vertices in different components of $G - U$, then we say U is a uv**-separating set**.

Remark. If uv is an edge in a graph of order n, then the graph has no uv-separating set, otherwise there always exists a uv-separating set of size $n-2$.

Definition 11.4. Given a uv-path $P : u = u_0, u_1, \ldots, u_r = v$, the vertices $u_1, \ldots, u_{r-1}$ are called the **internal vertices of** P. Two uv-paths P and Q are called **internally disjoint** if they don't share any internal vertices. uv-paths $P_1, P_2, \ldots, P_k$ are called **internally disjoint** if every two P_i and P_j with $i \neq j$ are internally disjoint.

The following theorem is one of many min-max theorems in Combinatorics:

Theorem 11.3 (Menger's Theorem)**.** *Let $u \neq v$ be nonadjacent vertices of a connected graph G. The cardinality of a minimum uv-separating set equals the maximum number of internally disjoint uv-paths in G.*

Theorem 11.4. *A nontrivial graph G is k-connected for some integer $k \geq 2$, if and only if for every two distinct vertices u and v of G there exists at least k internally disjoint uv-paths in G.*

Proof. Suppose G is k-connected for some $k \geq 2$ and let u and v be two distinct vertices of G. If u and v are not adjacent, then by Menger's Theorem there are k internally disjoint uv-paths, as desired.

Assume u and v are adjacent, and let $e = uv$. By Exercise 11.3, $G - e$ is $(k-1)$-connected. Therefore, by the previous case, $G - e$ contains $k - 1$ internally disjoint uv-paths. These along with u, v give us k internally disjoint uv-paths.

For the converse, assume G is not k-connected. Thus, there is a vertex-cut U of size less than k. Let u and v be vertices in different components of $G - U$. By Menger's Theorem, there are no more than $k - 1$ internally disjoint uv-paths, which is a contradiction. □

Theorem 11.5. *Let G be a k-connected graph and $S \subseteq V(G)$ be a set with $|S| = k$. If a graph H is obtained from G by adding a new vertex w to G and joining w to all vertices of S, then H is k-connected.*

Combining the above theorem with the Menger's Theorem we obtain the following:

Theorem 11.6. *If G is k-connected and $u, u_1, \ldots, u_k$ are distinct vertices of G, then for every j, $1 \leq j \leq k$, there is a uu_j-path P_j for which $P_1, P_2, \ldots, P_k$ are internally disjoint.*

We have previously proved that every two vertices of a nonseparable graph of order at least 3 lie on a common cycle. The following is a generalization of this theorem.

Theorem 11.7. *If G is k-connected for some integer $k \geq 2$, then every k vertices of G lie on a common cycle.*

11.2 Eulerian Graphs

Definition 11.5. An **Eulerian circuit** in a connected graph G is a circuit that traverses all edges of G. A graph is called **Eulerian** if it is connected and has an Eulerian circuit. An open trail that traverses all the edges of a connected graph G is called an **Eulerian trail**.

Remark. Note that if $x_1, x_2, \ldots, x_m, x_1$ is a circuit, then so are all of the following:

$$x_2, x_3, \ldots, x_m, x_1, x_2$$
$$x_3, x_4, \ldots, x_1, x_2, x_3$$
$$\vdots$$
$$x_m, x_1, \ldots, x_{m-2}, x_{m-1}, x_m$$

Theorem 11.8. *Let G be a nontrivial connected graph.*

(a) G is Eulerian if and only if all of its vertices are even.

(b) G has an Eulerian trail if and if the degrees of precisely two of its vertices are odd.

Example 11.1. Let G and H be two connected graphs. Find the necessary and sufficient condition for $G \times H$ to be Eulerian.

Solution. We know that $G \times H$ is connected since both G and H are connected. Note that a vertex (u, v) is connected to all vertices of the form (u, y) and (x, v), where x is a neighbor of u and y is a neighbor of v. Thus, the degree of every vertex (u, v) is $\deg_G u + \deg_H v$. By Theorem 11.8, $G \times H$ is Eulerian if and only if $\deg_G u + \deg_H v$ is even for all vertices u of G and v of H. This means the parity of $\deg_G u$ and $\deg_H v$ is the same for all vertices u of G and v of H. Thus, $G \times H$ is Eulerian if and only if either all vertices of G and H are even or all vertices of G and H are odd. □

11.3 Hamiltonian Graphs

Definition 11.6. A cycle in a graph G that contains every vertex of G is called a **Hamiltonian cycle** of G. A graph that has a Hamiltonian cycle is called a **Hamiltonian graph**. A path in a graph that contains every vertex of G is called a **Hamiltonian path**.

Example 11.2. For every integer $n \geq 3$, the graphs K_n and C_n are Hamiltonian.

Example 11.3. For what positive integers m and n, is the graph $K_{m,n}$ Hamiltonian?

Solution. Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ be two partite sets of $K_{m,n}$. Note that every cycle must alternate between vertices of X and Y. since, a Hamiltonian cycle contains all vertices of X and Y, we must have $m = n$. Now, suppose $m = n$. If $n = 1$, then we get the graph $K_{1,1}$ which is a tree and thus does not have any cycles and hence is not Hamiltonian. For $n > 2$, the cycle $x_1, y_1, \ldots, x_n, y_n, x_1$ is a Hamiltonian cycle. Thus, $\boxed{K_{m,n} \text{ is Hamiltonian if and only if } m = n > 1.}$ □

The above example can be generalized as follows:

Theorem 11.9. *If G is a Hamiltonian graph, then for every nonempty proper set S of vertices of G,*

$$k(G - S) \leq |S|.$$

Example 11.4. The Petersen graph is non-Hamiltonian.

Theorem 11.10 (Ore's Theorem)**.** *Let u, v be distinct nonadjacent vertices in a graph G of order n such that $\deg u + \deg v \geq n$. Then $G + uv$ is Hamiltonian if and only if G is Hamiltonian.*

Proof. If G is Hamiltonin, then any Hamiltonian cycle of G is also a Hamiltonian cycle of $G + uv$.

Suppose $G + uv$ is Hamiltonian, and let C be a Hamiltonian cycle of $G + uv$. If uv is not an edge of C, then C is a Hamiltonian cycle of G and we are done. So, suppose $C : u = u_1, u_2, \ldots, u_n = v, u$ is a Hamiltonian cycle of $G + uv$. The idea is to swap uv and another edge of C for two other edges. If uu_i and vu_{i-1} are edges of G, then $u = u_1, u_2, \ldots, u_{i-1}, v, u_{n-1}, \ldots, u_i, u$ is a Hamiltonian cycle of G. We will have to show there is some i for which u_i is adjacent to u and u_{i-1} is adjacent to v. Let $S = \{i \mid uu_i \in E(G)\}$, and $T = \{i \mid vu_{i-1} \in E(G)\}$. Note that $|S| = \deg u$ and $|T| = \deg v$. By assumption $|S| + |T| \geq n$. Note also that $1 \notin S$ and $1 \notin T$. Thus, $|S \cap T| \geq 1$. This completes the proof. □

Definition 11.7. Let G be a graph of order n. The **closure** of G, denoted by $C(G)$ is the graph obtained from G by recursively joining distinct nonadjacent vertices whose degree sum is at least n.

Theorem 11.11. *A graph is Hamiltonian if and only if its closure is Hamiltonian.*

Proof. By Theorem 11.10, if $\deg u + \deg v \geq n$, where u and v are nonadjacent, then $G + uv$ is Hamiltonian if and only if G is Hamiltonian. Repeated use of this theorem implies the result. □

Theorem 11.12. *Let G be a graph of order $n \geq 3$. If*

$$\deg u + \deg v \geq n$$

for each pair of distinct nonadjacent vertices of G, then G is Hamiltonian.

Proof. The closure of such a graph is the complete graph which is Hamiltonian. Therefore, by Theorem 11.11 we are done. □

Remark. The condition

"$\deg u + \deg v \geq n$ for each pair of distinct nonadjacent vertices of G"

is called the **Ore's Condition**.

Theorem 11.13. *Let G be a graph of order $n \geq 3$. If for every positive integer $j < n/2$, the number of vertices of G with degree at most j is less than j, then G is Hamiltonian.*

11.4 More Examples

Example 11.5. Let G be a connected graph and k, ℓ be two positive integers for which $k\ell \leq \text{diam}\,(G)$. Prove that $(G^k)^\ell = G^{k\ell}$.

Solution. By definition $V((G^k)^\ell) = V(G^k) = V(G)$ and $V(G^{k\ell}) = V(G)$.

Note that for any two vertices u, v in a graph H, $d_H(u, v) \leq k$ iff there is a sequence of (not necessarily distinct) vertices $u = u_0, u_1, \ldots, u_k = v$ for which $d_H(u_j, u_{j+1}) \leq 1$ for all $0 \leq j \leq k-1$. If $d_H(u, v) \leq k$, then this sequence can be created by appending a uv-geodesic by some v's if necessary, and if such a sequence exists the inequality $d_H(u, v) \leq k$ follows from the triangle inequality.

For vertices u, v in G we have uv is an edge of $G^{k\ell}$ iff there is a sequence of vertices $u = u_0, u_1, \ldots, u_{\ell k} = v$ for which $d_G(u_j, u_{j+1}) \leq 1$. By triangle inequality $d_G(u_0, u_k) \leq k, d_G(u_k, u_{2k}) \leq k, \ldots, d_G(u_{(\ell-1)k}, u_{\ell k}) \leq k$ which is equivalent to $d_{G^k}(u_{jk}, u_{(j+1)k}) \leq 1$, and this is equivalent to $d_{G^k}(u, v) \leq \ell$ which is true if and only if uv is an edge in $(G^k)^\ell$. Therefore, $(G^k)^\ell = G^{k\ell}$. □

Example 11.6. What is the necessary and sufficient condition on positive integers k and n for there to exist a graph G with two distinct nonadjacent vertices u and v with a minimum uv-separating set of size k?

Solution. Let G be such a graph. We know there must exist a uv-separating set U of size k. The set U along with u and v give us $k+2$ distinct vertices, which implies $n \geq k+2$. Assume $n \geq k+2$ and consider n distinct vertices $u, v, u_1, \ldots, u_{n-2}$. Create a graph on vertices $u, v, u_1, \ldots, u_{n-2}$ by including $(k-1)$ paths of form u, u_j, v for $j = 1, \ldots, k-1$, along with the path $u, u_k, \ldots, u_{n-2}, v$. Since these paths are internally disjoint each uv-separating set must be of size at least k. Furthermore, note that $u_1, \ldots, u_k$ is a uv-separating set. Thus, this graph satisfies the given conditions. Therefore, $n \geq k+2$ is the necessary and sufficient condition. □

Example 11.7. Prove that every Hamiltonian graph is 2-connected.

Solution. Suppose G is a Hamiltonian graph. If G is a complete graph, since G has a Hamiltonian cycle, its order must be at least 3, and thus $\kappa(G) \geq 2$, and thus G is 2-connected.
Suppose G is not complete and let v be a vertex of G. Let C be a Hamiltonian cycle of G, then $C - v$ is a spanning subgraph of $G - v$ which is a path. Therefore, $G - v$ is connected. Therefore, G is 2-connected. □

Example 11.8. Is the converse of Theorem 11.12 true?

Solution. The answer is no. If $n \geq 5$, then C_n is a 2-regular Hamiltonian graph that does not satisfy the Ore's condition. □

11.5 Exercises

11.5.1 Problems for Grading

The following problems must be submitted on Monday, May 11, 2020 at the beginning of the class. **Late submission will not be accepted.**

All proofs must be complete and solutions must be fully justified.

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

Exercise 11.1 (10 pts)**.** *Let $k \geq 2$ be an integer. Suppose G is a connected graph of order at least $k+1$. Prove that G^k is k-connected.*

(Hint: Use a method similar to Theorem 11.1.)

Exercise 11.2 (10 pts)**.** *Let G be a connected graph of diameter d and order n. Prove that $G, G^2, \ldots, G^d$ are all distinct graphs, and that $G^d \cong K_n$.*

Exercise 11.3 (10 pts)**.** *Let G be a k-connected graph for some integer $k \geq 2$ and let e be an edge of G. Prove that $G - e$ is $(k-1)$-connected.*

Note that the above exercise was used in the proof of Theorem 11.4, so you cannot use this theorem to do the exercise.

Exercise 11.4 (20 pts)**.** *Prove or disprove each of the following:*

(a) If G is a 2-connected graph, u, v are two distinct vertices of G, and P is a uv-path, then there is another uv-path that is internally disjoint from P.

(b) If u, v, w are three distinct vertices in a 2-connected graph, then there is a uv-path containing w.

Exercise 11.5 (10 pts)**.** *Let $a_1, a_2, \ldots, a_n$ be positive integers. Find the necessary and sufficient condition on integers $a_1, a_2, \ldots, a_n$ for $K_{a_1,a_2,\ldots,a_n}$ to be Eulerian.*

Exercise 11.6 (20 pts)**.** *(a) Let G be a 2-regular disconnected graph of order 19. Prove that $\overline{G}$ is Eulerian.*

(b) Let $r < n$ be two positive integers. Find the necessary and sufficient condition on r and n for which the complement of every r-regular disconnected graph of order n is Eulerian.

Exercise 11.7 (20 pts)**.** *Let $k \geq 2$ be an integer. A graph G is called minimally k-connected if it is k-connected and $G - e$ is not $k-$connected for every edge e of G.*

(a) Give examples of minimally k-connected graphs for $k = 2$ and $k = 3$.

(b) Prove that for every two positive integers $n > k \geq 2$, there is a minimally k-connected graph of order n.

Exercise 11.8 (10 pts)**.** *Use Menger's Theorem to prove Theorem 10.10: If G is a 3-regular graph, then $\kappa(G) = \lambda(G)$.*

Exercise 11.9 (5 pts)**.** *Find a Hamiltonian cycle or show none exists for $K_{3,4,7}$.*

Exercise 11.10 (10 pts)**.** *Let $n \geq 2$ be an integer and $1 \leq a_1 \leq a_2 \leq \ldots \leq a_n$ be a sequence of integers. Find the necessary and sufficient condition on $a_1, a_2, \ldots, a_n$ for the complete multipartite graph $K_{a_1,a_2,\ldots,a_n}$ to be Hamiltonian.*

11.5.2 Problems for Practice

Exercise 11.11. *Prove that every Eulerian graph is a union of edge-disjoint cycles.*

Solution. We will prove this by induction on the size of an Eulerian graph G. Since G is Eulerian the smallest possible size of G is 3, in which case $G \cong C_3$ is itself a cycle. Suppose G is an Eulerian graph Theorem 11.8, every vertex of G is even. Since G does not have any leaves, it cannot be a tree. Suppose C is a cycle of G. Consider the graph $H = G - E(C)$. Note that all components of H are Eulerian since for every vertex u, we have $\deg_H u = \deg_G u$ or $\deg_G u - 2$. Thus, by inductive hypothesis all nontrivial components of H are unions of edge-disjoint cycles. Those cycles along with C give us edge-disjoint cycles that cover all edges of G, as desired. □

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Page 129-130: 33, 36.

Page 139-140: 1, 4, 8.

12 Week 13

12.1 Matchings

Definition 12.1. A set of edged in a graph is called **independent** if no two of them share an endpoint. A **matching** is a set of independent edges. If $M = \{e_1, e_2, \ldots, e_k\}$ is a matching where $e_j = u_j w_j$, then we say M matches the set $\{u_1, u_2, \ldots, u_k\}$ to the set $\{w_1, w_2, \ldots, w_k\}$.

Definition 12.2. Let G be a graph. For any subset X of $V(G)$, the set $N(X)$ consists of all vertices that are adjacent to some vertex in X.

Theorem 12.1. *Let G be a bipartite graph with partite sets U and W such that $r = |U| \leq |W|$. Then G contains a matching of cardinality r if and only if for every subset X of U, we have $|N(X)| \geq |X|$.*

Definition 12.3. Let $S_1, S_2, \ldots, S_k$ be nonempty finite sets. We say $x_1, x_2, \ldots, x_k$ is a **system of distinct representatives** for $S_1, S_2, \ldots, S_k$ if $x_j \in S_j$ for every $j \leq k$.

Theorem 12.2. *A collection $\{S_1, S_2, \ldots, S_k\}$ of finite sets has a system of distinct representatives if and only if for each integer $r \leq k$, the union of any r of these sets has at least r elements.*

Definition 12.4. A vertex and an incident edge are said to **cover** each other. An **edge cover** of a graph G without isolated vertices is a set of edged of G that cover all vertices of G.

Definition 12.5. The **edge independence number** $\alpha'(G)$ of a graph G is the size of a maximum matching of G. The **edge covering number** $\beta'(G)$ of a graph G is the size of a minimum edge cover of G.

Theorem 12.3. *For every graph G of order n that has no isolated vertices,*

$$\alpha'(G) + \beta'(G) = n.$$

Similar notions can be defined for vertices and similar results hold.

Definition 12.6. A 1-regular spanning subgraph of a graph G is called a **1-factor** or a **perfect matching** of G.

Remark. If a graph has a 1-factor then its order is even.

Example 12.1. let n be a positive integer. The Petersen graph, C_n, P_n, and K_n all have 1-factors.

See page 195 for more examples.

Definition 12.7. A component of a graph G is called **odd** or **even** depending on whether its order is odd or even. For a graph G, the number $k_o(G)$ is the number of odd components of G.

Theorem 12.4. *A graph G contains a 1-factor if and only if $k_o(G - S) \leq |S|$, for every proper subset S of $V(G)$.*

12.2 Exercises

12.2.1 Problems for Grading

The following problems must be submitted on Monday, May 13, 2020 at the beginning of the class. **Late submission will not be accepted.**

All proofs must be complete and solutions must be fully justified.

Read and follow the directions carefully. If a problem is asking you to use a certain method, you must use that method to solve the problem.

Exercise 12.1 (10 pts)**.** *Let n be an even integer and G be a connected regular graph of order n for which $\overline{G}$ is also connected. Prove that either G or $\overline{G}$ is Hamiltonian.*

Exercise 12.2 (10 pts)**.** *Prove a graph G of order n without any isolated vertices has a perfect matching if and only if $\alpha'(G) = \beta'(G)$.*

Exercise 12.3 (10 pts)**.** *Using an idea similar to the one we used in the proof of Theorem 12.3, prove that if G is a graph of order n containing no isolated vertices, then $\alpha(G) + \beta(G) = n$.*

Exercise 12.4 (10 pts)**.** *A connected bipartite graph G has partite sets U and W, where $|U| = |W| = k \geq 2$. Prove that if every two vertex of U have distinct degrees in G, then G contains a perfect matching.*

Exercise 12.5 (10 pts)**.** *Prove that if G is a graph of order n having no isolated vertices, then*

$$\beta(G)(\Delta(G) + 1) \geq n.$$

Exercise 12.6 (10 pts)**.** *Let k be a non-negative integer.*

(a) For what values of k is there a graph G for which $|\alpha'(G) - \beta'(G)| = k$?

(b) For what values of k is there a graph G for which $|\alpha(G) - \beta(G)| = k$?

Exercise 12.7 (20 pts)**.** *Prove or disprove:*

(a) Every vertex cover of a graph contains a minimum vertex cover.

(b) Every vertex cover of a graph contains a minimal vertex cover.

(c) Every independent set of vertices is contained in a maximal independent set of vertices.

(d) Every independent set of vertices is contained in a maximum independent set of vertices.

Repeat all of the above when "vertex" is replaced by "edge".

12.2.2 Problems for Practice

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12.2.3 Challenge Problems

Exercise 12.8. *Let n be a positive integer and S be a set consisting of n distinct real numbers. What is the maximum number of pairs (a, b) of elements in S for which $1 < b - a < 2$?*

Exercise 12.9. *A group of $2n+1$ people have the property that for every group of n people there is somebody outside of this group that is friend with all n memebrs of this group. Prove that there is somebody who is friend with everybody.*

Exercise 12.10. *A party consists of $n + 1$ people in such a way that nobody is friend with all the other n people, every pair of strangers have exactly one common friend, and among every three people at least two of them are not friends. Prove that everybody has the same number of friends.*