Linear Algebra and Ordinary Differential Equations

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Notations

- $\in$ belongs to.
- $\forall$ for all.
- $\exists$ there exists or for some.
- $\operatorname{Im} f$ the image of function f.
- $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of nonnegative integers.
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ the set of positive integers.
- $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, \text{ and } n \neq 0\}$ the set of rational numbers.
- $\mathbb{R}$ the set of real numbers.
- $\mathbb{C}$ the set of complex numbers.
- $A \subseteq B$ set A is a subset of set B.
- $A \subsetneqq B$ set A is a proper subset of set B.
- $A \cup B$, the union of sets A and B.
- $A \cap B$, the intersection of sets A and B.
- $\bigcup\limits_{i=1}^{n} A_i$ the union of sets $A_1, A_2, \ldots, A_n$.
- $\bigcap\limits_{i=1}^{n} A_i$ the intersection of sets $A_1, A_2, \ldots, A_n$.
- $A_1 \times A_2 \times \cdots \times A_n$ the Cartesian product of sets $A_1, A_2, \ldots, A_n$.
- $\emptyset$ the empty set.
- $C(\mathbb{R})$ the vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.
- $C^n(\mathbb{R})$ the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$ whose n-th derivatives are continuous.
- $C^\infty(\mathbb{R})$ the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$ that are infinitely many times differentiable.
- $\mathrm{P}_n(\mathbb{F})$ the vector space of polynomials of degree at most n with coefficients in $\mathbb{F}$.
- $M_{m\times n}(\mathbb{F})$ the set of all $m \times n$ matrices with entries in $\mathbb{F}$.
- $M_n(\mathbb{F})$ the set of all $n \times n$ matrices with entries in $\mathbb{F}$.
- $\operatorname{Col} A$ the column space of matrix A.
- $\operatorname{Row} A$ the row space of matrix A.

- $W[Y_1, \ldots, Y_n]$ Wronskian of $Y_1, \ldots, Y_n$.
- $H(t)$, Heaviside function.
- $(f \star g)(t)$ Convolution of $f(t)$ and $g(t)$.
- $\mathcal{L}\{f(t)\}$ Laplace of $f(t)$.

Contents

These notes may contain typos or occasional errors. Feel free to bring them to my attention by sending me an email at ebrahimi@umd.edu

Chapter 1

Preliminaries and Review

1.1 Complex Numbers

1.1.1 Definition and Basic Operations

Definition 1.1. The set of **complex numbers** $\mathbb{C}$ is defined as

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\},$$

where i is a solution of the equation $i^2 = -1$. The form $a + bi$ for a complex number is called its **standard form**. Two complex numbers $a+bi$ and $c+di$ with $a, b, c, d \in \mathbb{R}$ are said to be equal if and only if $a = c$ and $b = d$. We say a and b are the **real** and the **imaginary** parts of the complex number $z = a+bi$, respectively. We denote these by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively.

The set $\mathbb{C}$ is equipped with two binary operations $+$ and . as follows:

- $\forall a, b, c, d \in \mathbb{R}\ \ (a + bi) + (c + di) = (a + c) + (b + d)i.$
- $\forall a, b, c, d \in \mathbb{R}\ \ (a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i.$ (Or $(a + bi)(c + di)$, without the dot.)

Note: Both real and imaginary parts of a complex number are *real*.

Definition 1.2. For a complex number $z = a + bi$, where a and b are real numbers, we define its **complex conjugate** as $\overline{z} = a - bi$ and its **absolute value** (or **norm**) as $|z| = \sqrt{a^2 + b^2}$.

Theorem 1.1 (Field properties of $\mathbb{C}$)**.** *For every $x, y, z \in \mathbb{C}$*

- *(Commutativity) $x + y = y + x$, and $xy = yx$.*
- *(Associativity) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$.*
- *(Additive Identity) $x + 0 = x$. (Here zero of $\mathbb{C}$ is given by $0 = 0 + 0i$.)*
- *(Additive Inverse) There is $t \in \mathbb{C}$ for which $x + t = 0$. (When $x = a + bi$, we have $t = -a + (-b)i$.)*

- *(Multiplicative Inverse) If $x \neq 0$, there is some $t \in \mathbb{C}$ for which $xt = 1$. (t is denoted by x^{-1} or $1/x$.)*

- *(Distributivity) $x(y + z) = xy + xz$.*

Theorem 1.2 (Properties of complex conjugate and norm)**.** *For every two complex numbers z and w, we have*

- $\overline{zw} = \overline{z}\,\overline{w}$.

- $|zw| = |z|\,|w|$.

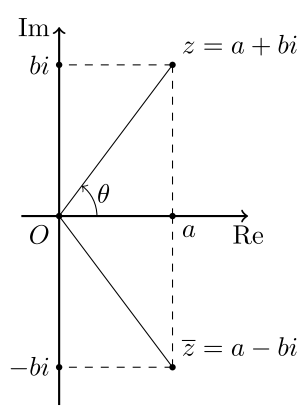
- $|z|^2 = z\,\overline{z}$.

- $|z + w| \leq |z| + |w|$. *(Triangle Inequality.)*

Example 1.1. Find the additive and multiplicative inverse of $3 + 2i$.

1.1.2 Geometry of $\mathbb{C}$

Each complex number $z = a + bi$ can be plotted on a plane called the **complex plane**. The horizontal axis consists of all real numbers and the vertical axis consists of all complex numbers with zero real part. If θ is the angle between $0z$ and the positive real axis, then $z = |z|(\cos\theta + i\sin\theta)$.



Theorem 1.3. *For every real number θ, we have $e^{i\theta} = \cos\theta + i\sin\theta$.*

Theorem 1.4. *Let x, y be two real numbers and n be an integer. Then,*

(a) $e^{ix}e^{iy} = e^{i(x+y)}$.

(b) (De Moivre's Formula) $(e^{ix})^n = e^{inx}$.

Example 1.2. Evaluate $\int e^x \cos x\, dx$.

Definition 1.3. For every complex number $z = a + bi$, with $a, b \in \mathbb{R}$ we define $e^z = e^a(\cos b + i\sin b)$.

Theorem 1.5. *For every two complex numbers z, w we have $e^{z+w} = e^z e^w$.*

1.1.3 More Examples

Example 1.3. Let $z = 2 + i, w = 1 - 3i$. Write down the complex numbers $z + w, z - w, zw$, and z/w in standard form.

Solution. $z+w = 3-2i$, $z-w = 1+4i$, $zw = 2+i-6i-3i^2 = 5-5i$. $z/w = z\overline{w}/|w|^2 = (2+i+6i+3i^2)/(1+9) = -0.1 + 0.7i$. □

Example 1.4. Find all real numbers a, b for which $a^2 + bi + 2i = (7 + 3i)(1 - i)$.

Solution. Writing the left hand side in standard form and multiplying the right hand side we obtain:

$$a^2 + (b + 2)i = 7 - 7i + 3i + 3 = 10 - 4i \Rightarrow a^2 = 10, \text{ and } b + 2 = -4.$$

The answer is $a = \pm\sqrt{10}$, and $b = -6$. □

Example 1.5. Evaluate $(1 + i)^{1000}$

Solution. Since we are finding large exponents of a complex number, De Moiver's formula would be helpful. So, we will first write down this complex number in polar form. $|1+i| = \sqrt{2}$. The angle between the segment from 0 to $1 + i$ and the positive real axis is $\pi/4$. This means $1 + i = \sqrt{2}e^{i\pi/4}$. Therefore, $(1 + i)^{1000} = 2^{500}e^{i250\pi} = 2^{500}(\cos(250\pi) + i\sin(250\pi)) = 2^{500}$. □

Example 1.6. Given a positive integer n, solve the equation $z^n = 1$ over complex numbers.

Solution. By taking the absolute value of both sides we obtain $|z|^n = 1$. Since $|z|$ is a nonnegative real number, we must have $|z| = 1$. Therefore, using the polar form we obtain $z = e^{i\theta}$ for some angle $\theta \in [0, 2\pi)$. This means, we must solve $e^{in\theta} = 1$. This holds if and only if $\cos(n\theta) = 1$ and $\sin(n\theta) = 0$. This is equivalent to $n\theta = 2k\pi$ for some integer k. Since $\theta \in [0, 2\pi)$, we have $k = 0, 1, \ldots, n - 1$. Therefore, all roots of $z^n = 1$ are $z = e^{2ik\pi/n}$ with $k = 0, 1, \ldots, n - 1$. □

Example 1.7. Prove that a complex number z satisfies $|z| = 1$ if and only if $z = e^{i\theta}$ for some real number θ.

Solution. ($\Rightarrow$) Suppose $|z| = 1$. By the polar form of z we know $z = |z|e^{i\theta} = e^{i\theta}$ for some $\theta \in [0, 2\pi)$.
($\Leftarrow$) Suppose $z = e^{i\theta}$ for some real number θ. Then, $z = \cos\theta + i\sin\theta$. Therefore, $|z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$. □

Example 1.8. Find infinitely many complex numbers z for which $e^z = 2$.

Solution. Letting $z = a + bi$, with real a, b, we obtain $e^z = e^a e^{ib}$. Taking the absolute value of both sides yields $|e^z| = |e^a|\ |(\cos\theta + i\sin\theta)| = e^a$. Therefore, if $z = a + bi$ satisfies $e^z = 2$, then $e^a = 2$ and thus

$a = \ln 2$. This reduces the equation to $e^{ib} = 1$. This is valid if and only if $\cos b = 1$ and $\sin b = 0$. Therefore, $z = \ln 2 + 2k\pi i$, where $k \in \mathbb{Z}$ yields all solutions of $e^z = 2$. □

1.1.4 Exercises

Exercise 1.1. *Find all real numbers a, b for which $a + 3bi + a^2 b = 2ab + ai + 2bi$.*

Exercise 1.2. *Using a method similar to the one we used in this chapter, evaluate $\int e^{2x} \sin(3x)\, dx$.*

Exercise 1.3. *Using the method of Mathematical Induction, prove the De Moivre's formula: $(e^{ix})^n = e^{inx}$ for every real number x and every integer n. Note that the cases where n is negative or positive should be dealt with separately.*

Exercise 1.4. *Prove that if z is a nonzero complex number, then there is a complex number w for which $z = e^w$. Prove that $e^w \neq 0$ for all complex numbers w.*

1.2 Vector Spaces, Subspaces and Bases

Definition 1.4. A nonempty set V (consisting of elements called **vectors**) is called a **vector space** over $\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), if there is a binary operation $+$ (assigning a vector $\mathbf{v} + \mathbf{w} \in V$ to every pair of vectors $\mathbf{v}, \mathbf{w} \in V$) and a scalar multiplication (assigning a vector $c\,\mathbf{v} \in V$ to every $c \in \mathbb{F}$ and $\mathbf{v} \in V$) that satisfy the following for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and all $c \in \mathbb{F}$:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

(iii) There is $\mathbf{e} \in V$ such that for all $\mathbf{x} \in V$, we have $\mathbf{x} + \mathbf{e} = \mathbf{x}$. (This element $\mathbf{e}$ is denoted by $\mathbf{0}$).

(iv) There is $\mathbf{z} \in V$ for which $\mathbf{v} + \mathbf{z} = 0$. (This element $\mathbf{z}$ is denoted by $-\mathbf{v}$.)

(v) $1\,\mathbf{v} = \mathbf{v}$.

(vi) For all $a, b \in \mathbb{F}$, we have $a(b\mathbf{v}) = (ab)\mathbf{v}$, and $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

(vii) For all $a, b \in \mathbb{F}$, we have $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

When $\mathbb{F} = \mathbb{R}$, we say the vector space V is a **real vector space**, and when $\mathbb{F} = \mathbb{C}$, we say the vector space V is a **complex vector space**.

Example 1.9. $\mathbb{R}^n$, $\mathbb{C}^n$, $M_n(\mathbb{R})$, $M_{m\times n}(\mathbb{R})$, $\mathrm{P}_n(\mathbb{R})$, $C(\mathbb{R})$, $C^n(\mathbb{R})$, and $C^\infty(\mathbb{R})$ are all real vector spaces. $\mathbb{C}^n$, $\mathrm{P}_n(\mathbb{C})$, and $M_n(\mathbb{C})$ are complex vector spaces.

Note: $C(\mathbb{R})$ is the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Similarly, $C^n(\mathbb{R})$ is the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ that have continuous n-th derivatives. $C^\infty(\mathbb{R})$ is the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ that have derivatives of all orders.

Most concepts that we discussed about real vector spaces are true for complex vector spaces. This is fundamentally because for the most part we only used the field properties of $\mathbb{R}$.

Definition 1.5. A subset W of a vector space V is called a **subspace** if W along with the same operations of V is itself a vector space.

Theorem 1.6 (Subspace Criterion)**.** *Let V be a vector space. A subset W of V is a subspace if and only if it satisfies both of the following:*

- *W contains the zero vector of V, and*
- *for all $\mathbf{x}, \mathbf{y} \in W$ and $c \in \mathbb{F}$, we have $\mathbf{x} + \mathbf{y} \in W$ and $c\mathbf{x} \in W$. [We say W is closed under vector addition and scalar multiplication.]*

Example 1.10. $\mathbb{R}^n$ is a subspace of the real vector space $\mathbb{C}^n$.

Definition 1.6. Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a subset of a vector space V, and $\mathbf{w}$ be a vector in V. We say $\mathbf{w}$ is a **linear combination** of S (or elements of S) if $\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ for some $c_1, \dots, c_n \in \mathbb{F}$. By definition, the only linear combination of the empty set is $\mathbf{0}$, the zero vector.

Definition 1.7. We say vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are **linearly dependent** if one of these vectors can be written as a linear combination of the others. Otherwise, we say $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are **linearly independent**.

Theorem 1.7. *The vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly dependent if and only if there are scalars $c_1, \dots, c_n \in \mathbb{F}$, not all zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$.*
In other words, vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent if and only if the following statement is true

$$\text{If } c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}, \text{ then } c_1 = c_2 = \cdots = c_n = 0.$$

Definition 1.8. We say a subset $\mathcal{S}$ of a vector space V is **spanning** (or **generating**) if every $\mathbf{v} \in V$ is a linear combination of some elements of $\mathcal{S}$.

Definition 1.9. We say a subset $\mathcal{B}$ of a vector space is a **basis** if $\mathcal{B}$ is both linearly independent and spanning.

Theorem 1.8. *Suppose V is a vector space with a basis consisting of n vectors, where n is an integer.*

(a) If m is an integer more than n, then every m vectors of V are linearly dependent.

(b) Every basis of V consists of precisely n vectors.

(c) Every n linearly independent vectors in V form a basis for V.

(d) Every n spanning vectors in V form a basis for V.

Definition 1.10. If V is a vector space that has a basis of size n we say V is n**-dimensional** and we write $\dim V = n$. If V has no basis that is finite, we say V is **infinite-dimensional**.

Example 1.11. Find the dimensions of $\mathbb{C}$ once as a real and once as a complex vector space.

Example 1.12 (Span of vectors)**.** Let $\mathcal{A}$ be a set of vectors in a vector space V, and let Span $\mathcal{A}$ be the set consisting of all vectors that are linear combinations of some vectors of $\mathcal{A}$. Then Span $\mathcal{A}$ is a subspace of V.

Definition 1.11. Let A be an $m \times n$ matrix. The **row space of** A denoted by Row (A) is the subspace of $\mathbb{F}^n$ spanned by the rows of A, and the **column space of** A denoted by Col (A) is the subspace of $\mathbb{R}^m$ spanned by the columns of A.

Example 1.13 (Row space and column space)**.** Row space and column space of every matrix are vector spaces.

1.2.1 More Examples

Example 1.14. Determine which of the following sets under their natural addition and scalar multiplication is a vector space.

(a) The set $C[a, b]$ of all continuous functions $f : [a, b] \to \mathbb{R}$.

(b) The set of all cubic polynomials.

(c) The set of functions of the form $a \sin t + b \cos t$, where $a, b \in \mathbb{R}$ are constants and $t \in \mathbb{R}$.

(d) The set $\mathrm{P}(\mathbb{F})$ consisting of all polynomials of any degree on a variable t with coefficients in $\mathbb{F}$.

(e) The set of all unit vectors in $\mathbb{R}^n$.

(f) The set of all positive real numbers.

Solution. (a) This is a vector space. First, note that if $f, g : [a, b] \to \mathbb{R}$ are continuous functions and $c \in \mathbb{R}$, then so are $f + g$ and cf. Next, we will check all properties of a vector space. Let $f, g, h : [a, b] \to \mathbb{R}$ be continuous and $c, d \in \mathbb{R}$ be constants. Note that the zero function 0 is continuous and that $-f$ is also continuous. Furthermore, we have the following:

$$
\begin{aligned}
&(f + g)(t) = f(t) + g(t) = g(t) + f(t) = (g + f)(t) \Rightarrow f + g = g + f \\
&((f + g) + h)(t) = (f + g)(t) + h(t) = f(t) + g(t) + h(t) = (f + (g + h))(t) \Rightarrow (f + g) + h = f + (g + h) \\
&(f + 0)(t) = f(t) + 0 = f(t) \Rightarrow f + 0 = f \\
&(f + (-f))(t) = f(t) - f(t) = 0 \Rightarrow f + (-f) = 0 \\
&(1f)(t) = f(t) \Rightarrow 1f = f \\
&((cd)f)(t) = (cd)f(t) = c(df(t)) = c((df)(t)) = (c(df))(t) \Rightarrow (cd)f = c(df) \\
&((c + d)f)(t) = (c + d)f(t) = cf(t) + df(t) = (cf)(t) + (df)(t) \Rightarrow (c + d)f = (cf) + (df) \\
&(c(f + g))(t) = c(f + g)(t) = c(f(t) + g(t)) = cf(t) + cg(t) = (cf)(t) + (cg)(t) = (cf + cg)(t)
\end{aligned}
$$

The last row implies $c(f + g) = cf + cg$. Therefore, this set satisfies all properties of a vector space.

(b) This is not a vector space, because it is not closed under addition. x^3 and $-x^3$ are both cubic but their sum is 0 which is not a cubic polynomial.

(c) This is a vector space. Note that $0 = 0 \sin t + 0 \cos t$ is in this set. Also, the set is closed under addition and scalar multiplication.

$$(a \sin t + b \cos t) + (c \sin t + d \cos t) = (a + c) \sin t + (b + d) \cos t, \text{ and } c(a \sin t + b \cos t) = ca \sin t + cb \cos t.$$

Therefore, this set is a subspace of $C(\mathbb{R})$.

(d) We will show that $\mathrm{P}(\mathbb{R})$ is a subspace of $C(\mathbb{R})$ and hence it is a vector space. First, note that every polynomial is continuous. Thus $\mathrm{P}(\mathbb{R}) \subseteq C(\mathbb{R})$. We have $0 \in \mathrm{P}(\mathbb{R})$. Also, if $p(t), q(t)$ are polynomials and $c \in \mathbb{F}$, then $p(t) + q(t)$ and $cp(t)$ are also polynomials. Therefore, this is a subspace of $C(\mathbb{R})$.

(e) This is not a vector space, since it is not closed under addition. For example $\mathbf{e}_1$ and $-\mathbf{e}_1$ are both unit vectors but their sum is not.

(f) This is not a vector space, because it is not closed under scalar multiplication. $(-1)2 = -2$ is not positive. □

Example 1.15. Using the definition or Theorem 1.7 determine if each of the following vectors are linearly independent.

(a) $(1, -1, 2), (2, 0, 3), (1, 1, 1)$ as vectors of $\mathbb{F}^3$.

(b) $1, t, \sin^2 t, \cos^2 t$ as elements of $C(\mathbb{R})$.

(c) $\mathbf{0}$ in a vector space V.

(d) $1, t^2, 2 + t + t^2$ in P_2.

Solution. (a) Suppose

$$c_1(1, -1, 2) + c_2(2, 0, 3) + c_3(1, 1, 1) = (0, 0, 0).$$

This yields the following system:

$$c_1 + 2c_2 + c_3 = 0$$
$$-c_1 + c_3 = 0$$
$$2c_1 + 3c_2 + c_3 = 0$$

To find all solutions to this system we need to row reduce the following matrix:

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix}$$

Applying the row operations $R_2 + R_1$ and $R_3 - 2R_1$ we obtain the following

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix}$$

The last two rows are multiples. We see that $c_2 = 1, c_3 = -1$ and $c_1 = -2c_2 - c_3 = -1$ are solutions. Therefore, these three vectors are linearly dependent.

(b) Note that $1 = 0t + \sin^2 t + \cos^2 t$. Therefore, by definition, these functions are linearly dependent.

(c) $\mathbf{0}$ is always linearly dependent by Theorem 1.7, since we can write $1\mathbf{0} = \mathbf{0}$.

(d) Suppose $c_1 + c_2t^2 + c_3(2 + t + t^2) = 0$ for some constants $c_1, c_2, c_3 \in \mathbb{F}$. This yields

$$(c_1 + 2c_3) + c_3t + (c_2 + c_3)t^2 = 0.$$

Since this equality holds for every t we must have $c_1 + 2c_3 = c_3 = c_2 + c_3 = 0$. Solving this system yields $c_1 = c_2 = c_3 = 0$ and thus these three polynomials are linearly independent. □

Definition 1.12. The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose j-th row is the j-th columns of A for all j.

Definition 1.13. A matrix A is called **symmetric** if $A = A^T$.

Example 1.16. Let V be the set of all 2×2 symmetric matrices with entries in $\mathbb{F}$. Prove that V is a vector space along with the usual matrix addition and scalar multiplication. Find its dimension.

Solution. Every 2×2 matrix A can be written as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ where } a, b, c, d \in \mathbb{F}.$$

Since $A = A^T$, we must have $b = c$. Therefore, every symmetric matrix is of the form

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad (*)$$

Clearly every such matrix is also symmetric. This means the set of symmetric matrices is the span of three matrices below:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad (**)$$

Thus, this set is a subspace of $M_2(\mathbb{F})$, by Example 1.12. Note that the three matrices in $(**)$ are linearly independent, since every linear combination of them is of the form $(*)$, and thus it is zero only when $a = b = d = 0$. So, the dimension of this vector space is 3. □

Example 1.17. What is the dimension of $C^\infty(\mathbb{R})$?

Solution. This vector space is infinite-dimensional. To prove that, we will show functions $1, t, t^2, \ldots, t^n$ are linearly independent for every positive integer n. This shows the dimension must be more than any positive integer n (See Theorem 1.8 part (a)) which means the dimension must be infinite. To prove that, assume

$$c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n = 0, \text{ for all } t \in \mathbb{R}.$$

Substituting $t = 0$ we obtain $c_0 = 0$. Differentiating we obtain

$$c_1 + 2c_2 t + \cdots + nc_n t^{n-1} = 0.$$

Substituting $t = 0$ again we obtain $c_1 = 0$. Repeating this we conclude all c_j's must be zero and thus $1, t, \ldots, t^n$ are linearly independent. $\square$

1.2.2 Exercises

Exercise 1.5. *Recall that* P_n *is the vector space consisting of all polynomials of degree not exceeding* n. *Let* $S = \{f_0, f_1, \ldots, f_n\}$ *be a set consisting of* $(n+1)$ *nonzero polynomials for which* $\deg f_j = j$ *for all* $j, 0 \leq j \leq n$. *Prove that* S *is a basis for* P_n. *Deduce that if* $f(t)$ *is a polynomial of degree* n, *then* $\{f(t), f'(t), \ldots, f^{(n)}(t)\}$ *is linearly independent.*

Hint: Use the definition of linear independence. If you use any other method, you must fully justify your proof using the theorems from this chapter. Do not assume P_n is the same as $\mathbb{F}^{n+1}$.

Exercise 1.6. *What is the dimension of* $\mathrm{P}(\mathbb{F})$, *the vector space of polynomials of any degree, as a vector space over* $\mathbb{F}$*?*

Exercise 1.7. *Let* V *be a complex vector space of dimension* n. *Prove that* V *is also a real vector space and its dimension is* $2n$.

Exercise 1.8. *Using axioms of vector space prove that for every vector* $\mathbf{v}$ *we have* $0\mathbf{v} = \mathbf{0}$ *and that* $(-1)\mathbf{v} = -\mathbf{v}$.

Exercise 1.9. *Prove that the set of all even continuous functions in* $C(\mathbb{R})$ *(i.e. those that satisfy* $f(-t) = f(t)$*) under the natural addition and scalar multiplication is a vector space.*

Exercise 1.10. *Let* S *be a set. Prove that the set* $\mathcal{F}(S, \mathbb{F})$ *consisting of all functions* $f : S \to \mathbb{F}$ *under their natural addition and scalar multiplication is a vector space over* $\mathbb{F}$.

Exercise 1.11. *Suppose* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *are linearly independent vectors in a vector space* V. *Let* $\mathbf{v}_{n+1} \in V$ *be a vector that does not belong to* $\mathrm{Span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. *Prove that* $\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}_{n+1}$ *are linearly independent.*

1.3 Linear Transformations and Matrices

Definition 1.14. Let V and W be two vector spaces over $\mathbb{F}$. A function $T : V \to W$ is called **linear** if

- (Additivity) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$, for all $\mathbf{u}, \mathbf{v} \in V$, and
- (Homogeneity) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u} \in V$ and $c \in \mathbb{F}$.

Theorem 1.9. *Let V, W be vector spaces over $\mathbb{F}$, and $T : V \to W$ be a function. The following are equivalent:*

(a) For every $\mathbf{u}, \mathbf{v} \in V$ and $c \in \mathbb{F}$ we have $T(\mathbf{u} + c\mathbf{v}) = T(\mathbf{u}) + cT(\mathbf{v})$.

(b) For every $\mathbf{u}, \mathbf{v} \in V$ and $a, b \in \mathbb{F}$ we have $T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$.

(c) T is linear.

Example 1.18. The following functions are all linear.

(a) $T : \mathbb{F}^n \to \mathbb{F}^m$ defined by $T(\mathbf{u}) = A\mathbf{u}$, where $A \in M_{m\times n}(\mathbb{F})$ is a fixed matrix.

(b) $S : C(\mathbb{R}) \to C(\mathbb{R})$, defined by $S(f)(x) = \int_0^x f(t)\, dt$.

(c) $L : C^1(\mathbb{R}) \to C(\mathbb{R})$ defined by $L(f)(x) = f'(x)$.

(d) $U : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by $U(f)(x) = f''(x) + (2x + 1)f'(x) - e^x f(x)$.

Given a linear transformation $T : V \to W$, recall that Ker T is the set of all $\mathbf{u} \in V$ for which $T(\mathbf{u}) = \mathbf{0}$ and Im T is the image of T.

Example 1.19. Solving the differential equation $y'' + (2x + 1)y' - e^x y = 0$ is the same as finding the kernel of the linear transformation U in the previous example.

Note that row reduction and echelon form work for matrices with complex entries as well as those with real entries. Given that, recall the following:

Theorem 1.10. *Let A be an $m \times n$ matrix. Suppose E is an echelon form of A obtained by row reducing A. Suppose E has r pivot columns. Then,*

(a) The dimensions of the column space and the row space of A are both r.

(b) The dimension of the kernel of A is $n - r$.

(c) (The Rank-Nullity Theorem) $\dim \mathrm{Ker}\ A + \dim \mathrm{Col}\ A = n$.

Definition 1.15. (a) A square matrix A is called **invertible** if there is a square matrix B for which $AB = BA = I$.

(b) The **rank** of a matrix is the dimension of its colum (or row) space.

(c) The **trace** of a square matrix A is the sum of its (main) diagonal entries and is denoted by tr A or trace A.

Theorem 1.11. *For a matrix $A \in M_n(\mathbb{F})$ the following are qequivalent:*

(a) A *is invertible.*

(b) $\mathrm{Col}\ A = \mathbb{F}^n$.

(c) $\mathrm{Ker}\ A = \{\mathbf{0}\}$.

Remark. To find the inverse of a square matrix A we form the augmented matrix $(A|I)$. Apply row operations until we obtain the matrix $(I|B)$. This matrix B is the inverse of A.

Definition 1.16. Let A be an $m \times n$ and B be an $n \times k$ matrix. The matrix AB is an $m \times k$ matrix whose j-th column is obtained from multiplying A by the j-th column of B. In other words, the (i, j) entry of AB is obtained by finding the dot product of the i-th row of A with the j-th column of B.

Remark. For every $m \times n$ matrix A and every column vector $\mathbf{v} \in \mathbb{R}^n$ the vector $A\mathbf{v}$ is a linear combination of columns of A with coefficients from entries of $\mathbf{v}$.

Theorem 1.12. *A function $T : \mathbb{F}^n \to \mathbb{F}^m$ is linear if and only if there is an $m \times n$ matrix A for which $T(\mathbf{v}) = A\mathbf{v}$ for every column vector $\mathbf{v} \in \mathbb{F}^n$. Furthermore, for a given linear mapping T the matrix A is unique, and the columns of A are given by $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$. In other words,*

$$A = (T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)) .$$

Example 1.20. Prove that the trace function $\mathrm{tr}\ : M_n(\mathbb{F}) \to \mathbb{F}$ is linear.

Theorem 1.13. *Let A and B be two square matrices. Then*

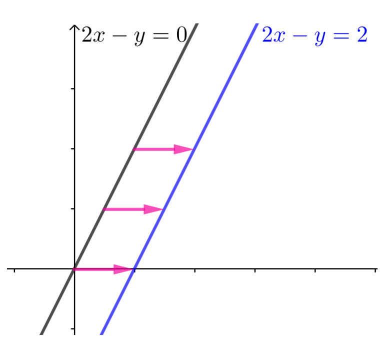
(a) $(AB)^T = B^T A^T$.

(b) If A and B are invertible, then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(c) If A is invertible, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.

Here we assume the matrix sizes are so that the multiplications are all defined.

Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}$ given by $T(x, y) = 2x - y$. The kernel of this transformation is the set consisting of all points (x, y) satisfying $2x - y = 0$, which is a line through the origin. At the same time, $T^{-1}(2)$ consists of all points on the line $2x - y = 2$. This is a line parallel to the kernel. In other words, $T^{-1}(2)$ is a translation of $\mathrm{Ker}\ T$ by the vector $(1, 0)$ as seen in the picture below:



The following theorem formalizes the above observation.

Theorem 1.14. *Let $T : V \to W$ be a linear transformation between vector spaces and let $\mathbf{w} \in W$. Then, either the inverse image $T^{-1}(\mathbf{w})$ is empty or*

$$T^{-1}(\mathbf{w}) = \mathbf{v} + \text{Ker } T = \{\mathbf{v} + \mathbf{u} \mid \mathbf{u} \in \text{Ker } T\},$$

for every $\mathbf{v} \in T^{-1}(\mathbf{w})$.

1.3.1 More Examples

Example 1.21. Let $T : \mathbb{F}^3 \to \mathbb{F}^2$ be defined by $T(a, b, c) = (2a + c, b - c)$. Using the above theorem, find $T^{-1}(2, 3)$.

Solution. First, note that T is linear, since

$$T(a, b, c) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

We will now find Ker T. By definition of kernel, $(a, b, c) \in$ Ker T if and only if $2a + c = b - c = 0$, which happends if and only if $b = c = -2a$. Thus,

$$\text{Ker } T = \{(a, -2a, -2a) \mid a \in \mathbb{F}\}.$$

We also see that $T(1, 3, 0) = (2, 3)$. Therefore,

$$T^{-1}(2, 3) = (1, 3, 0) + \text{Ker } T = \{(1 + a, 3 - 2a, -2a) \mid a \in \mathbb{F}\}.$$

□

Example 1.22. Describe the kernel and image of each of the linear mappings

(a) $S : C(\mathbb{R}) \to C(\mathbb{R})$, defined by $S(f)(x) = \displaystyle\int_0^x f(t)\, dt$.

(b) $L : C^1(\mathbb{R}) \to C(\mathbb{R})$ defined by $L(f)(x) = f'(x)$.

Solution. (a) If $f \in$ Ker S , then $S(f) = 0$. This means $\displaystyle\int_0^x f(t)\, \mathrm{d}t = 0$ for all $x \in \mathbb{R}$. Differentiating both sides we obtain $f(x) = 0$. Clearly 0 is in the kernel of S. Therefore, Ker $S = \{0\}$.

Suppose $F(x)$ is in the image of S. This means $F(x) = \displaystyle\int_0^x f(t)\, \mathrm{d}t$ for some continuous function $f : \mathbb{R} \to \mathbb{R}$. By the Fundamental Theorem of Calculus, we have $F'(x) = f(x)$. Therefore, F has continuous derivative and thus $F \in C^1(\mathbb{R})$. On the other hand $F(0) = \displaystyle\int_0^0 f(t)\, \mathrm{d}t = 0$. We will show that the image of S is the set of all functions $F \in C^1(\mathbb{R})$ that satisfy $F(0) = 0$. Since F is continuously differentiable its derivative F' is continuous. Note that by Fundamental Theorem of Calculus, $\displaystyle\int_0^x F'(t)\, \mathrm{d}t$ and F have the same derivatives. Therefore, they must differ by a constant. However, since both are zero at $x = 0$, we must have $F(x) = \int_0^x F'(t)\ \mathrm{d}t = S(F')(x)$.

(b) If $f \in \mathrm{Ker}\ L$, then $L(f) = 0$ and thus $f'(x) = 0$ for all $x \in \mathbb{R}$. Therefore, f is a constant function. Conversely, if f is constant, then $f'(x) = 0$ and thus $L(f)$=0. Therefore, Ker L is the set of all constant functions $c : \mathbb{R} \to \mathbb{R}$.

A function F is in the image of L iff $F = f'(x)$ for some continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$. We can take $f(x) = \int_0^x F(t)\ \mathrm{d}t$. By the Fundamental Theorem of Calculus $f'(x) = F(x)$. Since F is continous f is continuously differentiable. Therefore, the image of L is $C(\mathbb{R})$. □

Example 1.23. Prove that rank $(AB) \leq$ rank A and rank $(AB) \leq$ rank B for every two matrices A, B where AB is defined.

Solution. Let columns of A be $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Then, every column of AB is a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$. If $\mathbf{v}$ is a vector in the column space of AB, then it is a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Since $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are in Col A, the vector $\mathbf{v}$ is also in Col A. This shows Col $(AB) \subseteq$ Col A. Therefore, rank $(AB) \leq$ rank A.

The proof for rank $(AB) \leq$ rank B is similar. You'd just need to swap columns for rows in the argument above. □

Example 1.24. Let $A_1, \ldots, A_n$ be invertible square matrices of the same size. Prove that

$$(A_1 \cdots A_n)^{-1} = A_n^{-1} \cdots A_1^{-1}.$$

Solution. We will proceed by induction on n.
Basis step: For $n = 1$ both sides are A_1^{-1}.
Inductive step: Assume $(A_1 \cdots A_n)^{-1} = A_n^{-1} \cdots A_1^{-1}$. We have

$$(A_1 \cdots A_n A_{n+1})^{-1} = A_{n+1}^{-1}(A_1 \cdots A_n)^{-1} = A_{n+1}^{-1} A_n^{-1} \cdots A_1^{-1}.$$

Here, we are using the inductive hypothesis and the fact that $(AB)^{-1} = B^{-1}A^{-1}$. □

Example 1.25. Determine if each of the following is a linear transformation.

(a) $T : \mathrm{P}(\mathbb{F}) \to \mathrm{P}(\mathbb{F})$ given by $T(f)(t) = f(t^2)$.

(b) $S : C(\mathbb{R}) \to C(\mathbb{R})$ given by $S(f)(t) = f(\sin t)$.

(c) $U : C(\mathbb{R}) \to C(\mathbb{R})$ given by $U(f)(t) = \sin(f(t))$.

Solution. (a) This is linear. For every two polynomials f, g and constant c we have

$$T(f+cg)(t) = (f+cg)(t^2) = f(t^2) + cg(t^2) = T(f)(t) + cT(g)(t) \Rightarrow T(f+cg) = T(f) + cT(g).$$

Therefore, T is linear.

(b) This is linear. For every two continuous functions f, g and constant c we have

$$S(f+cg)(t) = (f+cg)(\sin t) = f(\sin t) + cg(\sin t) = S(f)(t) + cS(g)(t) \Rightarrow S(f+cg) = S(f) + cS(g).$$

Therefore, S is linear.

(c) U is not linear. $U(2t) = \sin(2t)$, while $2U(t) = 2\sin t$. We know $\sin(2t)$ and $2\sin t$ are not always the same (e.g. for $t = \pi/2$.) Therefore, $U(2t) \neq 2U(t)$. This shows U is not linear. □

Example 1.26. Suppose V is a real vector space and $T : V \to \mathbb{R}$ is a linear transformation for which $T(V) \subseteq [0, \infty)$. Prove that $T(\mathbf{v}) = 0$ for all $\mathbf{v} \in V$.

Solution. Let $\mathbf{v} \in V$. By assumption both $T(\mathbf{v})$ and $T(-\mathbf{v})$ are nonnegative. By linearity $T(-\mathbf{v}) = -T(\mathbf{v})$. Since both $T(\mathbf{v})$ and $-T(\mathbf{v})$ are nonnegative, we must have $T(\mathbf{v}) = 0$, as desired. □

1.3.2 Exercises

Exercise 1.12. *Determine if the transformation $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\mathbf{v}) = ||\mathbf{v}||^2$ is linear.*

Exercise 1.13. *Let A be an $m \times n$ matrix. Prove that there is an $n \times m$ matrix B for which $AB = I$ iff* Col $A = \mathbb{F}^m$*. Deduce that if for an $m \times n$ matrix A there is an $n \times m$ matrix B for which $AB = I$, then $m \leq n$.*

Exercise 1.14. *Let* P_3 *be the vector space of polynomials of degree at most 3 with real coefficients. Suppose* $T : \mathrm{P}_3 \to \mathrm{P}_3$ *is a linear transformation with*

$$T(t+1) = T(1) = T(t^2 - 1) = T(t^3 - 2t) = t.$$

(a) Find a basis for Ker T.

(b) Find $T^{-1}(-t)$. (i.e. all polynomials $p(t)$ for which $T(p(t)) = -t$.)

Exercise 1.15. *Let A be an $m \times n$ matrix. Define two functions $S, T : M_{n\times m}(\mathbb{F}) \to \mathbb{F}$ by $S(X) = \mathrm{tr}\ (AX)$, and $T(X) = \mathrm{tr}\ (XA)$.*

(a) Prove that S and T are both linear.

(b) Prove that S and T are the same over a basis of $M_{n\times m}(\mathbb{F})$.

(c) Deduce that $\text{tr }(AB) = \text{tr }(BA)$ *for every two matrices* A *and* B*, where both products* AB *and* BA *are defined.*

Exercise 1.16. *Prove that if* $A \in M_{m\times n}(\mathbb{F})$ *is a matrix with rank 1, then there are column vectors* $\mathbf{u} \in \mathbb{F}^m$*, and* $\mathbf{v} \in \mathbb{F}^n$ *for which* $A = \mathbf{u}\mathbf{v}^T$*. In other words,*

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix}.$$

Exercise 1.17. *Use the previous exercise to prove that if* $A \in M_n(\mathbb{F})$ *is a rank 1 matrix, then* $\text{tr } A^k = (\text{tr } A)^k$ *for all positive integers* k*.*

Exercise 1.18. *Find the matrix of reflection about the line* $y = mx$*, where* m *is a given real number.*

Hint: Let θ be the angle that this line makes with the positive x axis. This reflection can be written as a composition of three transformations: A rotation with angle $-\theta$, a reflection about the x-axis and finally a rotation with angle θ. You may assume rotations and reflections are linear transformations.

Exercise 1.19. *Find the inverse of each matrix:*

$$\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & 1 \\ 1 & 3 & 1 \end{pmatrix}$$

Exercise 1.20. *Suppose* A *is a* 45×23 *matrix with rank 19. Find the dimension of each of the following:*

$$\text{Ker } A, \text{Ker } A^T, \text{Col } (A), \text{Col } (A^T).$$

Exercise 1.21. *Prove that if* $A \in M_n(\mathbb{R})$*, then it is not possible for* $\text{Col } (A)$ *and* $\text{Ker } A^T$ *to be the same. With an example show this is possible if* A *has non-real complex entries.*

Exercise 1.22. *Let* A *be a given* $m \times n$ *matrix. Suppose* $A\mathbf{x} = \mathbf{0}$ *has the unique solution* $\mathbf{x} = \mathbf{0} \in \mathbb{F}^n$*. Prove that for every vector* $\mathbf{v} \in \mathbb{F}^n$ *the equation* $A^T\mathbf{x} = \mathbf{v}$ *has at least one solution* $\mathbf{x} \in \mathbb{F}^m$*.*

1.4 Determinants

All properties of determinant remain the same for matrices over $\mathbb{R}$ and $\mathbb{C}$. Below is a list of the most important properties of determinant:

Theorem 1.15 (Properties of Determinant)**.** *Let* A *and* B *be two square matrices of the same size. Then:*

(a) If B *is obtained from* A *by swapping two rows (or columns), then* $\det B = -\det A$*.*

(b) If B is obtained from A by adding a multiple of one row (or column) to another row (or column), then $\det B = \det A$.

(c) If B is obtained from A by multiplying a row (or column) by a scalar c, then $\det B = c \det A$,

(d) $\det A^T = \det A$.

(e) $\det(AB) = (\det A)(\det B)$.

(f) A is invertible if and only if $\det A \neq 0$.

(g) $\det A$ *can be obtained using co-factor expansion along a row or a column.*

Definition 1.17. Two vector spaces V and W over $\mathbb{F}$ are said to be **isomorphic** if there is a one-to-one and onto linear transformation $T : V \to W$.

Example 1.27. $\mathbb{F}^{n+1}$ and $\mathrm{P}_n(\mathbb{F})$ are isomorphic.

Theorem 1.16. *Two finite-dimensional vector spaces are isomorphic if and only if their dimensions are the same.*

1.4.1 More Examples

Example 1.28. Let $A \in M_n(\mathbb{F})$ and $c \in \mathbb{F}$. Prove $\det(cA) = c^n \det A$.

Solution. Suppose rows of A are $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Then rows of cA are $c\,\mathbf{a}_1, \ldots, c\,\mathbf{a}_n$. By properties of determinant, we have the following:

$$\det(cA) = \det \begin{pmatrix} c\,\mathbf{a}_1 \\ \vdots \\ c\,\mathbf{a}_n \end{pmatrix} = c \det \begin{pmatrix} \mathbf{a}_1 \\ c\,\mathbf{a}_2 \\ \vdots \\ c\,\mathbf{a}_n \end{pmatrix} = c^2 \det \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ c\,\mathbf{a}_3 \\ \vdots \\ c\,\mathbf{a}_n \end{pmatrix} = \cdots = c^n \det \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{pmatrix} = c^n \det A.$$

□

Example 1.29. A matrix A is called skew-symmetric if $A^T = -A$. Prove that if an $n \times n$ matrix A is skew-symmetric and n is odd, then A is not invertible.

Solution. By a theorem we know $\det(A^T) = \det A$. At the same time, we know $\det(-A) = (-1)^n \det A = -\det A$, since n is odd. Therefore, $\det A = -\det A$ and hence $\det A = 0$. This implies that A is not invertible. □

Example 1.30. Let A be the $n \times n$ matrix, whose entries above or on the main diagonal are all 1's and whose entries below the main diagonal are all a variable t. Find $\det A$ in terms of n and t.

Solution. The matrix A is as follows:

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ t & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t & t & \dots & 1 & 1 \\ t & t & \dots & t & 1 \end{pmatrix}_{n\times n}$$

We will apply the row operations $R_2 - tR_1, R_3 - tR_1, \dots, R_n - tR_1$ to obtain the following:

$$\det A = \det \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 0 & 1-t & \dots & 1-t & 1-t \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1-t & 1-t \\ 0 & 0 & \dots & 0 & 1-t \end{pmatrix}_{n\times n}$$

This is an upper triangular matrix and thus $\det A = (1-t)^{n-1}$. $\square$

Example 1.31. Evaluate the determinant of an $n \times n$ matrix whose off-diagonal entries are all 1 and whose diagonal entries are all t.

Solution. Let

$$E = \begin{pmatrix} t & 1 & \dots & 1 & 1 \\ 1 & t & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & t & 1 \\ 1 & 1 & \dots & 1 & t \end{pmatrix}_{n\times n}$$

Subtracting the first row from all other rows does not change the determinant and we obtain the following determinant:

$$\det E = \det \begin{pmatrix} t & 1 & \dots & 1 & 1 \\ 1-t & t-1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1-t & 0 & \dots & t-1 & 0 \\ 1-t & 0 & \dots & 0 & t-1 \end{pmatrix}$$

Adding all columns to the first we obtain the following:

$$\det E = \det \begin{pmatrix} t+n-1 & 1 & \dots & 1 & 1 \\ 0 & t-1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & t-1 & 0 \\ 0 & 0 & \dots & 0 & t-1 \end{pmatrix}$$

This is an upper triangular matrix. Thus, its determinant is the product of its diagonal entries. Therefore, $\det E = (t-1)^{n-1}(t+n-1)$. □

1.4.2 Exercises

Exercise 1.23. *For every matrix A with complex entries, let $\overline{A}$ be the matrix obtained by conjugating all entries of A. Prove the following for every two matrices A, B and every $c \in \mathbb{C}$. In each case assume the appropriate operation is defined.*

(a) $\overline{AB} = \overline{A}\,\overline{B}$.

(b) $\overline{A+B} = \overline{A} + \overline{B}$.

(c) $\overline{cA} = \bar{c}\,\overline{A}$.

(d) $\operatorname{tr} \overline{A} = \overline{\operatorname{tr} A}$.

(e) $\det \overline{A} = \overline{\det A}$.

Exercise 1.24. *Let n be a positive integer.*

(a) Prove that if n is odd, then there is no $n \times n$ matrix A with real entries that satisfies $A^2 + I = 0$.

(b) Prove that if n is even, then there is an $n \times n$ matrix A with real entries that satisfies $A^2 + I = 0$.

Exercise 1.25. *Find the determinant of an $n \times n$ matrix whose minor diagonal entries are $a_1, \dots, a_n$ and all of whose entries below the minor diagonal are zero. In other words, find the determinant of the matrix:*

$$\begin{pmatrix} * & * & \cdots & a_1 \\ * & \cdots & a_2 & 0 \\ \vdots & \iddots & & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix}$$

Exercise 1.26. *Let $A \in M_n(\mathbb{F})$. Prove that the set*

$$V = \{B \in M_n(\mathbb{F}) \mid AB = BA\}$$

along with natural matrix multiplication and scalar multiplication forms a vector space.

Exercise 1.27. *Let A be a square matrix. Prove that all of the following matrices*

$$\begin{pmatrix} A & * \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & * \\ 0 & A \end{pmatrix}, \begin{pmatrix} A & 0 \\ * & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ * & A \end{pmatrix}$$

have determinant equal to $\det A$. *In each case $*$ is an arbitrary matrix that makes the given matrix a square matrix.*

Exercise 1.28. *Let D_n be the determinant of the $n \times n$ matrix–shown below–whose main diagonal entries are all 1's, the entries immediately above the main diagonal (if any exists) are all -1's and the entries immediately below the main diagonal (if any exists) are all 1's, and whose all other entries (if any exists) are all 0's.*

$$\begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \end{pmatrix}$$

(a) Evaluate D_1, D_2, D_3, D_4 and D_5.

(b) Conjecture a recursion for D_n.

(c) Prove your claim in part (b). (Hint: Expand along the first column.)

Exercise 1.29. *Let D_n be the determinant of the $n \times n$ matrix–shown below–whose main diagonal entries are all 3's, the entries immediately above the main diagonal (if any exists) are all 2's and the entries immediately below the main diagonal (if any exists) are all 1's, and whose all other entries (if any exists) are all 0's.*

$$\begin{pmatrix} 3 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 3 \end{pmatrix}$$

(a) Evaluate D_1, D_2, D_3 and D_4.

(b) Conjecture a formula for D_n, for every n.

(c) Prove your claim in part (b) using induction.

For the next exercise you will need the following familiar theorem:

Theorem 1.17. *Suppose $p(x) = A_0 + A_1 x + \cdots + A_n x^n$ is a polynomial with complex coefficients $A_0, A_1, \ldots, A_n$. Suppose $p(x)$ has n distinct roots $r_1, \ldots, r_n \in \mathbb{C}$. Then*

$$p(x) = A_n (x - r_1) \cdots (x - r_n).$$

Exercise 1.30 (Vandermonde Determinant)**.** *In this exercise you will prove the Vandermonde Determinant using induction:*

$$\det \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix} = \prod_{0 \le j < k \le n} (c_k - c_j) \qquad (*)$$

(a) Prove $(*)$ for $n = 1$.

(b) Prove $(*)$ holds if $c_k = c_j$ for some $j \neq k$. For the rest of the problem assume c_j's are distinct.

(c) Instead of c_n in the last row use a variable x. Using co-factor expansion along the last row show that this determinant can be written as $A_0 + A_1 x + \cdots + A_n x^n$, where A_j's are constants depending on $c_0, \ldots, c_{n-1}$.

(d) Prove that the polynomial $p(x) = A_0 + A_1 x + \cdots + A_n x^n$ has n roots $x = c_0, c_1, \ldots, c_{n-1}$. Use this to show $p(x) = A_n (x - c_0) \cdots (x - c_{n-1})$. (Hint: Use Theorem 1.17.)

(e) Assuming $(*)$ is true for $n - 1$, find A_n. Use that to obtain a proof of the Vandermonde determinant using induction.

Exercise 1.31. *Let F_n be the Fibonacci sequence given by $F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n$ for all $n \ge 1$. Find a square matrix A for which $\det(A^n) = F_n$ for all $n \ge 1$, or show no such matrix exists.*

Exercise 1.32. *Suppose $A \in M_n(\mathbb{R})$ is such that $\lim\limits_{n \to \infty} \det(A^n) = 1$. Prove that A is invertible.*

Exercise 1.33. *Prove that there are no 5×5 matrices A, B for which $AB - BA = I$.*

Exercise 1.34. *Let $r \in \mathbb{F}$. Prove that for every integer $n \ge 2$, there is a matrix $A \in M_n(\mathbb{F})$ for which $\det A = r$ and A has no zero entries.*

Exercise 1.35. *Suppose a matrix A has positive rank r. Prove that there are r rows and r columns of A for which the submatrix formed by these r rows and r columns has a nonzero determinant.*

1.4.3 Challenge Problems

Exercise 1.36. *Prove that if z and w are two distinct complex numbers for which $|z| = |w| = 1$, then $\dfrac{1-zw}{z-w}$ is a real number.*

Exercise 1.37. *Find a formula for*

$$\sin x + \sin(2x) + \cdots + \sin(nx).$$

(Hint: Use $\sin\theta = \mathrm{Im} e^{i\theta}$.)

Solution. Since $\sin\theta = \mathrm{Im} e^{i\theta}$ for all $\theta \in \mathbb{R}$ we have:

$$\sin x + \sin(2x) + \cdots + \sin(nx) = \mathrm{Im}\left(\sum_{k=1}^{n} e^{ikx}\right).$$

We will now evaluate the imaginary part of the sum. The above sum is a geometric sum which is equal to

$$\frac{e^{ix} - e^{i(n+1)x}}{1 - e^{ix}}.$$

Now, we will use the fact that

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

$$\frac{e^{ix} - e^{i(n+1)x}}{1 - e^{ix}} = \frac{e^{i\frac{(n+2)x}{2}}(e^{-ni\frac{x}{2}} - e^{ni\frac{x}{2}})}{e^{i\frac{x}{2}}(e^{-i\frac{x}{2}} - e^{i\frac{x}{2}})} = e^{i\frac{(n+1)x}{2}}\frac{2i\sin(-n\frac{x}{2})}{2i\sin(-\frac{x}{2})} = \frac{\sin(\frac{nx}{2})e^{i(n+1)\frac{x}{2}}}{\sin(\frac{x}{2})}.$$

Thus, the answer is $\dfrac{\sin(nx/2)\sin((n+1)x/2)}{\sin(\frac{x}{2})}$. □

Exercise 1.38. *Consider the subset*

$$\mathcal{B} = \{1, \sin t, \cos t, \sin(2t), \cos(2t), \sin(3t), \cos(3t), \ldots\}$$

of the vector space $C(\mathbb{R})$.

(a) Prove that $\mathcal{B}$ is linearly independent.

(b) Prove that for every two positive integers n and m we have $\sin^n t \cos^m t \in \mathrm{Span}\ \mathcal{B}$.

Exercise 1.39. *Let V be a vector space over $\mathbb{F}$. Prove that V cannot be written as a union of finitely many proper subspaces of V.*

Exercise 1.40. *Consider a square matrix A whose entries in the j-th row from left to right form an arithmetic sequence with common difference d_j and first term x_j. Find $\det(A)$ in terms of x_j's and d_j's.*

Exercise 1.41. *Find the determinant of the $n \times n$ matrix whose entries from left to right and from top to bottom are $\cos 1, \cos 2, \ldots, \cos(n^2)$, where all angles are measured in radians.*

Exercise 1.42. *Generalize Exercises 1.28 and 1.29.*

Exercise 1.43. *Suppose $A, B \in M_n(\mathbb{R})$ satisfy $AB = BA$. Prove that $\det(A^2 + B^2) \geq 0$.*

Exercise 1.44. *Let $r \in \mathbb{F}$ and M be a positive real number. Prove that for every integer $n \geq 2$, there is a matrix $A \in M_n(\mathbb{F})$ for which $\det A = r$ and A has no entries with absolute value less than M.*

Exercise 1.45. *Suppose A, B are square matrices of the same size. Assume A is invertible. Prove there are infinitely many real numbers r for which $A + rB$ is also invertible.*

1.5 Summary

- Addition, subtraction, multiplication and division in $\mathbb{C}$ were discussed.
- Two complex numbers are the same whenever their real parts and imaginary parts are the same.
- Polar form of a complex number z is given as $|z|e^{i\theta}$, where $e^{i\theta} = \cos\theta + i\sin\theta$.
- $e^{a+bi} = e^a(\cos b + i\sin b)$.
- $e^z e^w = e^{z+w}$ for every $z, w \in \mathbb{C}$.
- When the scalars of a vector space are real numbers we say the vector space is a real vector space. When the scalars are complex numbers we say it is a complex vector space.
- Most properties of matrices, vector spaces, linear transformations, and determinants remain the same over $\mathbb{R}$ and $\mathbb{C}$.

Chapter 2

Diagonalization of a Matrix

2.1 Change of Coordinates

In this section all vector spaces are assumed to be finite-dimensional.

Definition 2.1. Let $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ be an ordered basis for a vector space V. The **coordinate vector** of a vector $\mathbf{v} \in V$ relative to $\mathcal{B}$ is a column vector $(c_1\ c_2\ \cdots\ c_n)^T$ for which $\mathbf{v} = \sum_{j=1}^{n} c_j \mathbf{b}_j$. This vector is denoted by $[\mathbf{v}]_{\mathcal{B}}$. The scalars $c_1, \ldots, c_n$ are called the **coordinates** of $\mathbf{v}$ in basis $\mathcal{B}$.

Example 2.1. Find the coordinate vector of $2t + 1$ in ordered basis $(1, 1 + t)$ for P_1.

Theorem 2.1. *Let $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be an ordered basis for the vector space V over $\mathbb{F}$. Then, the function $T : V \to \mathbb{F}^n$ defined by $T(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}$ is an isomorphism.*

Theorem 2.2. *Let V and W be two vector spaces with ordered bases $\mathcal{A} = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ and $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m)$, respectively, and let $T : V \to W$ be a linear transformation. Then, there is a unique $m \times n$ matrix A for which $[T(\mathbf{v})]_{\mathcal{B}} = A[\mathbf{v}]_{\mathcal{A}}$. Furthermore, this matrix A can be obtained by the following formula:*

$$A = \left([T(\mathbf{a}_1)]_{\mathcal{B}} \cdots [T(\mathbf{a}_n)]_{\mathcal{B}}\right).$$

Notation. The unique matrix A in the previous theorem is called the matrix of T relative to bases $\mathcal{A}$ and $\mathcal{B}$, and is denoted by $[T]_{\mathcal{B}\mathcal{A}}$. So, in shorts we have $[T(\mathbf{v})]_{\mathcal{B}} = [T]_{\mathcal{B}\mathcal{A}}[\mathbf{v}]_{\mathcal{A}}$.

Example 2.2. Consider the linear transformation $T : \mathbb{F}^2 \to \mathrm{P}_1$ defined by $T(a, b) = a + b + (a - b)t$. Write down the matrices of T relative to the following ordered bases:

(a) $(\mathbf{e}_1, \mathbf{e}_2)$ for $\mathbb{F}^2$ and $(1, t)$ for P_1.

(b) $((1, 1), (0, 1))$ for $\mathbb{F}^2$ and $(t, 1 - t)$ for P_1.

Theorem 2.3. *Suppose U, V, W are three vector spaces with ordered bases $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$, respectively. Let $S : U \to V$, and $T : V \to W$ be linear transformations. Then, $[T \circ S]_{\mathcal{C}\mathcal{A}} = [T]_{\mathcal{C}\mathcal{B}}[S]_{\mathcal{B}\mathcal{A}}$.*

Now, consider the identity transformation $I : V \to V$ defined by $I(\mathbf{v}) = \mathbf{v}$, from a vector space V to itself, and let $\mathcal{A}, \mathcal{B}$ be two ordered bases for V. We have $[\mathbf{v}]_{\mathcal{B}} = [I]_{\mathcal{BA}}[\mathbf{v}]_{\mathcal{A}}$. This matrix $[I]_{\mathcal{BA}}$ is called the matrix of **change of coordinates** from $\mathcal{A}$ to $\mathcal{B}$, because it allows us to change the coordinates of a vector from a basis $\mathcal{A}$ to a basis $\mathcal{B}$.

Theorem 2.4. *With the notations above,* $[I]_{\mathcal{BA}} = [I]_{\mathcal{AB}}^{-1}$.

The above theorem is especially useful when one of the bases is a standard basis. If $\mathcal{S}$ is the standard basis of $\mathbb{F}^n$, then $[I]_{\mathcal{SA}} = (\mathbf{a}_1 \cdots \mathbf{a}_n)$, and thus

$$[I]_{\mathcal{AS}} = (\mathbf{a}_1 \cdots \mathbf{a}_n)^{-1}.$$

Example 2.3. Write down $\binom{2}{3}$ in the ordered basis $(\binom{1}{2}, \binom{3}{5})$.

Example 2.4. Find the change of coordinate matrix from the ordered basis $(1, 1+t)$ to the ordered basis $(1+2t, 1-2t)$ of P_1. (You do not need to show these are bases of P_1.)

Let $\mathcal{A}, \mathcal{B}$ be ordered bases for a vector space V, and let $T : V \to V$ be a linear transformation. By what we discussed

$$[T]_{\mathcal{BB}} = [I]_{\mathcal{BA}}[T]_{\mathcal{AA}}[I]_{\mathcal{AB}} = [I]_{\mathcal{BA}}[T]_{\mathcal{AA}}[I]_{\mathcal{BA}}^{-1}.$$

We say $[T]_{\mathcal{BB}}$ and $[T]_{\mathcal{AA}}$ are similar matrices.

Definition 2.2. Two square matrices A, B of the same size are said to be **similar** if there is an invertible matrix P for which $A = PBP^{-1}$.

2.2 Eigenpairs and Diagonalization

Example 2.5. Evaluate $A^{100}\mathbf{v}$, where $A = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Definition 2.3. Suppose $T : V \to V$ is a linear transformation and $\mathbf{v} \in V$ is a nonzero vector for which $T(\mathbf{v}) = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{F}$. We say λ is an **eigenvalue**, $\mathbf{v}$ is an **eigenvector**, and the ordered pair $(\lambda, \mathbf{v})$ is called an **eigenpair** of T. Similarly we define eigenvalues, eigenvectors, and eigenpairs of a square matrix A.

Theorem 2.5. *λ is an eigenvalue of a square matrix A if and only if* $\det(A - \lambda I) = 0$.

Definition 2.4. Given a square matrix A, the polynomial $p(z) = \det(A - zI)$ is called the **characteristic polynomial of** A.

Remark. Note that if A is an $n \times n$ matrix, then $\det(zI - A) = (-1)^n \det(A - zI)$. Therefore, the roots of $\det(zI - A) = 0$ and $\det(A - zI) = 0$ are the same. In some textbooks, $\det(zI - A)$ is defined to be the characteristic polynomial of A.

Definition 2.5. A square matrix A is said to be **diagonalizable** if there is a diagonal matrix D and an invertible matrix P for which $A = PDP^{-1}$. In other words, a matrix is diagonalizable if it is similar to a diagonal matrix.

Theorem 2.6. *A matrix $A \in M_n(\mathbb{F})$ is diagonalizable if and only if there is a basis for $\mathbb{F}^n$ consisting of eigenvectors of A. Furthermore, if $(\lambda_1, \mathbf{v}_1), \ldots, (\lambda_n, \mathbf{v}_n)$ are eigenpairs of A whose eigenvectors form a basis for $\mathbb{F}^n$, then $A = PDP^{-1}$, where*

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}, \text{ and } P = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix}.$$

Notation. The diagonal matrix whose diagonal entries are $\lambda_1, \ldots, \lambda_n$ in this order is denoted by $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. In other words:

$$\mathrm{diag}(\lambda_1, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}.$$

Remark. The process of finding an invertible matrix P and a diagonal matrix D for which $A = PDP^{-1}$ is called **diagonalizing** matrix A.

Example 2.6. Diagonalize the matrix $A = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}$. Use that to find A^n for every n.

Remark. For every nonzero integer n we have $(PAP^{-1})^n = PA^nP^{-1}$.

Theorem 2.7. *Every two similar matrices have the same characteristic polynomial and therefore they have the same eigenvalues.*

Theorem 2.8 (Fundamental Theorem of Algebra)**.** *Every polynomial $p(x)$ of degree n with complex coefficients can be completely factored into linear terms. In other words, there are complex numbers $c, a_1, \ldots, a_n$ for which:*

$$p(x) = c(x - a_1) \cdots (x - a_n).$$

Theorem 2.9. *Eigenvectors corresponding to distinct eigenvalues are linearly independent. Furthermore, if an $n \times n$ matrix A has n distinct eigenvalues, then it is diagonalizable.*

Example 2.7. Show that the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

We will now look at some applications of diagonalization.

Suppose we want to define e^A for a square matrix A. We could use the power series expansion of e^x and define e^A by

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots.$$

However, we would need to first show this infinite sum converges. Then, we need to find a way to evaluate it.

Example 2.8. Evaluate e^A, where $A = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}$.

In general if A is diagonalizable we can find e^A fairly easily.

Theorem 2.10. *Suppose A is a diagonalizable matrix with $A = PDP^{-1}$, where $D = (\lambda_1, \ldots, \lambda_n)$. Then $e^A = P\ \mathrm{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})\ P^{-1}$.*

2.3 More Examples

Example 2.9. Suppose $(\lambda, \mathbf{v})$ is an eigenpair for a square matrix A with real entries. Prove that $(\overline{\lambda}, \overline{\mathbf{v}})$ is also an eigenpair for A.

Solution. By assumption $A\mathbf{v} = \lambda\mathbf{v}$. Note that since for every two complex numbers z, w we have $\overline{zw} = \overline{z}\,\overline{w}$ and $\overline{z+w} = \overline{z} + \overline{w}$, we will obtain the following:

$$\overline{A\mathbf{v}} = \overline{\lambda\mathbf{v}} \Rightarrow \overline{A}\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}} \Rightarrow A\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}.$$

Above we use the fact that all entries of A are real and thus $\overline{A} = A$. Since $\mathbf{v}$ is nonzero, $\overline{\mathbf{v}}$ is also nonzero and thus $(\overline{\lambda}, \overline{\mathbf{v}})$ is an eigenpair of A, as desired. $\square$

Example 2.10. Find the coordinate vector of $1 - t + 3t^2$ in each ordered basis of P_2 below. Assume these are bases.

(a) $\mathcal{A} = (1, t, t^2)$.

(b) $\mathcal{B} = (t, 1, t^2)$.

(c) $\mathcal{C} = (1 + t, 1 - t^2, t - t^2)$.

Solution. (a) By definition the answer is $(1 \quad -1 \quad 3)^T$.

(b) By definition the answer is $(-1 \quad 1 \quad 3)^T$.

(c) For simplicity let $p(t) = 1 - t + 3t^2$. We know $[p(t)]_{\mathcal{A}} = (1 \quad -1 \quad 3)^T$. In order to find $[p(t)]_{\mathcal{C}}$ we will find $[I]_{\mathcal{C}\mathcal{A}}$. Then use the fact that $[p(t)]_{\mathcal{C}} = [I]_{\mathcal{C}\mathcal{A}}[p(t)]_{\mathcal{A}}$. By Theorem 2.2 we have

$$[I]_{\mathcal{A}\mathcal{C}} = \big([1+t]_{\mathcal{A}}\ [1-t^2]_{\mathcal{A}}\ [t-t^2]_{\mathcal{A}}\big) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}.$$

Therefore, by Theorem 2.4 we have the following:

$$[I]_{\mathcal{C}\mathcal{A}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 \end{pmatrix}$$

The final answer is obtained by evaluating $[I]_{\mathcal{CA}}[p(t)]_{\mathcal{A}}$. The answer is $(3/2\ \ -1/2\ \ -5/2)^T$. $\square$

Example 2.11. Prove that a 2×2 matrix with complex entries is not diagonalizable if and only if it is similar to a matrix of the form

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix},$$

where $a, b \in \mathbb{C}$ and $b \neq 0$.

Solution. $(\Rightarrow)$ Assume A is a 2×2 matrix that is not diagonalizable. By Theorem 2.9 the two eigenvalues of A must be identical. Assume a is the only eigenvalue of A and let $\mathbf{v}$ be an eigenvector corresponding to a. Let $\mathbf{w} \in \mathbb{C}^2$ be a vector that is not a scalar multiple of $\mathbf{v}$. Since $A\mathbf{v} = a\mathbf{v}$, the matrix A in the basis $(\mathbf{v}, \mathbf{w})$ is of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

Since this matrix is similar to A, its only eigenvalue must be a. Therefore, $c = a$. On the other hand b cannot be zero, for otherwise A would be diagnozalizable.

$(\Leftarrow)$ Assume A is similar to a matrix of the form

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix},$$

where $a, b \in \mathbb{C}$ and $b \neq 0$. On the contrary assume A is diagonalizable. Since the eigenvalues of A are both a, we must have:

$$A = Q \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} Q^{-1} = QaIQ^{-1} = aI.$$

Therefore,

$$P \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} P^{-1} = aI \Rightarrow \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = P^{-1}aIP = aI = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

This implies $b = 0$, which is a contradiction. Therefore, A is not diagonalizable. $\square$

Example 2.12. Prove that if A and B are similar matrices, then $\det A = \det B$ and $\mathrm{tr}\ A = \mathrm{tr}\ B$.

Solution. Since A and B are similar, $B = PAP^{-1}$ for some invertible matrix P. By properties of determinant, we have

$$\det(PAP^{-1}) = (\det P)(\det A)(\det P^{-1}) = (\det P)(\det A)(\det P)^{-1} = \det A.$$

Also, by Exercise 1.15, we have $\mathrm{tr}\ (PAP^{-1}) = \mathrm{tr}\ (P^{-1}PA) = \mathrm{tr}\ A$. Therefore, $\det A = \det B$ and $\mathrm{tr}\ A = \mathrm{tr}\ B$, as desired. $\square$

Example 2.13. Determine which of the following matrices are similar.

$$\begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

Solution. Let's call these matrices A, B, C, D, E in order. We note that

$$\det A = \det C = 6, \quad \det B = \det D = \det E = 1.$$

Therefore, A and C may be similar and B, D, E may be similar. We notice tr $A = 4$ and tr $C = 5$ are not the same. Therefore, A and C are also not similar. Thus, A and C are not similar to any of the above matrices. Note that tr $B =$ tr $D =$ tr $E = 4$, so these three may be similar. The characteristic equations of B, D and E are all $z^2 - 4z + 6 = 0$ which has 2 distinct roots r, s. Thus, all matrices B, D, E are similar to the diagonal matrix diag(r, s). This means B, D, E are all similar. □

Definition 2.6. A square matrix A is said to be **nilpotent** if $A^k = 0$ for some positive integer k.

Example 2.14. Suppose a diagonalizable matrix A is also nilpotent. Prove that $A = 0$.

Solution. Since A is diagonalizable $A = PDP^{-1}$ for some diagonal matrix $D = \text{diag}(c_1, \ldots, c_n)$. By assumption, $A^k = 0$ for some positive integer k. Therefore,

$$(PDP^{-1})^k = 0 \Rightarrow PD^kP^{-1} = 0 \Rightarrow D^k = 0.$$

This implies

$$\text{diag}(c_1^k, \ldots, c_n^k) = 0 \Rightarrow c_1^k = \cdots = c_n^k = 0 \Rightarrow c_1 = \cdots = c_n = 0 \Rightarrow D = 0.$$

Therefore, $A = P0P^{-1} = 0$, as desired. □

Example 2.15. Prove that the set of all eigenvectors of a linear transformation $T : V \to V$ corresponding to a fixed eigenvalue λ along with the zero vector, is a subspace of V. Prove a similar result for a square matrix.

Solution. Let W be the set of all eigenvectors of T corresponding to λ along with the zero vector. We see that $\mathbf{x} \in W$ if and only if $T(\mathbf{x}) = \lambda\mathbf{x}$ or $\mathbf{x} = \mathbf{0}$, however $T(\mathbf{0}) = \mathbf{0} = \lambda\mathbf{0}$. Therefore, $\mathbf{x} \in W$ if and only if $T(\mathbf{x}) = \lambda\mathbf{x}$, which is equivalent to $(T - \lambda I)(\mathbf{x}) = \mathbf{0}$, which is equivalent to $\mathbf{x} \in$ Ker $(T - \lambda I)$. Therefore, $W =$ Ker $(T - \lambda I)$ and hence it is a subspace of V. A similar argument works for a square matrix. □

Example 2.16. Diagonalize each matrix or show the matrix is not diagonalizable.

$$A = \begin{pmatrix} -1 & -2 & 2 \\ -2 & -1 & 2 \\ -2 & -2 & 3 \end{pmatrix}, B = \begin{pmatrix} -2 & -4 & 5 \\ -2 & 0 & 1 \\ -3 & -3 & 5 \end{pmatrix}, C = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Solution. $\det(A - \lambda I) = -\lambda^3 + \lambda^2 + \lambda - 1$. We guess $\lambda = 1$ as a root. After performing long division we can factor this polynomial as $(\lambda - 1)(-\lambda^2 + 1)$. Therefore, the eigenvalues of A are $1, 1, -1$.

For $\lambda = 1$ we can find the eigenvectors by solving the following:

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -2 & 2 \\ -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \Rightarrow -2x - 2y + 2z = 0 \Rightarrow z = x + y.$$

This yields, two linearly independent eigenvectors for $\lambda = 1$: $(1\ 0\ 1)^T$ and $(0\ 1\ 1)^T$.

For $\lambda = -1$ we can find the eigenvectors by solving the following:

$$\begin{pmatrix} 0 & -2 & 2 \\ -2 & 0 & 2 \\ -2 & -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \Rightarrow \begin{cases} -2y + 2z = 0 \\ -2x + 2z = 0 \\ -2x - 2y + 4z = 0 \end{cases} \Rightarrow z = x = y.$$

This yields an eigenvector $(1\ 1\ 1)^T$ for $\lambda = -1$. Therefore, $A = PDP^{-1}$, where $D = \mathrm{diag}(1, 1, -1)$ and $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

$\det(B - \lambda I) = -\lambda^3 + 3\lambda^2 - 4$. By inspection a root of this polynomial can be obtained as $\lambda = -1$. After performing long division we obtain $-\lambda^3 + 3\lambda^2 - 4 = (\lambda + 1)(-\lambda^2 + 4\lambda - 4) = -(\lambda + 1)(\lambda - 2)^2$. The eigenvalues are $\lambda = -1, 2, 2$. Following the same process as before we can find an eigenvector for $\lambda = -1$. For $\lambda = 2$ we need to solve the following:

$$\begin{pmatrix} -4 & -4 & 5 \\ -2 & -2 & 1 \\ -3 & -3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \Rightarrow \begin{cases} -4x - 4y + 5z = 0 \\ -2x - 2y + z = 0 \\ -3x - 3y + 3z = 0 \end{cases}$$

After solving we obtain $z = 0$ and $y = -x$. Therefore, the eigenspace corresponding to $\lambda = 2$ is one-dimensional. This means we cannot find three linearly independent eigenvectors, which implies B is not diagonalizable.

$\det(C - \lambda I) = \lambda^2 - 4\lambda + 3$. The eigenvalues, thus, are $\lambda = 1, 3$, which are distinct and thus C is diagonalizable. After finding the eigenvectors we will get the following diagonalization of C:

$$C = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1}.$$

$\square$

Example 2.17. Find all scalars c for which the matrix A given below is not diagonalizable.

$$A = \begin{pmatrix} 1 & c \\ 2 & -1 \end{pmatrix}.$$

Solution. The characteristic polynomial is $(1-z)(-1-z) - 2c = z^2 - 1 - 2c$. The eigenvalues of A are then $z = \pm\sqrt{1+2c}$. If the eigenvalues are distinct, then by Theorem 2.9 the matrix A is diagonalizable. Suppose the two eigenvalues are identical. This means $1 + 2c = 0$, which implies $c = -1/2$. In this case the eigenvalues are both zero. If A were diagonalizable, then $A = P0P^{-1} = 0$, which is a contradiction, because A is not the zero matrix. Therefore, the answer is $c = -1/2$. □

Example 2.18. Find all scalars c for which $\lambda = 1$ is an eigenvalue of the matrix

$$A = \begin{pmatrix} 1 & c & -1 \\ c & 1 & 0 \\ 2 & 3 & -1 \end{pmatrix}$$

Solution. $\lambda = 1$ is an eigenvalue of A if and only if $\det(A - I) = 0$. This is equivalent to

$$\det \begin{pmatrix} 1-1 & c & -1 \\ c & 1-1 & 0 \\ 2 & 3 & -1-1 \end{pmatrix} = 0$$

Expanding along the second row we obtain

$$-c \det \begin{pmatrix} c & -1 \\ 3 & -2 \end{pmatrix} = 0 \Rightarrow c(-2c+3) = 0.$$

Therefore, the answer is $c = 0, 3/2$. □

Example 2.19. Show that the characteristic polynomial of an $n \times n$ matrix has degree n and its leading coefficient is $(-1)^n$.

Solution. We will prove this by induction on n. Suppose $A = (a_{ij}) \in M_n(\mathbb{F})$ and let B be the $(n-1)\times(n-1)$ upper left corner submatrix of A. In other words,

$$A = \begin{pmatrix} B & * \\ * & a_{nn} \end{pmatrix}.$$

This yields the following:

$$A - zI = \begin{pmatrix} B - zI & * \\ * & a_{nn} - z \end{pmatrix}.$$

Expanding along the last row we obtain

$$\det(A - zI) = (a_{nn} - z)\det(B - zI) + \sum_{j=1}^{n-1} (-1)^{j+n} a_{nj} \det A_{jn},$$

where A_{jn} is the matrix obtained by removing the j-th column and n-th row of $A - zI$. Since a_{nj} is a constant and A_{jn} is of size $(n-1)\times(n-1)$, none of the terms $a_{jn}\det A_{jn}$ contains a term of the form z^n. By inductive hypothesis, the leading term of $\det(B - zI)$ is $(-1)^{n-1}z^{n-1}$. Therefore, the leading term of $(a_{nn} - z)\det(B - zI)$ is $-z(-1)^{n-1}z^{n-1} = (-1)^n z^n$, as desired. $\square$

2.4 Exercises

Exercise 2.1. *Diagonalize each matrix or show it is not diagonalizable.*

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 5 & -4 \\ 6 & -5 \end{pmatrix}, \begin{pmatrix} 5 & 0 & -3 \\ 2 & 1 & -2 \\ 6 & 0 & -4 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise 2.2. *Suppose $T : \mathrm{P}_3 \to \mathrm{P}_2$ is a linear transformation for which*

$$T(t+2) = T(t^3 + 2t - 1) = T(t^2 - 1) = T(1) = t.$$

(a) Find the dimension of $\mathrm{Ker}\ T$ *and a basis for this subspace.*

(b) Find all polynomials $p(t)$ for which $T(p(t)) = 3t$.

Hint: Use Theorem 1.14.

Exercise 2.3. *Let A, P be two square matrices of the same size for which P is invertible. Using induction, prove that $(PAP^{-1})^n = PA^nP^{-1}$ for every nonzero integer n. Note that you would have to deal with the cases where n is negative and positive separately.*

Exercise 2.4. *Prove that the characteristic polynomial of any matrix $A \in M_2(\mathbb{F})$ is*

$$p(z) = z^2 - (\mathrm{tr}\ A)z + \det A.$$

Exercise 2.5. *Find the change of coordinate matrix from the basis $(t+1, 2t-1)$ for P_1 to the basis $(1, t+2)$. (You may assume these are bases of P_1.)*

Exercise 2.6. *Let n be a positive integer. For every $1 \le i, j \le n$ let $f_{ij}(x)$ be a differentiable function. Consider the $n \times n$ matrix $A(x)$ whose (i,j) entry is $f_{ij}(x)$. For example, when $n = 2$, we get the following matrix*

$$A(x) = \begin{pmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{pmatrix}$$

Let $F(x) = \det(A(x))$. In this problem we will find a formula for $F'(x)$.

For every i with $1 \le i \le n$, let $A_i(x)$ be the matrix obtained from $A(x)$ by replacing the i-th row of $A(x)$ with the derivative of the i-th row of $A(x)$, keeping everything else intact. In other words, the i-th row of $A_i(x)$ is $(f'_{i1}(x)\ f'_{i2}(x)\ \ldots\ f'_{in}(x))$. For example, when $n = 2$, we get the following matrices

$$A_1(x) = \begin{pmatrix} f'_{11}(x) & f'_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{pmatrix}, \text{ and } A_2(x) = \begin{pmatrix} f_{11}(x) & f_{12}(x) \\ f'_{21}(x) & f'_{22}(x) \end{pmatrix}$$

(a) When $n = 2$, prove that

$$F'(x) = \det(A_1(x)) + \det(A_2(x)).$$

(b) Using induction on n, prove that

$$F'(x) = \det(A_1(x)) + \det(A_2(x)) + \cdots + \det(A_n(x)).$$

Exercise 2.7. *Construct a matrix A for which $A^3 = 0$ but $A^2 \neq 0$.*

Exercise 2.8. *Find all eigenpairs of the linear transformation $T : \mathrm{P}_2 \to \mathrm{P}_2$ defined by*

$$T(at^2 + bt + c) = (a - b)t + (c - b), \forall a, b, c \in \mathbb{F}.$$

You may assume T is linear.

Exercise 2.9. *Suppose a square matrix A satisfies $A^2 - 2A + 7I = 0$. Prove that:*

(a) A is invertible.

(b) A has no real eigenvalues!

Exercise 2.10. *Let F_n be the sequence of Fibonacci numbers $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$.*

(a) Prove that $\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = A \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Deduce $\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A^n \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$ for all $n \geq 1$.

(b) Diagonalize A and evaluate A^n.

(c) Find an explicit formula for F_n.

Exercise 2.11. *Suppose $A, B \in M_n(\mathbb{R})$ are invertible matrices. Is it true that $A + iB$ must be invertible?*

Exercise 2.12. *Suppose A is a 2×2 matrix with eigenvalues 1 and 2. Find two sequences a_n, b_n for which $A^n = a_n A + b_n I$ for all positive integers n.*

Exercise 2.13. *Suppose the list of all eigenvalues of an $n \times n$ matrix is*

$$\lambda_1, \lambda_2, \ldots, \lambda_n.$$

Prove that for every $r \in \mathbb{F}$, the list of all eigenvalues of $A + rI$ is

$$\lambda_1 + r, \lambda_2 + r, \ldots, \lambda_n + r.$$

Hint: Write down the characteristic polynomial of A in terms of λ_j's.

Exercise 2.14. *Consider a 2×2 matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with complex entries. Find the necessary and sufficient condition on a, b, c, d for which A is not diagonalizable.

Hint: Find the characteristic polynomial. Under what condition are the eigenvalues identical? If the eigenvalues are identical and A is diagonalizable, show $A = \lambda I$ for some $\lambda \in \mathbb{C}$.

Exercise 2.15. *Consider the transformation $T : \mathbb{F}^2 \to \mathbb{F}^2$ given by $T(a, b) = (2a + b, 3a - b)$.*

(a) Prove T is linear.

(b) Find the matrix of T in the standard basis.

(c) Find the matrix of T in the ordered basis $((1, 2), (-1, 3))$.

Exercise 2.16. *Determine which of the following matrices are similar:*

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -2 & -1 \\ -3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$$

Exercise 2.17. *Consider the function $T : \mathbb{C} \to \mathbb{C}$ defined by $T(a + bi) = (3a - b) + bi$, for all $a, b \in \mathbb{R}$.*

(a) Is T linear, if $\mathbb{C}$ is considered a real vector space?

(b) Is T linear, if $\mathbb{C}$ is considered a complex vector space?

(c) In each case above, if T is linear find its matrix in the standard basis.

Exercise 2.18. *Consider the matrix*

$$A = \begin{pmatrix} a & b - a \\ 0 & b \end{pmatrix}.$$

(a) Evaluate A^2, A^3, and A^4.

(b) Guess and prove a formula for A^n for every $n \in \mathbb{Z}^+$, once using induction and once using diagonalization.

Exercise 2.19. *Let* $A = \begin{pmatrix} -2 & 3 \\ -6 & 7 \end{pmatrix}$

(a) Find a diagonalization of A.

(b) Using part (a) find four different matrices B for which $B^2 = A$. You do not need to simplify your answers.

Exercise 2.20. *Given a square matrix A, prove that*

(a) $\det A$ *is the product of eigenvalues of A.*

(b) $\operatorname{tr} A$ *is the sum of eigenvalues of A.*

Exercise 2.21. *Suppose A and B are two square matrices of the same size that have the same eigenvalues.*

(a) By an example show that A and B do not have to be similar.

(b) Suppose all eigenvalues of A are distinct. Prove that A and B are similar.

Exercise 2.22. *Suppose $A \in M_2(\mathbb{F})$ has eigenvalues λ_1, λ_2. Prove that every column of $A - \lambda_1 I$ is either zero or it is an eigenvector of A corresponding to λ_2.*

Hint: Use Cayley-Hamilton Theorem.

Exercise 2.23. *State the converse of Theorem 2.7, and by an example show it is false.*

2.5 Challenge Problems

Exercise 2.24. *Let $A \in M_n(\mathbb{C})$ be an invertible matrix with n distinct eigenvalues. Prove that there are precisely 2^n matrices B for which $B^2 = A$.*

Exercise 2.25. *Let $A \in M_n(\mathbb{R})$ be a matrix all of whose eigenvalues are real. Suppose A as a matrix in $M_n(\mathbb{C})$ is diagonalizable. Is it true that A is diagonalizable in $M_n(\mathbb{R})$?*

Exercise 2.26. *Suppose $A, B \in M_n(\mathbb{R})$ are matrices that are similar in $M_n(\mathbb{C})$. Is it true that A and B must be similar as matrices of $M_n(\mathbb{R})$?*

Exercise 2.27. *Prove that if A, B are square matrices of the same size for which $A + B = AB$, then $AB = BA$.*

Exercise 2.28. *Let F_n be the Fibonacci sequence given by $F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$. For every positive integer k, find a matrix $A \in M_k(\mathbb{R})$ for which*

$$\lim_{n\to\infty} \frac{\det(A^n)}{F_n} = 1,$$

or show no such matrix exists.

Exercise 2.29. *Suppose $A_1, \ldots, A_n$ are diagonalizable matrices that commute pairwise. Prove that they can all be simultaneously diagonalized. In other words, there is an invertible matrix P for which PA_jP^{-1} is diagonal, for every j.*

2.6 Summary

- The matrix of a linear transformation T relative to bases $\mathcal{A}$ and $\mathcal{B}$ is given by

$$[T]_{\mathcal{B}\mathcal{A}} = \left([T(\mathbf{a}_1)]_{\mathcal{B}} \cdots [T(\mathbf{a}_n)]_{\mathcal{B}}\right).$$

- The matrix of change of coordinates can often be found using $[I]_{\mathcal{B}\mathcal{A}} = [I]_{\mathcal{B}\mathcal{S}}[I]_{\mathcal{S}\mathcal{A}}$ and the fact that $[I]_{\mathcal{B}\mathcal{S}} = [I]_{\mathcal{S}\mathcal{B}}^{-1}$.

- The matrix of a linear transformation $T : V \to V$ in two different bases of V are similar. (Similar means $B = PAP^{-1}$.)

- To find eigenvalues of A, solve $\det(A - \lambda I) = 0$.
- To find eigenvectors of A solve $A\mathbf{v} = \lambda\mathbf{v}$ after having found an eigenvalue λ.
- A matrix is diagonalizable iff there is a basis of eigenvectors: $A = PDP^{-1}$, where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, and $P = (\mathbf{v}_1 \cdots \mathbf{v}_n)$.
- Eigenvectors corresponding to distinct eigenvalues are linearly independent. Thus, if an $n \times n$ matrix has n distinct eigenvalues, then it is diagonalizable.

Chapter 3

Jordan Canonical Form

3.1 Triangularization

Not all matrices can be diagonalized. Fortunately, we can do something close to diagonalization!

Definition 3.1. A square matrix A is said to be **upper triangular** if all of its entries below the main diagonal are zero.

Theorem 3.1 (Block Multiplication)**.** *Suppose matrices A and B are given as block matrices below*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \; B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where A_{jk}, B_{jk} are themselves matrices. Then, assuming all appropriate multiplications and additions are defined we obtain

$$AB = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

Theorem 3.2. *Every square matrix with complex entries is similar to an upper triangular matrix with complex entries.*

Example 3.1. Write the following matrix in the form PTP^{-1}, where T is upper triangular.

$$\begin{pmatrix} 0 & 0 & -1 \\ 5 & 2 & 3 \\ 2 & 0 & 3 \end{pmatrix}$$

Example 3.2. Give an example of a matrix with real entries that is not similar to an upper triangular matrix with real entries.

Definition 3.2. Given a polynomial $p(x) = a_m x^m + \cdots + a_1 x + a_0$ and an $n \times n$ matrix A we define

$$p(A) = a_m A^m + \cdots + a_1 A + a_0 I.$$

Theorem 3.3 (Cayley-Hamilton Theorem)**.** *Let A be a square matrix, and $p(x)$ be the characteristic polynomial of A. Then, $p(A) = 0$.*

3.2 Jordan Canonical Form

We have so far shown that every matrix is similar to an upper triangular matrix, but can we write these upper triangular matrices in a more specific form? Let's look at an example.

Example 3.3. Consider the matrix $A = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

(a) Find all eigenpairs of A and show A is not diagonalizable.

(b) Find a basis for Ker $(A-2I)^2$.

(c) Find a basis for Ker $(A-2I)^n$ for all n.

(d) Use that to write a matrix similar to A in an "almost diagonal" form.

Theorem 3.4. *Let λ be an eigenvalue of an $n \times n$ matrix A. Then, there is an integer $k \leq n$ for which*

$$\text{Ker } (A-\lambda I) \subsetneqq \text{Ker } (A-\lambda I)^2 \subsetneqq \cdots \subsetneqq \text{Ker } (A-\lambda I)^k = \text{Ker } (A-\lambda I)^{k+1} = \cdots .$$

Definition 3.3. For an eigenvalue λ of an $n \times n$ matrix A, every nonzero vector in Ker $(A-\lambda I)^n$ is called a **generalized eigenvector** of A corresponding to eigenvalue λ. The vector space Ker $(A-\lambda I)^n$ is called the **generalized eigenspace** of A corresponding to eigenvalue λ.

Example 3.4. Suppose $(\lambda, \mathbf{v})$ is an eigenpair of a square matrix A and $p(x)$ is a polynomial. Prove that $p(A)\mathbf{v} = p(\lambda)\mathbf{v}$.

Theorem 3.5. *Generalized eigenvectors correspoding to distinct eigenvalues are linearly independent.*

Definition 3.4. The multiplicity of a root r of a polynomial $p(z)$ is m if

$$p(z) = (z-r)^m q(z),$$

for some polynomial $q(z)$ with $q(r) \neq 0$.

Theorem 3.6. *The dimension of the generalized eigenspace corresponding to the eigenvalue λ for a square matrix A is the same as the multiplicity of λ as a root of the characteristic polynomial of A.*

The above theorem implies that given an $n \times n$ matrix A we can find a basis for $\mathbb{F}^n$ consisting of generalized eigenvectors of A by finding a basis for each generalized eigenspace and putting all of these bases together.

Definition 3.5. A matrix is said to be in **Jordan canonical form** (or Jordan form for short) if it is a block matrix of the form

$$\begin{pmatrix} B_1 & 0 & \cdots & 0 & 0 \\ 0 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & B_n \end{pmatrix},$$

where each B_j, called a **Jordan block**, is a matrix with an eigenvalue λ_j on its main diagonal, 1's immediately above the main diagonal, and zeros everywhere else. In other words:

$$B_j = \begin{pmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & \cdots & 0 & \lambda_j \end{pmatrix}.$$

Example 3.5. Write down the following matrix in Jordan form:

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 5 & -1 \\ -4 & 13 & -2 \end{pmatrix}$$

Theorem 3.7. *Every matrix in $M_n(\mathbb{C})$ is similar to a matrix in Jordan form. Furthermore, this matrix in Jordan form is unique up to a permutation of Jordan blocks.*

Example 3.6. How many 5×5 nonsimilar matrices in Jordan form are there all of whose eigenvalues are 0?

Example 3.7. Find the number of nonsimilar 6×6 matrices in Jordan form whose eigenvalues are $1, 2, 2, 3, 3, 3$.

Theorem 3.8. *Let A be a matrix in $M_n(\mathbb{C})$ and J be a matrix in Jordan form that is similar to A. Then, for every positive integer k, the number of Jordan blocks of J with size at least $k \times k$ corresponding to an eigenvalue λ is given by*

$$\dim \operatorname{Ker}\, (A - \lambda I)^k - \dim \operatorname{Ker}\, (A - \lambda I)^{k-1}.$$

Here $(A - \lambda I)^0 = I$ and thus its kernel has dimension zero.

Using the above theorem, we can find the Jordan form of any matrix rather easily, however finding the matrix P in $A = PJP^{-1}$ is more difficult.

Example 3.8. The characteristic polynomial of a matrix A is $p(z) = z^6(z-1)^4$. Suppose

$$\dim \operatorname{Ker} A = 1, \text{ and } \dim \operatorname{Ker}\, (A - I) = 3.$$

Find a matrix in Jordan form that is similar to A.

To find P in $A = PJP^{-1}$, where J is in Jordan form, start with a vector $\mathbf{v}_k$ in Ker $(A - \lambda I)^k$ that does not lie in Ker $(A - \lambda I)^{k-1}$. Evaluate vectors $\mathbf{v}_{k-1} = (A - \lambda I)\mathbf{v}_k$, $\mathbf{v}_{k-2} = (A - \lambda I)\mathbf{v}_{k-1}, \ldots$, $\mathbf{v}_1 = (A - \lambda I)\mathbf{v}_2$. Repeat this until you get enough vectors. These $\mathbf{v}_j$'s give us columns of matrix P.

3.3 Applications of Jordan Canonical Form

Theorem 3.9. *For any square matrix A with complex entries, there are matrices D and N of the same size for which all of the following hold:*

- *D is diagonalizable and N is nilpotent;*
- *The eigenvalues of D and A are the same;*
- *$DN = ND$; and*
- *$A = D + N$.*

Previously we defined e^A for any square matrix A by

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!},$$

but we never proved this sum in fact converges. Here we will prove that and we will also define $f(A)$ for a class of functions called analytic functions.

Definition 3.6. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **analytic** if there is a sequence a_n of real numbers for which

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \text{ for all } x \in \mathbb{R}.$$

Theorem 3.10. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is analytic, i.e. $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then, $a_n = \dfrac{f^{(n)}(0)}{n!}$ and*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \text{ for all } z \in \mathbb{C}.$$

Definition 3.7. Given a sequence of matrices $A_n = (a_{ij,n})$, we define the matrix $A = \lim_{n\to\infty} A_n$ to be the matrix whose (i,j) entry is the limit of the sequence of (i,j) entries of A_n. In other words

$$\lim_{n\to\infty} (a_{ij,n}) = (\lim_{n\to\infty} a_{ij,n}).$$

When $A_n(t)$ is a sequence of matrices whose entries are functions of t, then their limit $A(t)$ is defined the same way for every real number t.

The following theorem can be easily proved using the above definition and properties of limit.

Theorem 3.11. *Let j, k, ℓ be three positive integers and let A_n, B_n be two sequences of $j \times k$ matrices, and C_n be a sequence of $k \times \ell$ matrices. Let $a_n \in \mathbb{F}$ be a sequence of scalars. Suppose*

$$\lim_{n\to\infty} A_n = A, \lim_{n\to\infty} B_n = B, \lim_{n\to\infty} C_n = C, \text{ and } \lim_{n\to\infty} a_n = a.$$

Then,

- $\lim_{n\to\infty} (A_n + B_n) = A + B$.
- $\lim_{n\to\infty} (A_n C_n) = AC$.
- $\lim_{n\to\infty} (a_n A_n) = aA$.

Now, suppose $f(t)$ is an analytic function. By Theorem 3.9, we can write $A = D + N$, where D is diagonalizable and N is nilpotent and $ND = DN$. We will define $f(A) = \sum_{k=0}^{\infty} a_k A^k$, where $\sum_{k=0}^{\infty} a_k t^k$ is the Taylor series for $f(t)$. We will need to show $\sum_{k=0}^{\infty} a_k A^k$ converges for every matrix A.

Let $p_m(t) = \sum_{k=0}^{m} a_k t^k$ be the m-th partial sum of the Taylor series for $f(t)$.

Since D is diagonalizable we can write $D = S \operatorname{diag}\{\lambda_1, \dots, \lambda_n\} S^{-1}$. Since, p_m is a polynomial, we have:

$$p_m(D) = S \operatorname{diag}\{p_m(\lambda_1), \dots, p_m(\lambda_n)\} S^{-1}$$

As $m \to \infty$, we have $p_m(\lambda_j) \to f(\lambda_j)$ for every j. Therefore,

$$f(D) = \lim_{m\to\infty} p_m(D) = S \operatorname{diag}\{f(\lambda_1), \dots, f(\lambda_n)\} S^{-1}.$$

From calculus, we know the Taylor polynomial of p_m centered at x_0 is given by

$$p_m(x) = \sum_{k=0}^{\infty} \frac{p_m^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Substituting $x = A$ and $x_0 = D$, we see that

$$p_m(A) = \sum_{k=0}^{\infty} \frac{p_m^{(k)}(D)}{k!} N^k.$$

Since N is nilpotent, $N^n = 0$ for some positive integer n. Therefore,

$$p_m(A) = \sum_{k=0}^{n-1} \frac{p_m^{(k)}(D)}{k!} N^k,$$

since $N^n = N^{n+1} = \cdots = 0$. As $m \to \infty$, $p_m^{(k)}(D) \to f^{(k)}(D)$. Therefore, we have:

$$f(A) = \lim_{m\to\infty} p_m(A) = \sum_{k=0}^{n-1} \frac{f^{(k)}(D)}{k!} N^k.$$

This means the $p_m(A)$ converges and thus the power series for $f(A)$ is convergent.

If we substitute $f(x) = e^x$, we obtain: $e^A = e^D \sum_{k=0}^{n-1} \frac{N^k}{k!}$.

(Note: We define $A^0 = I$ for every square matrix A.)

We summarize this in the following two theorems.

Theorem 3.12. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is an analytic function, with its Taylor series given as $f(t) = \sum_{k=0}^{\infty} a_k t^k$. Then, for every matrix $A \in M_n(\mathbb{C})$, the series $f(A) = \sum_{k=0}^{\infty} a_k A^k$ converges. Furthermore, if $A = D + N$ with $DN = ND$, D diagonalizable, and N nilpotent, then $f(A) = \sum_{k=0}^{n-1} \frac{f^{(k)}(D)}{k!} N^k$.*

Theorem 3.13. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is an analytic function, $D = P\text{diag}(\lambda_1, \ldots, \lambda_n)P^{-1}$ is a diagonalizable matrix. Then,*

$$f(D) = P\text{diag}(f(\lambda_1), \ldots, f(\lambda_n))P^{-1}.$$

Example 3.9. Evaluate $\sin A$ where $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$.

Example 3.10. Evaluate e^B, where $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

3.4 More Examples

Example 3.11. Suppose a matrix A with complex entries satisfies the equation $A^3 - A = 0$. Prove that A is diagonalizable.

Solution. First, note that in order to show A is diagonalizable it is enough to show the Jordan form J similar to A is diagonal. Note that since $A = PJP^{-1}$ we have $PJ^3P^{-1} = PJP^{-1}$, and thus $J^3 = J$. Note that using block multiplication of matrices we conclude that if B is a Jordan block of J then $B^3 = B$. So, we will have to show if B is a Jordan block for which $B^3 = B$ then B is diagonal. After calculation we can see that the $(1, 2)$ entry of B^3 is 3λ which must be the same as the $(1, 2)$ entry of B which is 1. However by comparing the diagonal entries of B and B^3 we conclude that $\lambda^3 = \lambda$. This is impossible. $\square$

Example 3.12. Suppose $A = PBP^{-1}$ for three square matrices A, B, P. Prove that $e^A = Pe^BP^{-1}$.

Solution. By definition $e^A = \lim_{m\to\infty} p_m(A)$, where $p_m(z) = \sum_{k=0}^{m} \frac{z^k}{k!}$. Substituting $A = PBP^{-1}$ we obtain the following:

$$p_m(A) = \sum_{k=0}^{m} \frac{(PBP^{-1})^k}{k!} = \sum_{k=0}^{m} \frac{PB^kP^{-1}}{k!} = P\left(\sum_{k=0}^{m} \frac{B^k}{k!}\right)P^{-1} = Pp_m(B)P^{-1}.$$

By properties of limit we have the following:

$$e^A = \lim_{m\to\infty} p_m(A) = \lim_{m\to\infty} Pp_m(B)P^{-1} = P\left(\lim_{m\to\infty} p_m(B)\right)P^{-1} = Pe^BP^{-1}.$$

Therefore, $e^A = Pe^BP^{-1}$. $\square$

Example 3.13. Find a matrix in Jordan form that is similar to the following matrix:

$$A = \begin{pmatrix} -6 & 5 & -3 & 9 \\ -1 & 2 & 0 & 1 \\ 4 & -4 & 4 & -4 \\ -5 & 3 & -2 & 8 \end{pmatrix}.$$

Solution. The characteristic polynomial of A is $\det(A - zI) = z^4 - 8z^3 + 23z^2 - 28z + 12$. By inspection, we see that $z = 1$ is a root. After performing long division we obtain the following:

$$z^4 - 8z^3 + 23z^2 - 28z + 12 = (z - 1)(z^3 - 7z^2 + 16z - 12).$$

By inspection, we find $z = 2$ as a root of $z^3 - 7z^2 + 16z - 12 = 0$. Repeating this process we find out that the four eigenvalues of A are $1, 2, 2, 3$. For the eigenvalue $z = 2$, the eigenspace Ker $(A - 2I)$ is one-dimensional (and is generated by $(1\ 1\ 2\ 1)^T$). Thus, A is not diagonalizable and thus the Jordan block corresponding to eigenvalue 2 must be 2×2. Therefore, the matrix in Jordan form that is similar to A is

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

□

Example 3.14. Suppose a 2×2 matrix A satisfies tr $A = 0$. Prove that $A^2 = cI$ for some scalar c.

Solution. Since A is 2×2 and tr $A = 0$, the matrix A must be of the form

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

The characteristic polynomial of A is $p(z) = (a - z)(-a - z) - bc = z^2 - a^2 - bc$. By the Cayley-Hamilton Theorem, we must have $p(A) = 0$ and thus $A^2 = (a^2 + bc)I$, as desired. □

Example 3.15. Find an explicit formula for A^n, where n is a positive integer, and

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 3 & -3 & 7 \\ 2 & -3 & 6 \end{pmatrix}.$$

Solution. The characteristic polynomial of this matrix is $\det(A - zI) = -z^3 + 4z^2 - 5z + 2$. By inspection we can find a root of this polynomial to be $z = 1$. Dividing by $z - 1$ and factoring we obtain $(z - 1)^2(2 - z)$. For $z = 2$ we find $\mathbf{v}_1 = (1\ 2\ 1)^T$ as an eigenvector. For $z = 1$ we see that Ker $(A - I)$ is one-dimensional and is generated by $(0\ 1\ 1)^T$. We also obtain

$$(A - 2I)^2 = \begin{pmatrix} 0 & 1 & -1 \\ 3 & -4 & 7 \\ 2 & -3 & 5 \end{pmatrix}^2 = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 1 & -1 & 2 \end{pmatrix}.$$

The vector (x, y, z) is in Ker $(A - I)^2$ if and only if

$$x - y + 2z = 0, \text{ and } 2x - 2y + 4z = 0.$$

Solving for x we obtain $x = y - 2z$. Thus, elements of Ker $(A - I)^2$ are of the form

$$\begin{pmatrix} y - 2z \\ y \\ 2z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

Now, we will choose a vector in Ker $(A-I)^2$ that does not belong to Ker $(A-I)$. We set $\mathbf{v}_3 = (1\ 1\ 0)^T$, and

$$\mathbf{v}_2 = (A-I)\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

Since $A\mathbf{v}_1 = 2\mathbf{v}_1$, $A\mathbf{v}_2 = \mathbf{v}_2$ and $A\mathbf{v}_3 = \mathbf{v}_2 + \mathbf{v}_3$ we have the following decomposition:

$$A = PJP^{-1}, \text{ where } P = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \text{ and } J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We know $A^n = PJ^nP^{-1}$. By block multiplication of matrices $J^n = \begin{pmatrix} 2^n & 0 \\ 0 & B^n \end{pmatrix}$, where $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Note that $B = I + E$, where I is the 2×2 identity matrix and $E^2 = 0$. By the binomial theorem we have $B^n = I + nE + \binom{n}{2}E^2 + \cdots = I + nE$. Therefore,

$$A^n = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix}^{-1}.$$

□

Example 3.16. Find all $A \in M_n(\mathbb{C})$ satisfying $A^2 = A$.

Solution. First, note that if λ is an eigenvalue of A, then $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero $\mathbf{v}$ and thus $A^2\mathbf{v} = \lambda^2\mathbf{v}$, which implies $\lambda^2 = \lambda$, since $A = A^2$. Therefore, $\lambda = 0, 1$. We will now write A in Jordan form: $A = PJP^{-1}$.

$$A^2 = A \iff PJ^2P^{-1} = PJP^{-1} \iff J^2 = J.$$

By block multiplication of matrices $B^2 = B$ for every Jordan block of J. If B is of size more than 1×1, then the $(1,2)$ entry of B^2 is 2λ, while the $(1,2)$ entry of B is 1. Therefore, $\lambda = 1/2$, which is a contradiction. Therefore, all Jordan blocks of J are 1×1, and thus A is diagonalizable. This means $A^2 = A$ if and only if $A = PDP^{-1}$, where D is a diagonal matrix whose diagonal entries are 0 and 1. □

Example 3.17. Suppose A is a diagonalizable matrix all of whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_n$. Prove that

$$(A - \lambda_1 I) \cdots (A - \lambda_n I) = 0.$$

Solution. For simplicity let $f(x) = (x-\lambda_1)\cdots(x-\lambda_n)$. Suppose $A = PDP^{-1}$, where D is a diagonal matrix. We know $f(A) = Pf(D)P^{-1}$, since $A^j = PD^jP^{-1}$ for every positive integer j. Therefore, it is enough to show $f(D) = 0$. We will now prove this by induction on the size of D. If D is 1×1, then $D = (\lambda_1) = \lambda_1 I$, and thus $f(D) = 0$, as desired.

Write D as

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & B \end{pmatrix},$$

where B is a diagonal $(n-1) \times (n-1)$ diagonal matrix. Since the diagonal entries of B are $\lambda_2, \ldots, \lambda_k$ and possibly λ_1, by inductive hypothesis we will have $f(B) = 0$. On the other hand $f(\lambda_1) = 0$. Using block multiplication of matrices,

$$f(D) = \begin{pmatrix} f(\lambda_1) & 0 \\ 0 & f(B) \end{pmatrix} = 0.$$

Therefore, $f(D) = 0$, as desired. $\square$

3.5 Exercises

Exercise 3.1. *Prove that for every two square matrices A, B we have*

$$\det \begin{pmatrix} A & * \\ 0 & B \end{pmatrix} = \det \begin{pmatrix} A & 0 \\ * & B \end{pmatrix} = \det A \det B.$$

Here $*$ is an arbitrary matrix with an appropriate size.

Hint: Use block multiplication of matrices:

$$\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} A & C \\ 0 & I \end{pmatrix}.$$

Exercise 3.2. *Suppose all eigenvalues of a matrix A in $M_n(\mathbb{R})$ are real. Prove that there is an upper triangular matrix T and an invertible matrix P with real entries for which $A = PTP^{-1}$.*

Hint: Use the same proof as in Theorem 3.2.

Exercise 3.3 (Cayley-Hamilton for 2×2 matrices)**.** *Let $A = (a_{ij})$ be a 2×2 matrix.*

(a) Find the characteristic polynomial $p(z) = \det(zI - A)$.

(b) Algebraically verify the Cayley-Hamilton Theorem for A.

Exercise 3.4. *An incorrect "proof" of the Cayley-Hamilton Theorem is provided below:*

> *"The characteristic polynomial is $p(z) = \det(A - zI)$. Substituting $z = A$ we obtain $p(A) = \det(A - AI) = \det(A - A) = 0$. This shows $p(A) = 0$, as desired."*

What is the flaw in this "proof"?

Provide two square matrices A, B of the same size for which $\det(A - BI) = 0$, but $p(B) \neq 0$, where $p(z) = \det(A - zI)$ is the characteristic polynomial of A.

Exercise 3.5. *Suppose A is an invertible matrix. Use the Cayley-Hamilton Theorem to prove A^{-1} can be written as a polynomial of A.*

Hint: Write down the characteristic polynomial $p(z) = c_0 + c_1 z + \cdots + c_n z^n$. Then, use $p(A) = 0$. What is c_0?

Exercise 3.6. *Recall that a square matrix A is called* **nilpotent** *if $A^k = 0$ for some positive integer k. Suppose A is a nilpotent $n \times n$ matrix.*

(a) Prove that 0 is the only eigenvalue of A.

(b) Use part (a) and the Cayley-Hamilton Theorem to show $A^n = 0$.

Exercise 3.7. *Prove there is no matrix $A \in M_2(\mathbb{C})$ for which*

$$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Exercise 3.8. *Suppose $A \in M_n(\mathbb{C})$ has a single eigenvalue. Prove that there is a complex number c for which $A - cI$ is nilpotent.*

Exercise 3.9. *Let A be a square matrix.*

(a) Consider the matrices $I, A, A^2, \ldots$. Using the fact that $M_n(\mathbb{F})$ is a finite dimensional vector space over $\mathbb{F}$, prove that there are constants c_i, not all zero, for which $c_0 I + \cdots + c_k A^k = 0$, for some $k \leq n^2$. Deduce that there is a nonzero polynomial $f(z)$ for which $f(A) = 0$. (Do not use the Cayley-Hamilton Theorem for this part.)

(b) Prove that if $f(A) = 0$ for a polynomial f and λ is an eigenvalue of A, then $f(\lambda) = 0$.

(c) Suppose $g(z) \neq 0$ is a monic polynomial with the smallest degree for which $g(A) = 0$. (Such a polynomial exists by part (a).) Let $p(z)$ be the characteristic polynomial of A. Prove that $p(z)$ is divisible by $g(z)$. (Hint: Using long division write $p(z) = g(z)q(z) + r(z)$, where $q(z)$ and $r(z)$ are the quotient and remainder when $p(z)$ is divided by $g(z)$.)

(d) Use a method similar to the one in part (c) to show that if $h(A) = 0$ for some polynomial $h(z)$, then $g(z)$ divides $h(z)$, where $g(z)$ is the polynomial in part (c). Use that to prove such a polynomial $g(z)$ is unique.

The polynomial $g(z)$ in the above exercise is called the **minimal polynomial** of A.

Exercise 3.10. *Let*

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}.$$

Evaluate $A^7 - A^6 - A^5 + A^4 + 3A - I$.

Exercise 3.11. *In each of the following determine if A is invertible. If it is, find an expression for A^{-1} as a polynomial of A.*

(a) A is a 3×3 matrix with eigenvalues $1+i, -1+2i, 0$.

(b) A is a 4×4 matrix with eigenvalues $1+i, 1-i, 2, -2$.

Exercise 3.12. *Show that a matrix is diagonalizable if and only if its Jordan form J is diagonal.*

Exercise 3.13. *Is it true that if a square matrix A satisfies $A^3 = A^2$, then A is diagonalizable?*

Exercise 3.14. *Suppose $A, B \in M_2(\mathbb{C})$ for which $(AB)^2 = 0$. Prove that $(BA)^2 = 0$.*

Hint: Prove $\det(BA) = \mathrm{tr}\ (BA) = 0$ and then use the Cayley-Hamilton Theorem.

Exercise 3.15. *Two $(n+1) \times (n+1)$ matrices A and B with complex entries are given. Assume the list of eigenvalues of both A and B is*

$$1, 2, \ldots, n-2, n-1, n, n.$$

Suppose further that neither A nor B is diagonalizable. Prove that A and B are similar matrices.

Hint: Find Jordan forms of A and B.

Exercise 3.16. *Given nonzero numbers $a_1, \ldots, a_n$ find the matrix in Jordan form that is similar to the $(n+1) \times (n+1)$ matrix (shown below) whose entries immediately below the main diagonal are $a_1, \ldots, a_n$, and all of its other entries are zero.*

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_n & 0 \end{pmatrix}.$$

Definition. For every positive integer n, the number of sequences of positive integers $a_1 \leq a_2 \leq a_2 \leq \cdots \leq a_k$ satisfying $a_1 + a_2 + \cdots + a_k = n$ is denoted by $p(n)$.

The answer to the next problem could be in terms of the function $p(n)$ defined above.

Exercise 3.17. *How many $n \times n$ matrices J in Jordan form are there that have a single given eigenvalue λ? How about if J were to have two distinct given eigenvalues λ_1 and λ_2?*

Exercise 3.18. *Suppose B is a Jordan block with eigenvalue λ, i.e. a square matrix of the form:*

$$B = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

(a) Prove that B is similar to B^T.

(b) Using part (a) and Jordan canonical form, prove that every matrix in $M_n(\mathbb{C})$ is similar to its transpose.

Exercise 3.19. *Suppose λ is an eigenvalue of a matrix A, and that λ is a root of the characteristic polynomial of A with multiplicity k. Furthermore assume $\dim \operatorname{Ker}\, (\lambda I - A) = 1$. Prove that $\dim \operatorname{Ker}\, (\lambda I - A)^j = j$ for all $j \leq k$.*

Exercise 3.20. *Let A, B be two square matrices of the same size.*

(a) Prove that $\det(AB + I) = \det(BA + I)$.

(b) Deduce that the characteristic polynomials of AB and BA are the same. Deduce, AB and BA have the same eigenvalues.

(c) With an example show that AB and BA may not have the same eigenvectors.

Hint: For part (a) consider both products of the following block matrices:

$$\begin{pmatrix} A & I \\ I & -B \end{pmatrix}, \text{ and } \begin{pmatrix} B & I \\ I & 0 \end{pmatrix}.$$

Exercise 3.21. *Suppose the list of all eigenvalues of an $n \times n$ matrix is*

$$\lambda_1, \lambda_2, \ldots, \lambda_n.$$

Prove that for every positive integer k, the list of all eigenvalues of A^k is

$$\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k.$$

Exercise 3.22. *Write the matrix A in the form PJP^{-1}, where J is in Jordan form:*

$$A = \begin{pmatrix} 3 & -1 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 12 & -5 \end{pmatrix}.$$

Use this to find e^A. You could leave your answers as products of matrices.

Exercise 3.23. *Let A be an $n \times n$ matrix. Recall that A can be written as $A = D + N$, where D is diagonalizable, N is nilpotent and $ND = DN$. Also, recall that eigenvalues of A and D are the same.*

(a) Using the fact that N is nilpotent, prove that for any positive integer m, we have $A^m = \sum_{k=0}^{n-1} \binom{m}{k} D^{m-k} N^k$.

(b) Suppose every eigenvalue λ of A satisfies $|\lambda| < 1$. Prove that D^m approaches the zero matrix as $m \to \infty$.

(c) Prove that A^m approaches the zero matrix as $m \to \infty$.

Hint: You may use the fact that exponential decay is faster than polynomial growth.

Exercise 3.24. *Suppose A is a square matrix satisfying $A^4 = A$. Prove that A is diagonalizable.*

Exercise 3.25. *Find* $\cos(A)$ *and* e^A *if A is each of the following matrices.*

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Exercise 3.26. *Suppose λ is an eigenvalue of an $n \times n$ matrix A for which* $\text{Ker } (A - \lambda I)^{n-1} \neq \text{Ker } (A - \lambda I)^n$. *Prove that A is similar to a single Jordan block.*

Exercise 3.27. *Prove that for every analytic function f, two square matrices A, P of the same size, with P being invertible, we have $f(PAP^{-1}) = Pf(A)P^{-1}$.*

Exercise 3.28. *Prove that for every block matrix*

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix},$$

where B, C are square matrices, and every analytic function $f : \mathbb{R} \to \mathbb{R}$ we have

$$f(A) = \begin{pmatrix} f(B) & 0 \\ 0 & f(C) \end{pmatrix}.$$

Exercise 3.29. *Consider the matrices*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Evaluate e^A, e^B and e^{A+B}. Show that $e^A e^B \neq e^{A+B}$.

Exercise 3.30. *Prove Theorem 3.11.*

3.6 Challenge Problems

Exercise 3.31. *Suppose A, B are $m \times n$ and $n \times m$ matrices, respectively, where $n \leq m$. Let $p(z), q(z)$ be the characteristic polynomials of AB and BA, respectively. Prove that $p(z) = z^{m-n}q(z)$.*

Exercise 3.32. *Determine all complex numbers λ for which there is a square matrix A with λ as its eigenvalue such that $A^2 = A^T$.*

Exercise 3.33. *Let A be an $n \times n$ matrix whose entries are all ± 1 and that whose rows are pairwise orthogonal. Prove that if A has an $a \times b$ submatrix all of whose entries are 1, then $ab \leq n$.*

Exercise 3.34. *Prove that if for a square matrix A we know* $\text{tr } (A^k) = 0$ *for all positive integers k, then A is nilpotent.*

Exercise 3.35. *Suppose $A, B \in M_2(\mathbb{F})$ satisfy $A = AB - BA$. Prove that $A^2 = 0$.*

Exercise 3.36. *Suppose $p(z)$ is a polynomial with complex coefficients. We know the number of roots of $p(z)$ in $\mathbb{C}$ does not exceed its degree. Let n be a positive integer. How many matrices $A \in M_n(\mathbb{C})$ exist that satisfy $p(A) = 0$?*

Exercise 3.37. *Suppose A, B, C, D are matrices of size $m \times m$, $m \times n$, $n \times m$, and $n \times n$, respectively. Prove that if D is invertible, then*

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C) \det D.$$

Exercise 3.38. *Prove that for every square matrix A we have* $\sin^2 A + \cos^2 A = I$.

Hint: First prove this identity for a diagonalizable and a nilpotent matrix.

3.7 Summary

- Every matrix can be upper triangularized over $\mathbb{C}$.
- The Cayley-Hamilton Theorem states that every square matrix satisfies its characteristic equation.
- To find the Jordan form of a matrix A:
  - Find all eigenvalues of A.
  - For each eigenvalue λ, find the dimension d_k of Ker $(A - \lambda I)^k$ for $k = 1, 2, \ldots$ until they level off.
  - The number of Jordan blocks of size at least $k \times k$ is evaluated by the formula $d_k - d_{k-1}$, with $d_0 = 0$.
  - After determining how many Jordan blocks of each size we have, we can determine the blocks of the Jordan form of A corresponding to eigenvalue λ.
  - Repeat this for every eigenvalue and create a block matrix in Jordan form.
- To find P in $A = PJP^{-1}$, start with a vector $\mathbf{v}_k \in$ Ker $(A - \lambda I)^k$ and repeatedly evaluate $\mathbf{v}_{j-1} = (A - \lambda I)\mathbf{v}_j$. Columns of P are $\mathbf{v}_j$'s.
- To find $f(A)$ for a square matrix A and an analytic function f:
  - Write $A = D + N$, using the Jordan form.
  - Evaluate the least n for which $N^n = 0$.
  - Find $f(D), f'(D), \ldots, f^{(n-1)}(D)$, using the formula: $f(D) = S \operatorname{diag}\{f(\lambda_1), \ldots, f(\lambda_n)\} S^{-1}$.
  - $f(A) = \sum_{j=0}^{n-1} \dfrac{f^{(j)}(D)}{j!} N^j$.
  - When $f(x) = e^x$, use the formula $e^A = e^D \sum_{j=0}^{n-1} \dfrac{N^j}{j!}$.

Chapter 4

Ordinary Differential Equations

4.1 Introduction

A differential equation is an equation involving derivatives of one or more functions. If there are some partial derivatives in the equation we call the equation a **partial differential equation** (PDE), and if there are no partial derivatives we say it is an **ordinary differential equation** (ODE). The **order** of a differential equation is the largest integer n for which the equation involves the n-th derivative of one of the functions we are solving for.

Example 4.1. Determine if each of the following is an ODE or a PDE. Find the order of each equation.

(a) $\left(\dfrac{dx}{dt}\right)^2 + x\sin t = \cos x.$

(b) $\dfrac{\partial y}{\partial t}\dfrac{\partial y}{\partial s} + y\dfrac{\partial z}{\partial t} = \sin(st).$

(c) $y'' + ty' + y = \cos t.$

In this class we will only focus on ordinary differential equations. Note that the domain and range of all solutions to differential equations are assumed to be subsets $\mathbb{R}$.

The main questions that we are trying to answer are the following:

- What is the general solution of an ODE?
- Can we find solutions satisfying certain initial values?
- How many solutions are there satisfying given initial values?
- If finding an explicit formula for a solution is not possible, can we approximate the solution?
- What are all solutions that are constant?
- Are all solutions bounded? Are there any bounded solutions?

- Are all solutions periodic? Are there any periodic solutions?
- What is the long term behavior of solutions?
- How do solutions change when we change their initial values?

4.2 Explicit First Order Equations

An equation of the form $\dfrac{dy}{dt} = f(t)$ is called a (first order) **explicit differential equation.**

Example 4.2. Find all solutions of the differential equation $\dfrac{dy}{dt} = \dfrac{1}{t^2 - t}$.

Theorem 4.1 (Existence and Uniqueness Theorem for Explicit Equations)**.** *Suppose $f(t)$ is continuous over an open interval (a, b). Then, for every $t_0 \in (a, b)$ and every real number y_0, there is a unique solution to the initial value problem*

$$\frac{dy}{dt} = f(t),\ \ y(t_0) = y_0.$$

This solution is given by

$$y(t) = y_0 + \int_{t_0}^{t} f(s)\ \mathrm{d}s.$$

4.3 First Order Linear Equations

An n-th order linear differential equation in **normal form** (i.e. with the leading coefficient of 1) is an equation of the form:

$$y^{(n)} + a_n(t)y^{(n-1)} + \cdots + a_2(t)y' + a_1(t)y = f(t).$$

$f(t)$ is called **forcing** and $a_i(t)$'s are called **coefficients**. This equation is often written as $L[y] = f(t)$, where L is the **differential operator** given by

$$L = D^n + a_n(t)D^{n-1} + \cdots + a_2(t)D + a_1(t).$$

Here, we write D instead of $\dfrac{\mathrm{d}}{\mathrm{d}t}$.

We would also like to explore **initial value problems** (i.e. equations along with initial values in a specific format) or IVP's of the form:

$$\begin{cases} y^{(n)} + a_n(t)y^{(n-1)} + \cdots + a_2(t)y' + a_1(t)y = f(t). \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \\ \vdots \\ y^{(n-1)}(t_0) = y_{n-1} \end{cases}$$

Example 4.3. Solve the equation: $y' + y = e^t$.

To solve an equation of the form $\dfrac{dy}{dt}+a(t)y = f(t)$, we find a function $A(t)$ for which $A'(t) = a(t)$. Multiplying both sides by $e^{A(t)}$ we can rewrite the equation as

$$\frac{d}{dt}(e^{A(t)}y) = e^{A(t)}f(t).$$

Theorem 4.2 (Existence and Uniqueness Theorem for First Order Linear Equations)**.** *Suppose $a(t)$ and $f(t)$ are continuous over an open interval (a,b). Let $t_0 \in (a,b)$ and y_0 be a real number. Then, the initial value problem given below has a unique solution defined over (a,b).*

$$\frac{dy}{dt} + a(t)y = f(t), \quad y(t_0) = y_0.$$

4.4 Separable Equations

A first order equation is called **separable** if it can be written in the form

$$\frac{dy}{dt} = f(t)g(y).$$

The name "separable" refers to the fact that we can separate the variables and write the differential equation in the form

$$\frac{dy}{g(y)} = f(t)\ dt.$$

Solutions can then be obtained by simply integrating both sides.

Example 4.4. Solve the equation $\dfrac{dy}{dt} = 2ty^2 + 3t^2y^2$. Can you find a solution that satisfies $y(1) = 0$?

Definition 4.1. A solution to a differential equation is called **stationary** or **equilibrium** or a **fixed point** or a **critical point** if it is constant.

All stationary solutions of the separable equation $\dfrac{dy}{dt} = f(t)g(y)$ are found by solving $g(y) = 0$ for y.

Example 4.5. Find all solutions of $\dfrac{dy}{dt} = ty^2 - ty,\ y(1) = 2$.

4.5 Change of Variables

Example 4.6. Solve the equation $y' = \dfrac{e^{y+t} - y - t}{y+t}$.

An equation of the form $\dfrac{dy}{dt} = f(at + by + c)$ can be transformed into a separable equation by substituting $u = at + by + c$.

Example 4.7. Solve the equation $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{y-t}{y+t}$.

To solve an equation of the form $y' = f(y/t)$ we use the substitution $u = y/t$. This yields $y = ut$, which implies

$$y' = u't + u \Rightarrow u't + u = f(u) \Rightarrow u' = \frac{f(u) - u}{t}.$$

This equation is separable that can be solved using the method discussed earlier.

One common example is equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{ay + bt}{cy + dt} \qquad (*)$$

for constants a, b, c, d. For these we can use the substitution $u = y/t$.

Example 4.8. Solve the equation $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{y - t + 1}{y + t - 3}$.

To solve equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{ay + bt + m}{cy + dt + n}$$

for constants a, b, c, d, m, n first choose $T = t + r$, $Y = y + s$ for constants r, s. Find r, s in such a way that the equation turns into one of the form $(*)$. Then, use the substitution $u = Y/T$.

4.6 Exact Equations and Integrating Factors

Suppose the solution to a differential equation is given by an implicit equation $\phi(t, y) = $ constant. This is equivalent to $\dfrac{d\phi(t, y)}{dt} = 0$. Using the chain rule we obtain $\phi_t + \phi_y \dfrac{dy}{dt} = 0$.

Definition 4.2. An equation of the form $M(t, y) + N(t, y)y' = 0$ is called **exact** over an open rectangle $R = (a, b) \times (c, d)$ in the ty-plane, if there is a function $\phi(t, y)$ for which $\phi_t = M$ and $\phi_y = N$ over R.

All solutions of an exact equation are of the form $\phi(t, y) = c$. The name *exact* refers to the fact that the left hand side is exactly the derivative of one function.

Example 4.9. Solve the equation $e^x y + 2x + (2y + e^x)\dfrac{dy}{dx} = 0$.

Remark. Sometimes equations of the form $M(t, y) + N(t, y)\dfrac{\mathrm{d}y}{\mathrm{d}t} = 0$ are written as $M(t, y)\ \mathrm{d}t + N(t, y)\ \mathrm{d}y = 0$.

Theorem 4.3. *Suppose $M(t, y)$ and $N(t, y)$ are continuous over the rectangle $R = (a, b) \times (c, d)$ in the ty-plane. If $\phi(t, y)$ is a function satisfying $\phi_t = M$ and $\phi_y = N$ over R, then the general solution to the differential equation $M(t, y) + N(t, y)y' = 0$ over R is given by $\phi(t, y) = C$, where C is a constant.*

Question. How do we know which equations are exact?

From multivariable calculus we know $\phi_{ty} = \phi_{yt}$, assuming second partials of ϕ are continuous. Therefore, in order for an equation $M + N\dfrac{dy}{dt} = 0$ to be exact we need to make sure $M_y = N_t$. The following theorem shows that under certain conditions, the converse is also true.

Theorem 4.4. *Let $M(t, y), N(t, y)$ be continuous and have continuous first partials over an open rectangle $R = (a, b) \times (c, d)$. Then, there is a function $\phi(t, y)$ defined over R for which $\phi_t = M$ and $\phi_y = N$ if and only if $M_y = N_t$.*

Example 4.10. Solve $(xy^2 + y + e^x) + (x^2 y + x)\dfrac{dy}{dx} = 0$.

Example 4.11. Solve the initial value problem $3t^2 y + 8ty^2 + (t^3 + 8t^2 y + 12y^2)\dfrac{dy}{dt} = 0,\ y(2) = 1$.

Example 4.12. Solve the equation $2ty + (2t^2 - e^y)\dfrac{dy}{dt} = 0$.

When the equation is not exact one possible remedy is to multiply both sides by a factor $\mu(t, y)$ in such a way that the equation becomes exact. Such a factor μ is called an **integrating factor**.

Example 4.13. Solve the equation: $4xy + 3y^3 + (x^2 + 3xy^2)\dfrac{dy}{dx} = 0$.

Example 4.14. Find all functions $M(t, y)$ with continuous first partials for which t is an integrating factor of the equation

$$M(t, y) + t\frac{dy}{dt} = 0.$$

4.7 More Examples

Example 4.15. Consider the differential equation $y' = f(t)$, where $f(t) = \begin{cases} 2t - 1 & \text{if } t > 0 \\ 1 & \text{if } t < 0 \end{cases}$ Find all continuous solutions y to this differential equation.

Solution. Integrating we obtain $y = t^2 - t + C_1$ for $t > 0$, and $y = t + C_2$ for $t < 0$. Since this function is continuous, we must have

$$\lim_{t \to 0^+} t^2 - t + C_1 = \lim_{t \to 0^-} t + C_2 = y(0).$$

Therefore, $C_1 = C_2 = y(0)$. This means all solutions are of the following form:

$$f(t) = \begin{cases} t^2 - t + C & \text{if } t > 0 \\ t + C & \text{if } t \le 0 \end{cases}$$

where C is a constant. □

Example 4.16. Consider the linear transformation $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ given by $T(f)(t) = f'(t)$. Find all eigenpairs of this transformation.

Solution. $(\lambda, f(t))$ is an eigenpair iff $T(f)(t) = \lambda f(t)$, which means $f'(t) = \lambda f(t)$. Note that since f and f' are real valued functions and $f(t) \neq 0$, the scalar λ must be real. The equation $f'(t) - \lambda f(t) = 0$ is a first order differential equation with integrating factor $e^{-\lambda t}$. This yields $e^{-\lambda t} f(t) = c$ is a constant. Therefore, $f(t) = ce^{-\lambda t}$ yields all eigenvector of T, where $c \neq 0$ is a constant. This means all eigenpairs of T are of the form

$$(\lambda, ce^{\lambda t}),$$

where $c, \lambda \in \mathbb{R}$ are constants and $c \neq 0$. □

Example 4.17. Solve each of the following differential equations.

(a) $y' + 2ty = 0$.

(b) $ty' + y = \sin t$.

(c) $\dfrac{y'}{\cos t} + y = 1$.

(d) $y'' + y' = 0$

Solution. (a) Integrating $2t$ we obtain t^2. Thus, one integrating factor is e^{t^2}. This yields

$$\frac{d}{dt}\left(e^{t^2}y\right) = 0 \Rightarrow e^{t^2}y = C \Rightarrow y = Ce^{-t^2}.$$

(b) The left hand side is already the derivative of ty, so we can rewrite the equation as

$$\frac{d(yt)}{dt} = \sin t \Rightarrow yt = -\cos t + C \Rightarrow y = -\frac{\cos t - C}{t}.$$

(c) Multiplying by $\cos t$ we obtain $y' + y\cos t = \cos t$. An integrating factor is $e^{\sin t}$. This yields the equation

$$\frac{d}{dt}\left(e^{\sin t}y\right) = e^{\sin t}\cos t \Rightarrow e^{\sin t}y = e^{\sin t} + C \Rightarrow y = 1 + Ce^{-\sin t}.$$

(d) This is not a first order equation, but if you think of $z = y'$ as a new function, then it becomes a first order linear equation. An integrating factor is e^t. This yields

$$\frac{d}{dt}\left(e^t y'\right) = 0 \Rightarrow e^t y' = C \Rightarrow y' = Ce^{-t} \Rightarrow y = -Ce^{-t} + D,$$

where C, D are two constants. □

Example 4.18. Discuss the long term behavior of solutions, i.e. the limit of each solution as $t \to \infty$.

(a) $y' + \alpha y = \alpha$, where α is a constant.

(b) $y' + 2ty = 1$.

Solution. (a) The integrating factor is $e^{\alpha t}$. Therefore, we can rewrite the equation as

$$\frac{d}{dt}\left(e^{\alpha t}y\right) = \alpha e^{\alpha t} \Rightarrow e^{\alpha t}y = e^{\alpha t} + C \Rightarrow y = 1 + Ce^{-\alpha t},$$

where $C = y(0) - 1$.

When $\alpha = 0$, we have $y = 1 + C$ is a constant.
When $\alpha > 0$, we see that $e^{-\alpha t} \to 0$ as $t \to \infty$. Therefore, $y \to 1$.
When $\alpha < 0$, $e^{-\alpha t} \to \infty$ as $t \to \infty$. Therefore, depending on if $C = 0$, $C < 0$ or $C > 0$, the solution stays at 1, tends to $-\infty$, or tends to ∞ as $t \to \infty$.

(b) The integrating factor is e^{t^2}. The equation then becomes

$$\frac{d}{dt}\left(e^{t^2}y\right) = e^{t^2} \Rightarrow e^{t^2}y = C + \int_0^t e^{s^2}\ ds \Rightarrow y = Ce^{-t^2} + \int_0^t e^{s^2-t^2}\ ds.$$

As $t \to \infty$, we have $t^2 \to -\infty$ and thus $e^{-t^2} \to 0$. The integral above cannot be evaluated, but we can estimate this integral. Note that since we are looking for the limit of y as $t \to \infty$ we may assume $0 \le s \le t$. Note also that when $x \ge 0$, the Taylor series for e^x yields

$$e^x = 1 + x + \cdots \ge 1 + x \Rightarrow e^{-x} \le \frac{1}{1+x} \Rightarrow e^{s^2-t^2} \le \frac{1}{1+t^2-s^2} \Rightarrow \int_0^t e^{s^2-t^2}\ ds \le \int_0^t \frac{1}{1+t^2-s^2}\ ds.$$

The integral can now be evaluated using the method of partial fractions. For simplicity set $a = \sqrt{1+t^2}$. Note that $a > t \ge s \ge 0$.

$$\int_0^t \frac{1}{a^2-s^2}\ ds = \frac{1}{2a}\int_0^t \frac{1}{a+s} + \frac{1}{a-s}\ ds = \frac{1}{2a}\ln\left(\frac{a+s}{a-s}\right)\Big]_{s=0}^{s=t} = \frac{1}{2a}\left(\ln\left(\frac{a+t}{a-t}\right) - \ln 1\right) = \frac{1}{2a}\ln\left(\frac{a+t}{a-t}\right).$$

Substituting $a = \sqrt{1+t^2}$ we see the following:

$$\frac{a+t}{a-t} = \frac{(a+t)^2}{a^2-t^2} = (a+t)^2 \Rightarrow \ln\left(\frac{a+t}{a-t}\right) = 2\ln(a+t) \le 2\ln(2a).$$

Therefore,

$$\int_0^t \frac{1}{a^2-s^2}\ ds \le \frac{2\ln(2\sqrt{1+t^2})}{2\sqrt{1+t^2}} = \frac{\ln(2\sqrt{1+t^2})}{\sqrt{1+t^2}}.$$

As $t \to \infty$, so does $\sqrt{1+t^2}$. Since $\ln x$ grows slower than x, we have $\dfrac{\ln(2\sqrt{1+t^2})}{\sqrt{1+t^2}} \to 0$ as $t \to \infty$. Therefore, by the squeeze theorem applies to

$$0 \le \int_0^t e^{s^2-t^2}\ ds \le \frac{\ln(2\sqrt{1+t^2})}{\sqrt{1+t^2}},$$

we conclude that $y \to 0$ as $t \to \infty$. □

Example 4.19. Consider the differential equation $y' + \alpha y = t$. For which constants α does this equation have at least one periodic solution?

Solution. Suppose y is a periodic solution. This means there is a positive constant p for which $y(t+p) = y(t)$ for all $t \in \mathbb{R}$. This means $y'(t+p) = y'(t)$. Therefore,

$$y'(t+p) + \alpha y(t+p) = y'(t) + \alpha y(t).$$

Since $y(t)$ is a solution to the given differential equation, the left hand side is $t+p$, while the right hand side is t. Therefore, $p = 0$, which implies no such constant α exists. □

Example 4.20. Prove that the equation $y' + y = 2\sin t$ has a unique periodic solution.

Solution. The integrating factor is e^t. This yields the equation

$$\frac{d}{\mathrm{d}t}\left(e^t y\right) = 2e^t \sin t \Rightarrow e^t y = e^t(\sin t - \cos t) + C \Rightarrow y = \sin t - \cos t + Ce^{-t}.$$

Note that by the Extreme Value Theorem, a periodic continuous function must also be bounded. As $t \to -\infty$, e^{-t} approaches infinity. Therefore, if C is nonzero, the function would be unbounded and thus not periodic. Therefore, the only solution where y is peridic is $y = \sin t - \cos t$, which is clearly periodic with period 2π. □

Example 4.21. Solve each differential equation.

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{te^y}{1+t^2}$.

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \cos(y+2t) - 2$.

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{ty+t}{y^2+ty^2}$.

(d) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = ty^2 - t + 2y^2 - 2$.

(e) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{y+2t}{2y+t}$.

(f) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{y+2t+1}{2y+t-1}$.

Solution. (a) This is a separable equation. First, note that e^y cannot be zero and thus, there are no stationary solutions.

Rearranging we have

$$\int e^{-y}\,\mathrm{d}y = \int \frac{t}{1+t^2}\,\mathrm{d}t \Rightarrow -e^{-y} = \frac{1}{2}\ln(1+t^2) + C \Rightarrow -2e^{-y} = \ln(1+t^2) + C.$$

(b) We will use the change of variable $u = y + 2t$. This yields

$$u' = y' + 2 = \cos u \Rightarrow \sec u\,\mathrm{d}u = \mathrm{d}t \Rightarrow \ln|\sec u + \tan u| = t + C \Rightarrow |\sec u + \tan u| = e^C e^t.$$

Since e^C can be any positive constant we can drop the absolute value and write $\sec(y+2t)+\tan(y+2t) = Ce^t$ with $C \neq 0$.

The stationary solutions of the equation $u' = \cos u$ are those satisfying $\cos u = 0$ or $u = \pi k + \dfrac{\pi}{2}$. Therefore, the solutions are

$$\sec(y+2t) + \tan(y+2t) = Ce^t \text{ with } C \neq 0, \text{ and } y + 2t = \pi k + \frac{\pi}{2}, \text{ with } k \in \mathbb{Z}.$$

(c) The equation is separable and can be written as $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{t}{1+t}\dfrac{y+1}{y^2}$. This means the only stationary solution is $y = -1$.

Rearranging and using long division we obtain:

$$\frac{y^2\ \mathrm{d}y}{y+1} = \frac{t}{1+t}\ \mathrm{d}t \Rightarrow \int \left(y - 1 + \frac{1}{y+1}\right)\ \mathrm{d}y = \int \left(1 - \frac{1}{1+t}\right)\ \mathrm{d}t \Rightarrow \frac{y^2}{2} - y + \ln|y+1| = t - \ln|1+t| + C.$$

Therefore, the solutions are

$$y = -1, \text{ and } y + t + C = \frac{y^2}{2} + \ln\left|\frac{y+1}{1+t}\right|.$$

(d) The equation can be written as $\frac{\mathrm{d}y}{\mathrm{d}t} = (t+2)(y^2-1)$. This means it is a separable equation. The stationary solutions must satisfy $y^2 = 1$. Thus, there are two stationary solutions $y = \pm 1$.

Separating the variables we find the rest of the solutions:

$$\frac{\mathrm{d}y}{y^2-1} = (t+2)\ \mathrm{d}t \Rightarrow \frac{1}{2}\int \left(\frac{1}{y-1} - \frac{1}{y+1}\right)\ \mathrm{d}y = \frac{t^2}{2} + 2t \Rightarrow \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = \frac{t^2}{2} + 2t + C.$$

Here, we used partial fractions to integrate $\frac{1}{y^2-1}$.

(e) This is a function of y/t since

$$\frac{y+2t}{2y+t} = \frac{y/t+2}{2y/t+1}.$$

Setting $u = y/t$ we obtain $ut = y$. Differentiating we have

$$u + u't = y' = \frac{u+2}{2u+1} \Rightarrow u' = \frac{2-2u^2}{t(2u+1)}.$$

This is a separable equation. Its stationary solutions are obtained by solving $2 - 2u^2 = 0$, which yields $u = \pm 1$. The nonstationary solutions are obtained as follows:

$$\frac{(2u+1)\ \mathrm{d}u}{1-u^2} = \frac{2\ \mathrm{d}t}{t} \Rightarrow -\frac{3}{2}\ln|1-u| - \frac{1}{2}\ln|1+u| = 2\ln|t| + C.$$

The integration on the left is obtained using partial fractions.

(f) Setting $Y = y + r$, $T = t + s$ we have $\frac{\mathrm{d}Y}{\mathrm{d}T} = \frac{\mathrm{d}y}{\mathrm{d}t}$, $y = Y - r$, $t = T - s$. Substituting we obtain the following:

$$\frac{\mathrm{d}Y}{\mathrm{d}T} = \frac{Y - r + 2T - 2s + 1}{2Y - 2r + T - s - 1}.$$

In order to homogenize this we need $r + 2s = 1$ and $2r + s = -1$. This yields $r = -1, s = 1$. This yields the equation

$$\frac{\mathrm{d}Y}{\mathrm{d}T} = \frac{Y + 2T}{2Y + T}.$$

By the previous part its solutions are

$$-\frac{3}{2}\ln|1 - Y/T| - \frac{1}{2}\ln|1 + Y/T| = 2\ln|T| + C, \text{ and } Y/T = \pm 1.$$

Substituting back Y, T in terms of y, t we obtain the solutions. □

Example 4.22. Solve each of the following equations:

(a) $(t^2 + y^2 + 2t)\ \mathrm{d}t + 2ty\ \mathrm{d}y = 0$.

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -\dfrac{y\sin(ty)}{t\sin(ty) + y}$.

(c) $\sin^2 y \cos y\ \mathrm{d}y + \tan^2 x\ \mathrm{d}x = 0$.

Solution. (a) $(t^2 + y^2 + 2t)_y = 2y$, and $(2ty)_t = 2y$. Since these are the same, the equation is exact. We can find the solutions by solving the system

$$\begin{cases} \phi_t = t^2 + y^2 + 2t \\ \phi_y = 2ty \end{cases}$$

The first equation yields $\phi = \dfrac{t^3}{3} + ty^2 + t^2 + f(y)$. Substiuting this into the second equation we obtain $2ty + f'(y) = 2ty$. Thus $f(y) = 0$ works. Therefore, the general solution is

$$\frac{t^3}{3} + ty^2 + t^2 = C.$$

(b) This equation can be written as

$$(t\sin(ty) + y)\frac{\mathrm{d}y}{\mathrm{d}t} + y\sin(ty) = 0 \qquad (*)$$

We see $(t\sin(ty) + y)_t = \sin(ty) + ty\cos(ty)$ and $(y\sin(ty))_y = \sin(ty) + yt\cos(ty)$. Therefore, the equation $(*)$ is exact. Its general solution may be obtained by solving the system:

$$\begin{cases} \phi_y = t\sin(ty) + y \\ \phi_t = y\sin(ty) \end{cases}$$

The first equation yields $\phi = -\cos(ty) + y^2/2 + f(t)$. Substituting into the second equation we obtain $y\sin(ty) + f'(t) = y\sin(ty)$. Thus, $f(t) = 0$ is a solution. The general solution is:

$$-\cos(ty) + \frac{y^2}{2} = C.$$

(c) This equation is both exact and separable and can be solved using either method. □

Example 4.23. Show the following equations are not exact. In each case find an integrating factor and solve. When necessary, the form of an integrating factor is given.

(a) $(1 + 3t^2\sin y)\ \mathrm{d}t - t\cot y\ \mathrm{d}y = 0$.

(b) $(y + ty^2)\ \mathrm{d}t - t\ \mathrm{d}y = 0$.

(c) $(t^3y^2 + y)\ \mathrm{d}t + (t^2y^3 + t)\ \mathrm{d}y = 0$; $\mu = \omega(ty)$.

(d) $(2\sin t + (t + y)\cos t)\ \mathrm{d}t + 2\sin t\ \mathrm{d}y = 0$; $\mu = \omega(t + y)$.

Solution. (a) $(1+3t^2\sin y)_y = 3t^2\cos y \neq (-t\cot y)_t = -\cot y$. Thus, the equation is not exact.

Let μ be an integrating factor. We must have

$$(\mu+3\mu t^2\sin y)_y = (-\mu t\cot y)_t \Rightarrow \mu_y + 3\mu_y t^2\sin y + 3\mu t^2\cos y = -\mu_t t\cot y - \mu\cot y.$$

Setting $\mu_y = 0$ we obtain the following:

$$3\mu t^2\cos y = -\mu' t\cot y - \mu\cot y \Rightarrow \mu(3t^2\cos y + \cot y) = -\mu' t\cot y$$

This implies

$$\frac{\mu}{\mu'} = \frac{-t\cot y}{3t^2\cos y + \cot y}.$$

This is impossible since the right hand side is a function of both t and y but the left hand side is a function of t, only.

Setting $\mu_t = 0$, thus assuming μ is a function of y, only, we obtain the following:

$$\mu'(1+3t^2\sin y) = -\mu(\cot y + 3t^2\cos y) \Rightarrow \frac{\mu'}{\mu} = -\frac{\cot y + 3t^2\cos y}{1+3t^2\sin y} = -\frac{\cos y + 3t^2\cos y\sin y}{\sin y(1+3t^2\sin y)} = \frac{-\cos y}{\sin y}.$$

Integrating we obtain $\ln|\mu| = -\ln|\sin y| = \ln|\csc y|$. Therefore, $\mu = \csc y$ is one integrating factor.

(b) $(y+ty^2)_y = 1+2ty \neq (-t)_t = -1$. Thus, the equation is not exact.

Let μ be an integrating factor. We must have

$$(y\mu + ty^2\mu)_y = (-t\mu)_t \Rightarrow \mu + y\mu_y + 2ty\mu + ty^2\mu_y = -\mu - t\mu_t.$$

Setting $\mu_y = 0$ we obtain the following:

$$\mu + 2ty\mu = -\mu - t\mu' \Rightarrow \mu(2+2ty) = -t\mu'.$$

The left is a function of both t and y, while the right side is a function of t, only. So, this is impossible. We will now try setting $\mu_t = 0$. This yields:

$$\mu + y\mu_y + 2ty\mu + ty^2\mu_y = -\mu \Rightarrow \mu(2+2ty) + y\mu'(1+ty) = 0 \Rightarrow 2\mu + y\mu' = 0.$$

This equation is separable and yields $\ln|\mu| = -2\ln|y| = \ln|y^{-2}|$. Therefore, $\mu = 1/y^2$ is one integrating factor. Therefore, the following equation is exact.

$$\left(\frac{1}{y} + t\right)\,\mathrm{d}t - \frac{t}{y^2}\,\mathrm{d}y = 0.$$

The general solution is obtained by solving the system:

$$\begin{cases} \phi_t = \dfrac{1}{y} + t \Rightarrow \phi = \dfrac{t}{y} + \dfrac{t^2}{2} + f(y) \\ \phi_y = -\dfrac{t}{y^2} \end{cases}$$

Substituting into the second equation we obtain:

$$-\frac{t}{y^2} + f'(y) = -\frac{t}{y^2} \Rightarrow f(y) = 0 \text{ works}$$

The general solution, therefore, is

$$\frac{t}{y} + \frac{t^2}{2} = C.$$

(c) $(t^3y^2 + y)_y = 2t^3y \neq (t^2y^3 + t)_t = -2ty^3 - 1$. Thus, the equation is not exact.

Let $\mu = \omega(ty)$ be an integrating factor. By the Chain Rule, we have $\mu_t = y\omega'(ty)$ and $\mu_y = t\omega'(ty)$. We also have

$$(t^3y^2\mu + y\mu)_y = (t^2y^3\mu + t\mu)_t \Rightarrow 2t^3y\mu + t^3y^2\mu_y + \mu + y\mu_y = 2ty^3\mu + t^2y^3\mu_t + \mu + t\mu_t.$$

Substituting what we found above, we obtain the following:

$$(2t^3y - 2ty^3)\omega = (-t^4y^2 - ty + t^2y^4 + ty)\omega' \Rightarrow 2ty(t^2 - y^2)\omega = -t^2y^2(t^2 - y^2)\omega' \Rightarrow 2\omega(ty) = -ty\omega'(ty).$$

This means we need to solve $2\omega(x) = -x\omega'(x)$. This separable equation has a solution $\omega(x) = x^{-2}$. Therefore, an integrating factor is $\mu = (ty)^{-2}$. Therefore, the equation below is exact:

$$\left(t + \frac{1}{t^2y}\right) \, \mathrm{d}t + \left(y + \frac{1}{ty^2}\right) \, \mathrm{d}y = 0.$$

The solution satisfies

$$\begin{cases} \phi_t = t + \dfrac{1}{t^2y} \Rightarrow \phi = \dfrac{t^2}{2} - \dfrac{1}{ty} + f(y) \\ \phi_y = y + \dfrac{1}{ty^2} \end{cases}$$

Substituting into the second equation we obtain

$$\frac{1}{ty^2} + f'(y) = y + \frac{1}{ty^2} \Rightarrow f(y) = \frac{y^2}{2} \text{ is one solution.}$$

The general solution is, therefore,

$$\frac{t^2}{2} - \frac{1}{ty} + \frac{y^2}{2} = C.$$

(d) $(2\sin t + (t + y)\cos t)_y = \cos t \neq (2\sin t)_t = 2\cos t$. Thus, the equation is not exact.

Let $\mu = \omega(t + y)$ be an integrating factor. By the Chain Rule we have $\mu_t = \omega'(t + y)$ and $\mu_y = \omega'(t + y)$. We have the following:

$$(2\mu\sin t + (t + y)\mu\cos t)_y = (2\mu\sin t)_t \Rightarrow 2\mu_y\sin t + \mu\cos t + (t + y)\mu_y\cos t = 2\mu_t\sin t + 2\mu\cos t.$$

Using $\mu_t = \mu_y = \omega'$ we will obtain the following:

$$(t+y)\omega' \cos t = \omega \cos t \Rightarrow (t+y)\omega' = \omega \Rightarrow x\omega'(x) = \omega(x) \Rightarrow \omega(x) = x \text{ works.}$$

Therefore, $t+y$ is an integrating factor, which means the following equation is exact:

$$\left(2(t+y)\sin t + (t+y)^2 \cos t\right)\, dt + 2(t+y)\sin t\, dy = 0$$

The solution can be obtained by solving the system below:

$$\begin{cases} \phi_t = 2(t+y)\sin t + (t+y)^2 \cos t \\ \phi_y = 2(t+y)\sin t \Rightarrow \phi = \dfrac{(t+y)^2 \sin t}{2} + f(t) \end{cases}$$

Substituting in the first equation we obtain

$$2(t+y)\sin t + (t+y)^2\cos t + f'(t) = 2(t+y)\sin t + (t+y)^2 \cos t \Rightarrow f(t) = 0 \text{ works.}$$

Therefore, the solution is given by $\dfrac{(t+y)^2 \sin t}{2} = C$ or $(t+y)^2 \sin t = C$. □

Example 4.24. Suppose $\phi(t,y)$ has first partial derivatives over a rectangle $(a,b)\times(c,d)$ in the ty-plane. Prove that $\phi(t,y) = f(t) + g(y)$ for two differential functions f and g if and only if $\phi_{ty} = 0$.

Solution. First, assume $\phi(t,y) = f(t) + g(y)$. We have $\phi_t = f'(t)$, and thus $\phi_{ty} = 0$.

Now, assume $\phi_{ty} = 0$. The equality $\phi_{ty} = 0$ implies $\phi_t = f(t)$ is independent of y for all $y \in (c,d)$, and hence a function of t, only. By integrating again we obtain $\phi(t,y) = \displaystyle\int f(t)\, dt + g(y)$ for some function g for all $t \in (c,d)$, as desired. □

Example 4.25. Show that every equation of the form $f(t) + g(y)\dfrac{dy}{dt} = 0$ is exact.

Note: This means all separable equations can be written in the form of an exact equation.

Solution. We note that $\dfrac{\partial f(t)}{dy} = \dfrac{\partial g(y)}{dt} = 0$, and thus this equation is exact. □

Example 4.26. Show that a first order linear equation $\dfrac{dy}{dt} + a(t)y - f(t) = 0$, where $a(t), f(t)$ are continuous, is exact if and only if $a(t) = 0$. Show that there is always an integrating factor that turns this equation into an exact equation.

Solution. Suppose $\dfrac{dy}{dt} + a(t)y - f(t) = 0$ is exact. We need to have

$$\frac{\partial 1}{\partial t} = 0 = \frac{\partial(a(t)y - f(t))}{\partial y} = a(t).$$

Now, suppose μ is an integrating factor. We need to have $\mu_t = a(t)\mu + a(t)y\mu_y - f(t)\mu_y$. Taking $\mu_y = 0$ we obtain $\mu_t = a(t)\mu$. We realize that $\mu = e^{A(t)}$ is a solution if $A'(t) = a(t)$. $\square$

Example 4.27. Find all constants c for which the equation

$$2t \ \mathrm{d}t + (t + cy) \ \mathrm{d}y = 0$$

has an integrating factor of the form $\mu = \omega`(t + y)$. For each of these constants solve the equation.

Solution. Suppose $\mu = \mu(t + y)$ is an integrating factor. We must have the following:

$$(2t\mu)_y = (t\mu + cy\mu)_t \Rightarrow 2t\mu_y = \mu + t\mu_t + cy\mu_t.$$

By the chain rule, we have $\mu_t = \mu_y = \omega'(t + y)$. This yields the following:

$$2t\omega' = \omega + t\omega' + cy\omega' \Rightarrow (t - cy)\omega' = \omega.$$

Since ω and thus ω' are functions of $t + y$, the function $t - cy$ must also be a function of $t + y$. This function can be written as $t - cy = t + y - (1 + c)y$. It is a function of $t + y$ if and only if $c = -1$. When $c = -1$ we have $(t + y)\omega' = \omega$. One solution is $\mu = t + y$. This yields

$$\begin{cases} 2t(t + y) = \phi_t \\ (t - y)(t + y) = \phi_y \end{cases}$$

The first equation yields $\phi = \dfrac{2t^3}{3} + t^2y + f(y)$. Substituting this into the second equation we obtain

$$t^2 - y^2 = t^2 + f'(y) \Rightarrow f(y) = -\frac{y^3}{3} \text{ is one solution.}$$

Therefore, the general solution is $\dfrac{2t^3 - y^3}{3} + t^2y = c$. $\square$

Example 4.28. Suppose $M(t, y)$ and $N(t, y)$ are continuous over a rectangle $R = (a, b) \times (c, d)$ and they have continuous partials over R. Assume, further that $M^2 + N^2 \neq 0$ over R. Prove $1/(M^2 + N^2)$ is an integrating factor of $M \ \mathrm{d}t + N \ \mathrm{d}y = 0$ if $M_t = N_y$ and $M_y = -N_t$.

Solution. By definition, for $1/(M^2 + N^2)$ to be an integrating factor the equation

$$\frac{M}{M^2 + N^2} \ \mathrm{d}t + \frac{N}{M^2 + N^2} \ \mathrm{d}y = 0$$

must be exact. By Theorem 4.4 this equation is exact if and only if

$$\left(\frac{M}{M^2 + N^2}\right)_y = \left(\frac{N}{M^2 + N^2}\right)_t.$$

By the quotient rule this is equivalent to

$$\frac{M_y(M^2 + N^2) - (2MM_y + 2NN_y)M}{(M^2 + N^2)^2} = \frac{N_t(M^2 + N^2) - (2MM_t + 2NN_t)N}{(M^2 + N^2)^2}.$$

Eliminating the denominator and combining like terms, this equality is equivalent to

$$M_y(N^2 - M^2) - 2MNN_y = N_t(M^2 - N^2) - 2MNM_t.$$

Since by assumption $M_y = -N_t$ and $N_y = M_t$ the result follows. $\square$

4.8 Exercises

Solutions to some differential equations may be implicit.

Exercise 4.1. *Draw a Venn Diagram for the following "sets":*

1. *All ODE's.*
2. *Separable Equations.*
3. *Linear Equations.*
4. *Explicit Equations.*
5. *Autonomous Equations.*
6. *Exact Equations.*
7. *Equations with Integrating Factors.*

Exercise 4.2. *Prove each function is a solution to the corresponding differential equation:*

(a) $y = 2t^{3/2}$ *with* $t > 0$*;* $2t^{1/2}y'' + t^{-1/2}y' - 6 = 0$.

(b) $y = \sin(t^2)$*;* $ty'' - y' + 4t^3y = 0$.

(c) $y = t^4 + 17t^3 + 14t$*;* $y^{(5)} = 0$.

Exercise 4.3. *Find all solutions of the following differential equations:*

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \cos^3 t \sin t$.

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \tan^2 t$.

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{2}{t^2 - 1}$.

(d) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{2t}{t^4 + 1}$.

(e) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{1}{\sqrt{1 - t^2}}$.

(f) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = e^{2t - e^t}$.

Exercise 4.4. *Find a continuous solution $y : \mathbb{R} \to \mathbb{R}$ to the initial value problem and prove this solution is unique.*

$$y' = \begin{cases} (t-1)y & \text{if } t > 0 \\ (1-t)y & \text{if } t < 0 \end{cases} \qquad y(0) = 2.$$

Exercise 4.5. *Let $p, f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Prove that for every $y_0 \in \mathbb{R}$ there is a unique continuous solution $y : \mathbb{R} \to \mathbb{R}$ to the initial value problem*

$$y' + p(t)y = \begin{cases} f(t) & \text{if } t > 0 \\ g(t) & \text{if } t < 0 \end{cases} \qquad y(0) = y_0$$

Exercise 4.6. *Solve each of the following initial value problems:*

(a) $\dfrac{dy}{dt} = \dfrac{t^2+1}{t^3-t}, y(2) = 1.$

(b) $\dfrac{dy}{dt} = \sin^4 t, y(0) = 1.$

(c) $\dfrac{dy}{dt} = \tan t, y(\pi) = 1.$

(d) $\dfrac{dy}{dt} = \sqrt{3t-1}, y(1) = 2,\ t > 1/3.$

(e) $\dfrac{dy}{dt} = te^t, y(0) = 1.$

Exercise 4.7. *Let y be the solution to the initial value problem $\dfrac{dy}{dt} = \sin(t^3) + 2,\ y(-1) = 5$. Evaluate $y(1)$.*

Exercise 4.8. *Solve each differential equation:*

(a) $ty' - 2y = 1/t$, *with* $t < 0$.

(b) $y' \cos t + y = \sin t$, *with* $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Exercise 4.9. *Find all real constants c or show no such constant c exists, for which the differential equation $y' + cy = t$ has at least one solution that satisfies $y(0) = 1, y(1) = -1$.*

Exercise 4.10. *Find all bounded solutions of each equation:*

(a) $(t+1)y' - y + 1 = 0.$

(b) $y' = 2t + 2ty.$

Exercise 4.11. *Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) = 2 + \int_2^x (t - tf(t))dt$, for all $x \in \mathbb{R}$.*

Exercise 4.12. *Find the general solution of each equation:*

(a) $ty' + \sec y = 0.$

(b) $(1+t^2)y' + (1-y^2)t = 0.$

(c) $y' = \cos y \sin^2 t.$

(d) $y' = y^2 - (a+b)y + ab$, *where* $a, b \in \mathbb{R}$ *are constants.*

Exercise 4.13. *Solve each of the following equations:*

(a) $\dfrac{dy}{dt} = (y-t)^2$

(b) $\dfrac{dy}{dt} = \dfrac{e^{t+y}}{t+y} - 1.$

(c) $y' - 4t^2 = 4yt + y^2.$

(d) $y' = \dfrac{y+t}{y+t+1}.$

Exercise 4.14. *Find an integrating factor for the following equation, given the integrating factor is of the form* $\mu = t^m y^n$*.*

$$(y - y^2) + ty' = 0.$$

Exercise 4.15. *Find all stationary and nonstationary solutions of the equation* $\dfrac{dy}{dt} = yt - y - t + 1.$

Exercise 4.16. *Solve the initial value problem* $(t^2+1)y' + y^2 + 1 = 0,\ y(3) = 2.$ *Your final answer must be explicit and simplified.*

Exercise 4.17. *Prove that if* y_1, y_2 *are solutions to* $y' + a(t)y = f(t)$*, then* $y_1 - y_2$ *is a solution to* $y' + a(t)y = 0.$

Exercise 4.18. *Find all solutions to each equation satisfying the given condition:*

(a) $t^2 y' \sin y = 1, \lim\limits_{t\to\infty} y(t) = \pi.$

(b) $y' + 2y = 5\cos t$*, and* y *is periodic.*

(c) $y' - 2ty = 0$*, and* y *is bounded.*

(d) $y' = \dfrac{y+t}{y+t+1}$*, and* $y(0) = 1.$

Exercise 4.19. *Determine* $\lim\limits_{t\to\infty} y(t)$*, for all solutions of the differential equation* $y' + y\cos t = \cos t$*. Find your answer in terms of* $y(0)$*.*

Exercise 4.20. *Solve the initial value problem* $(t^2+y^2)\dfrac{dy}{dt} + (3t^2y + 2ty + y^3) = 0, y(0) = 1.$

Exercise 4.21. *Find the general solution to each equation:*

(a) $(3ty^2 + 2y)\ \mathrm{d}t + (2t^2y + t)\ \mathrm{d}y = 0.$

(b) $y\cos t\ \mathrm{d}t + (y\sin t + \sin t + 1)\ \mathrm{d}y = 0.$

(c) $(y\cos t + y^2)\ \mathrm{d}t + (3\sin t + 4yt)\ \mathrm{d}y = 0.$

(d) $(7y + 8ty^3)\ \mathrm{d}t + (t + 3t^2y^2)\ \mathrm{d}y = 0.$

(e) $(t^2y + y + 1)\ \mathrm{d}t + (t + t^3)\ \mathrm{d}y = 0.$

Exercise 4.22. *Let $f(t)$ and $g(y)$ be continuous functions. Show that the equation*

$$\frac{f(t)}{y} + 1 + (g(y) + t/y)\frac{dy}{dt} = 0$$

is not generally exact. Find an integrating factor and use that to find a general solution for this equation.

Exercise 4.23. *Determine all constants c, for which the differential equation $(t^2 + y^2) + \frac{ct^3 + t^2}{y}\frac{dy}{dt} = 0$ has an integrating factor $\mu = \frac{1}{t^2y^2}$. For all such constants c, solve the resulting equation.*

Exercise 4.24. *Find all curves of the form $y = f(x)$ on the xy-plane that intersect the x-axis at an angle of $\frac{\pi}{4}$ and satisfy the differential equation $xy' + y = 2$.*

Exercise 4.25. *Prove that the IVP*

$$y'' = e^{t^2},\ y(0) = 1,\ y'(1) = -1$$

has a unique solution.

Exercise 4.26. *Let $f : I \to \mathbb{R}$ be a continuous function, where I is an open interval, $t_0, t_1 \in I$, and $y_0, y_1 \in \mathbb{R}$. Prove that there is a unique function y defined over I for which*

$$y'' = f(t),\ y(t_0) = y_0,\ y'(t_1) = y_1.$$

Exercise 4.27. *Let $f : I \to \mathbb{R}$ be a continuous function, where I is an open interval, $t_0, t_1 \in I$ be distinct real numbers, and y_0, y_1 be two real numbers. Prove that there is a unique function y defined over I for which*

$$y'' = f(t),\ y(t_0) = y_0,\ y(t_1) = y_1.$$

Exercise 4.28. *Solve each second order IVP.*

(a) $y'' = t^2 + \sin t,\ y(0) = 1,\ y'(0) = 0.$

(b) $y'' = y',\ y(0) = y'(0) = 2.$

(c) $y'' + 1 = (y' + t)^2,\ y(0) = 1, y'(0) = 2.$

(d) $y'' = (y')^2, y(0) = y'(0) = 1.$

Hint: Substitute $z = y'$.

Exercise 4.29. *Find all real constants y_0, y_1 for which the equation*

$$y'' = e^{t^2}, y(0) = y_0, y(1) = y_1$$

has a unique solution defined over $\mathbb{R}$.

Exercise 4.30. *Suppose $M(t, y)$ and $N(t, y)$ have continuous first partials over a rectangle R. Prove that the equation $M(t, y)\ \mathrm{d}t + N(t, y)\ \mathrm{d}y = 0$ has a C^1 integrating factor of the form $\mu(y)$ if and only if $\frac{N_t - M_y}{M}$ only depends on y.*

Exercise 4.31. *Suppose $M(t,y)$ and $N(t,y)$ have continuous first partials over a rectangle R. Prove that the function $\mu(t,y)$ with continuous first partials is an integrating factor for the equation $M(t,y)\ \mathrm{d}t+N(t,y)\ \mathrm{d}y = 0$ if and only if*

$$\mu\left(M_y - N_t\right) = N\mu_t - M\mu_y$$

on R.

In the next exercise we will prove that each first order IVP can be turned into one with initial time $t_0 = 0$.

Exercise 4.32. *Consider the IVP*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t,y),\ \ y(t_0) = y_0.$$

Set $z(s) = y(s+t_0)$. Prove that the above IVP is equivalent to the following IVP

$$\frac{\mathrm{d}z}{\mathrm{d}s} = f(s+t_0, z),\ \ z(0) = y_0.$$

Exercise 4.33. *Suppose $M(t,y)$ and $N(t,y)$ are continuous and have continuous first partials on the rectangle R given by $|t-t_0| < a, |y-y_0| < b$. Assume $M_y = N_t$ on R. Prove that the solution to the equation*

$$M(t,y) + N(t,y)y' = 0$$

is given by

$$\int_{y_0}^{y} N(t,u)\ \mathrm{d}u + \int_{t_0}^{t} M(u,y_0)\ \mathrm{d}u = C,$$

where C is a constant.

Exercise 4.34. *Suppose the differential equation*

$$M(t,y)\ \mathrm{d}t + N(t,y)\ \mathrm{d}y = 0 \qquad (*)$$

is exact and has a nonconstant integrating factor $\mu(t,y)$. Prove that $\mu(t,y) = C$ is a solution to $(*)$.

4.9 Challenge Problems

Exercise 4.35. *Suppose $f:\mathbb{R}\to\mathbb{R}$ is a continuous function. Prove that all solutions of the differential equation $y' = f(t)$ are periodic with period $L > 0$ if and only if f is periodic with period L and $\displaystyle\int_0^L f(t)\ \mathrm{d}t = 0$.*

Exercise 4.36. *Solve the initial value problem $y^2 + 2yy' + 2t + 2 = 2e^t$, $y(0) = 2$.*

Exercise 4.37. *Suppose $a, f:\mathbb{R}\to\mathbb{R}$ are continuous functions and c is a positive constant for which*

$$\lim_{t\to\infty} f(t) = 0,\ \ \textit{and}\ \forall\ t\in\mathbb{R}\ a(t)\geq c.$$

Let $y(t)$ be a solution to the differential equation $y' + a(t)y = f(t)$. Prove that

$$\lim_{t\to\infty} y(t) = 0.$$

Exercise 4.38. *Let $a : \mathbb{R} \to \mathbb{R}$ be a continuous function. prove that all solutions of the equation $y' + a(t)y = 0$ are periodic with period L if and only if $a(t)$ is periodic with period L and that* $\int_0^L a(t)\ \mathrm{d}t = 0$.

Exercise 4.39. *Solve the equation* $\dfrac{dy}{dt} = -\dfrac{2y + 3ty^2}{2t + 4t^2y^2}$.

Definition 4.3. Let k be a positive integer. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be **homogeneous** of degree k, if

$$f(tx_1, tx_2, \ldots, tx_n) = t^k f(x_1, x_2, \ldots, x_n),$$

for all $t, x_1, \ldots, x_n \in \mathbb{R}$.

Exercise 4.40. *Suppose $P(x, y)$ and $Q(x, y)$ are homogeneous functions of the same degree with continuous partial derivatives. Prove that $\frac{1}{xP+yQ}$ is an integrating factor for the equation*

$$P(x, y)\ \mathrm{d}x + Q(x, y)\ \mathrm{d}y = 0.$$

Exercise 4.41. *Solve each of the following:*

(a) $(t - ty)\ \mathrm{d}t + (t^2 + y)\ \mathrm{d}y = 0.$

(b) $(t^2 + y^2 + 1)\ \mathrm{d}t - 2ty\ \mathrm{d}y = 0.$

(c) $t^2y'y + ty' + ty^2 + y - ty = 0.$

Exercise 4.42. *Suppose $M(t, y)$ and $N(t, y)$ have continuous partials over a rectangle R. Assume both t and y are integrating factors for the equation*

$$M(t, y)\ \mathrm{d}t + N(t, y)\ \mathrm{d}y = 0.$$

Prove that all solutions of this equation are either lines of the form $y = Ct$ for a constant C, or satisfy $tN(t, y) = 0$.

Exercise 4.43. *Consider the differential equation*

$$y' = f(t, y) \qquad (*)$$

over a rectangle $R = (a, b) \times (c, d)$ in the ty-plane, where $f, f_t, f_y, f_{ty} = f_{yt}$ are all continuous. Assume $f \neq 0$ on R. Prove that the equation $(*)$ is separable if and only if $f f_{ty} = f_t f_y$.

Exercise 4.44. *Solve the initial value problem*

$$y'' + (\cos t)y' - (\sin t)y = -\sin t,\ \ y(0) = 1,\ \ y'(0) = 1.$$

4.10 Summary

- An explicit IVP $\dfrac{dy}{dt} = f(t),\ y(t_0) = y_0$ has a unique solution as long as $f(t)$ is continuous. The solution can be found by integrating both sides from t_0 to t and using the initial condition y_0 as the constant.

- To solve a linear equation $\dfrac{dy}{dt} + a(t)y = f(t)$:
  - Keep in mind that the goal is to write the left hand side as the derivative of one function.
  - Find $A(t)$ for which $A'(t) = a(t)$.
  - Rewrite the equation as $\dfrac{d}{dt}\left(e^{A(t)}y\right) = e^{A(t)}f(t)$. Then integrate both sides.
- Existence and Uniqueness Theorem for linear first order equations requires the coefficient $a(t)$ and the forcing $f(t)$ to be continuous.
- To solve a separable equation of the form $\dfrac{dy}{dt} = f(t)g(y)$:
  - Find all stationary solutions by solving $g(y) = 0$.
  - For nonstationary solutions: separate the variables and rewrite the equation as $\dfrac{dy}{g(y)} = f(t)dt$. Then integrate both sides.
- There are three common types of equations that require change of variables:
  1. Equations of the form $y' = f(ay + bt + c)$ can be solved by the change of variable $u = ay + bt + c$.
  2. Equations of the form $y' = (ay + bt)/(cy + dt)$ can be solved by the change of variable $u = y/t$.
  3. For equations of the form $y' = (ay+bt+m)/(cy+dt+n)$ we first do a translation $Y = y+r, T = t+s$ to determine which constants r, s change this equation into one of the form #2 above. After finding r, s we proceed with the change of variable $u = y/t$. Note that some problems that might look like #3 are actually instances of #1. So make sure you check for #1 first.
- An equation $M + N\dfrac{dy}{dt} = 0$ is exact if $M_y = N_t$.
- To solve an exact equation $M + N\dfrac{dy}{dt} = 0$ we will find $\phi(t, y)$ for which $\phi_t = M$, and $\phi_y = N$. The solutions then are given by $\phi(t, y) = c$.
- To solve equations using the integrating factor method:
  - First check if the equation is exact.
  - If it is not, multiply both sides by μ and set up the equation $(\mu M)_y = (\mu N)_t$.
  - Find an appropriate μ. Generally, finding μ is not easy and there is no method that always works. Test if $\mu_y = 0$ would yield a function of t for μ, or if $\mu_t = 0$ would yield a function of y for μ.
  - Multiply both sides of the equation by μ, and solve the resulting equation using the method for exact equations.

Chapter 5

Existence and Uniqueness Theorems

5.1 Existence and Uniqueness for Linear Equations

Recall that a differential equation of the form

$$\frac{d^n y}{dt^n} + a_n(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_2(t)\frac{dy}{dt} + a_1(t)y = f(t)$$

is called an n-th order linear differential equation in standard or normal form (i.e. the leading coefficient is 1). When $f(t) = 0$, we say the equation is **homogeneous**, otherwise we say it is **nonhomogeneous**.

Theorem 5.1 (Exietence and Uniqueness Theorem for Linear Equations)**.** *Let I be an open interval and let $a_j(t)$, $1 \le j \le n$, and $f(t)$ be continuous over I. Then, for every $t_0 \in I$ and every $y_0, y_1, \ldots, y_{n-1} \in \mathbb{R}$, the initial value problem*

$$\begin{cases} \dfrac{d^n y}{dt^n} + a_n(t)\dfrac{d^{n-1}y}{dt^{n-1}} + \cdots + a_2(t)\dfrac{dy}{dt} + a_1(t)y = f(t) \\ y(t_0) = y_0 \\ \vdots \\ y^{(n-1)}(t_0) = y_{n-1} \end{cases}$$

has a unique solution over I.

Example 5.1. Find the largest interval I for which the Existence and Uniqueness Theorem guarantees a unique solution $y(t)$ with $t \in I$ to the IVP exists:

$$ty'' + \frac{\tan t}{t-3}y' - y = e^t,\ y(1) = 2,\ y'(1) = 4.$$

Example 5.2. Prove that $\sin(t^2)$ cannot be a solution to a second order homogeneous linear differential equation whose coefficients are continuous over $(-1, 1)$.

5.2 Picard Iterates

Some initial value problems have multiple solutions despite the fact that all functions involved are continuous.

Example 5.3. Solve the initial value problem $y' = y^2, y(0) = 1$. Show this solution is not defined over $\mathbb{R}$.

Example 5.4. Find two solutions for the initial value problem

$$\frac{dy}{dt} = 3y^{2/3}, \quad y(0) = 0.$$

We would like to know in what circumstances a first order initial value problem is guaranteed to have a unique solution.

First, note that solving most differential equations is impossible, so we need to show the existence of a solution without actually solving the equation, but how could this be done? We have seen this idea being used previously when dealing with series.

Example 5.5. Prove that the function $f(x) = \sum\limits_{n=1}^{\infty} \dfrac{\sin(nx)}{n^2}$ is defined for every $x \in \mathbb{R}$.

Similar to the above example we need to follow the following steps to prove a first order initial value problem has a solution:

- Construct a sequence of functions $y_n(t)$ that approximate the solution.
- Prove that this sequence converges to a function $y(t)$ as n approaches infinity.
- Show that $y(t)$ is a solution to the IVP.

The IVP $y' = f(t, y), y(t_0) = y_0$ can be written as $y(t) = y_0 + \displaystyle\int_{t_0}^{t} f(s, y(s))ds$. Let us call the right hand side $L(t, y)$. The first guess for a solution would naturally be $y = y_0$. Our next estimate for the solution will be evaluated by applying L to get $y_1 = L(t, y_0)$, and the next approximation for the solution would be $y_2 = L(t, y_1)$, and so on.

Definition 5.1. Given an initial value problem $\dfrac{dy}{dt} = f(t, y),\ y(t_0) = y_0$, the **Picard iterates** associated to this IVP are functions y_n that are defined recursively by:

- y_0 is the constant given by the initial value.
- For every $n \geq 0$, we have $y_{n+1} = y_0 + \displaystyle\int_{t_0}^{t} f(s, y_n(s))ds$.

Example 5.6. Compute the Picard iterates for the IVP $y' = y,\ y(0) = 1$, and show they converge to the solution to the given IVP.

Theorem 5.2 (Existence and Uniqueness Theorem)**.** *Suppose $f(t, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous over a rectangle R on the ty-plane given by $t_0 \leq t \leq t_0 + a$, $|y - y_0| \leq b$. Let M be the maximum value of $|f(t, y)|$ over R, and let $\alpha = \min(a, b/M)$. Then the initial value problem*

$$\frac{dy}{dt} = f(t, y),\ y(t_0) = y_0 \qquad (*)$$

has a unique solution $y(t)$ defined over $[t_0, t_0 + \alpha]$. Furthermore, $|y(t) - y_0| \leq b$ for all $t \in [t_0, t_0 + \alpha]$. A similar result holds if the interval for t is changed to $[t_0 - \alpha, t_0]$ or $[t_0 - \alpha, t_0 + \alpha]$.

Remark. M does not have to be the exact value of the maximum of f over R. It is enough to find a value of M for which $|f(t,y)| \leq M$ for all $(t,y) \in R$.

Example 5.7. Consider the IVP $\dfrac{dy}{dt} = t + e^{-y^2}$, $y(0) = 0$. Show that there is a solution defined over $[0, 0.5]$ and that the solution satisfies $|y(t)| \leq 1$ for all $t \in [0, 0.5]$.

The proof of Theorem 5.2 comes in two parts: (1) The existence of a solution, and (2) The uniqueness of the solution. The proof of existence requires us to show Picard Iterates approach a solution. We will skip the proof of existence, however we will prove the uniqueness using some facts from calculus.

Definition 5.2. Let I be an interval. A function $f : I \to \mathbb{R}$ is said to be **piecewise continuous** if f has only finitely many points of discontinuity inside each bounded interval.

Theorem 5.3. *Suppose two piecewise continuous function f, g defined over an interval $[a,b]$ satisfy*

$$f(x) \leq g(x) \text{ for all } x \in [a,b].$$

Then,

$$\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx.$$

A similar result holds if the interval $[a,b]$ is replaced by $[a,\infty)$

Example 5.8. Using the above theorem, prove that for every piecewise continuous function $f(x)$ over an interval $[a,b]$ we have

$$\left|\int_a^b f(x)\ dx\right| \leq \int_a^b |f(x)|\ dx.$$

Prove a similar result if $[a,b]$ is replaced by $[a,\infty)$.

Recall also that the Mean Value Theorem implies that if $f(t,y)$ is continuous and f_y exists, then for every $y_1 < y_2$ there exists a real number $c \in (y_1, y_2)$ for which $f(t,y_1) - f(t,y_2) = f_y(t,c)(y_1 - y_2)$.

Proof. (Uniqueness) Suppose $z_1(t), z_2(t)$ are two solutions to $(*)$. Then,

$$|z_1(t) - z_2(t)| = \left|\int_{t_0}^t [f(s,y(s)) - f(s,z(s))]ds\right| \leq \int_{t_0}^t |f(s,y(s)) - f(s,z(s))|ds = \int_{t_0}^t |(z_1(s) - z_2(s))f_y(s,c)|ds,$$

for some c between $y(s)$ and $z(s)$. (Note that c depends on s, but that is unimportant.) The existence of c is guaranteed by the Mean Value Theorem.

Since $|f_y|$ is continuous over R and R is closed, by the Extreme Value Theorem there is a real number L for which $|f_y| \leq L$ over R. Thus, the integral above does not exceed $L\displaystyle\int_{t_0}^t |z_1(s) - z_2(s)|ds$. To summarize, we have shown

$$|z_1(t) - z_2(t)| \leq L\int_{t_0}^t |z_1(s) - z_2(s)|ds$$

for all $t \in [t_0, t_0 + \alpha]$.

Let $W(t) = \int_{t_0}^{t} |z_1(s) - z_2(s)| ds$. We know $W'(t) \leq LW(t)$. We will multiply both sides by an integrating factor to obtain:

$$e^{-Lt}W'(t) - Le^{-Lt}W(t) \leq 0 \Rightarrow \frac{d}{dt}(e^{-Lt}W(t)) \leq 0.$$

Integrating both sides from t_0 to t and applying Theorem 5.3, we obtain $e^{-Lt}W(t) - e^{-Lt_0}W(t_0) \leq 0$. Since $W(t_0) = \int_{t_0}^{t_0} |z_1(s) - z_2(s)| ds = 0$ this implies $W(t) \leq 0$. However $W(t) \geq 0$, since $|z_1(s) - z_2(s)| \geq 0$. Therefore, $W(t) = 0$. Differentiating we get $|z_1(t) - z_2(t)| = 0$ or $z_1 = z_2$, as desired. $\square$

Example 5.9. Show that the following initial value problem has a unique solution over $[0, \frac{\sqrt{2}}{4+2\sqrt{2}}]$:

$$y' = e^{-t^2} + y^2, \; y(0) = 1.$$

Example 5.10. Show that the following initial value problem has a unique solution over $[0, \infty)$:

$$\frac{dy}{dt} = e^{-y^2} + t^4, \; y(0) = 1.$$

5.3 More Examples

Example 5.11. Find the largest interval for which a unique solution to each IVP is guaranteed to exist.

(a) $ty' + y = \tan t, \; y(1) = -1.$

(b) $y'' + \ln(t-1)y' + \sqrt{10 - t^2}\; y = 1, \; y(2) = 4.$

Solution. (a) This IVP written in normal form is:

$$y' + \frac{y}{t} = \frac{\tan t}{t}, \; y(1) = -1.$$

We will use the Existence and Uniqueness Theorem for linear equations. For that we need all coefficients and the forcing to be continuous. Therefore, we need $t \neq 0$ and $t \neq k\pi + \pi/2$ for $k \in \mathbb{Z}$. We also need the initial value $t_0 = 1$ to be inside the interval of definition. Therefore, the answer is $(0, \pi/2)$.

(b) Similar to above, we need $t - 1 > 0$ and $10 - t^2 \geq 0$. This yields $t > 1$ and $-\sqrt{10} \leq t \leq \sqrt{10}$. We also need the initial value $t_0 = 2$ to be inside the interval of definition. Therefore, the answer is $(1, \sqrt{10})$. $\square$

Example 5.12. Consider the initial value problem $y' = ty$ and $y(0) = 1$. Find all Picard iterates of this IVP and show they converge to the solution.

Solution. This is a linear equation with integrating factor $e^{-t^2/2}$. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t^2/2}y\right) = 0 \Rightarrow e^{-t^2/2}y = C \Rightarrow y = Ce^{t^2/2}.$$

Using the initial condition we obtain $C = 1$, and thus the solution is $y = e^{t^2/2}$.

Picard iterates are

$$y_0 = 1$$

$$y_1 = 1 + \int_0^t s \, \mathrm{d}s = 1 + \frac{t^2}{2}$$

$$y_2 = 1 + \int_0^t s(1 + \frac{s^2}{2}) \, \mathrm{d}s = 1 + \frac{t^2}{2} + \frac{t^4}{2 \cdot 4}$$

$$y_3 = 1 + \int_0^t s(1 + \frac{s^2}{2} + \frac{s^4}{2 \cdot 4}) \, \mathrm{d}s = 1 + \frac{t^2}{2} + \frac{t^4}{2 \cdot 4} + \frac{t^6}{2 \cdot 4 \cdot 6}$$

Using induction on n we will prove

$$y_n = 1 + \frac{t^2}{2} + \frac{t^4}{2 \cdot 4} + \frac{t^6}{2 \cdot 4 \cdot 6} + \cdots + \frac{t^{2n}}{2 \cdot 4 \cdots (2n)}.$$

The base case was proved above. For the inductive step we have

$$y_{n+1} = 1 + \int_0^t s y_n(s) \, \mathrm{d}s = 1 + \int_0^t s + \frac{s^3}{2} + \cdots + \frac{s^{2n+1}}{2 \cdot 4 \cdots (2n)} = 1 + \frac{t^2}{2} + \frac{t^4}{2 \cdot 4} + \cdots + \frac{t^{2n+2}}{2 \cdot 4 \cdots (2n+2)}.$$

This completes the proof of the claim above. y_n can be rewritten as

$$y_n = 1 + \frac{t^2}{2} + \frac{t^4}{2 \cdot 4} + \cdots + \frac{t^{2n}}{2 \cdot 4 \cdots (2n)} = 1 + \frac{t^2}{2 \cdot 1!} + \frac{t^4}{2^2 \cdot 2!} + \cdots + \frac{t^{2n}}{2^n n!}$$

This is the n-th partial sum of the Taylor series for $e^{t^2/2}$. This means y_n tends to $e^{t^2/2}$ as $n \to \infty$. $\square$

Example 5.13. Suppose $y = y_0$ is a stationary solution to a first order equation $y' = f(t, y)$, where f is continuous over $\mathbb{R}^2$. Prove that all Picard iterates to the IVP $y' = f(t, y), \; y(t_0) = y_0$ are the same, i.e. $y_n(t) = y_0$ for all n and for all $t \in \mathbb{R}$. Is the converse true?

Solution. Since $y = y_0$ is a solution to the given equation, we obtain $0 = f(t, y_0)$ for all $t \in \mathbb{R}$. By definition

$$y_1 = y_0 + \int_{t_0}^t f(s, y_0) ds = y_0 + \int_{t_0}^t 0 ds = y_0.$$

Therefore, $y_1 = y_0$. By repeating the same argument we obtain $y_n = \cdots = y_1 = y_0$. Thus, $y_n(t) = y_0$ for all n.

Now, assume $y_n(t) = y_0$ for all n. Therefore,

$$y_1 = y_0 \Rightarrow y_0 = y_0 + \int_{t_0}^t f(s, y_0) ds \Rightarrow \int_{t_0}^t f(s, y_0) ds = 0,$$

for all $t \in \mathbb{R}$. Differentiating with respect to t we conclude that $f(t, y_0) = 0$ and thus $y = y_0$ is a stationary solution to the equation $y' = f(t, y)$. $\square$

Example 5.14. Prove each equation has a unique solution over the given interval.

(a) $y' = e^{-y^2} + e^{-t}, \; y(0) = 2$ given $0 \le t$.

(b) $y' = e^{y^2+t^2}, y(0) = 0$ given $0 \le t \le 1/e^2$.

(c) $y' = y^2 + e^{-t}, y(1) = 2$ given $|t-1| \le (\sqrt{5}-2)/2$.

Solution. (a) First note that $e^{-y^2} + e^{-t}$ as well as its partial derivative with respect to y are both continuous. Let a, b be two positive constants and assume $0 \le t \le a$ and $|y-2| \le b$. We have

$$|e^{-y^2} + e^{-t}| \le e^0 + e^0 = 2 \Rightarrow M = 2 \Rightarrow \alpha = \min(a, b/2).$$

Setting $b = 2a$ we conclude that there is unique solution to the IVP over the interval $[0, a]$. Since a is arbitrary with a method similar to the one used in Example 5.10 we can show there is a unique solution over $[0, \infty)$.

(b) The function $e^{y^2+t^2}$ and its partial with respect to y are both continuous. For two positive constants a, b, if $|t| \le a$ and $|y| \le b$, then

$$e^{y^2+t^2} \le e^{a^2+b^2} \Rightarrow M = e^{a^2+b^2} \Rightarrow \alpha = \min(a, b/e^{a^2+b^2}).$$

Setting $a = b = 1$ we conclude that there is a unique solution with $t \in [0, 1/e^2]$.

(c) Similar to above $y^2 + e^{-t}$ and its partial with respect to y are continuous. Let a, b be positive constants and assume $|t-1| \le a, |y-2| \le b$. We have

$$|y^2 + e^{-t}| \le (b+2)^2 + e^{a-1} \Rightarrow M = (b+2)^2 + e^{a-1}.$$

For simplicity choose $a = 1$, this yields $\alpha = \min(1, \frac{b}{(b+2)^2+1})$. Solving $\frac{b}{(b+2)^2+1} = (\sqrt{5}-2)/2$ we obtain

$$2b = (\sqrt{5}-2)(b^2+4b+5) \Rightarrow (\sqrt{5}-2)b^2 + (4\sqrt{5}-10)b + 5(\sqrt{5}-2) = 0 \Rightarrow b^2 - 2\sqrt{5}b + 5 = 0 \Rightarrow b = \sqrt{5}.$$

This completes the proof. $\square$

Example 5.15. Consider the initial value problem

$$y' = t^2 + y^2, y(0) = 0.$$

(a) Prove that this equation has a unique solution on $|t| \le \frac{1}{\sqrt{2}}$.

(b) Let y be the unique solution to this IVP. Prove that $-y(-t)$ is also another solution to this IVP. Deduce the solution to this IVP must be an odd function.

Solution. (a) First, note that $t^2 + y^2$ as a polynomial is continuous and has continuous partials. Therefore, we may apply the Existence and Uniqueness Theorem.

Let a, b be two positive real numbers. Consider the rectangle R given by $|t| \le a, |y| \le b$. Over this rectangle we have

$$|t^2 + y^2| \le a^2 + b^2 \Rightarrow M = a^2 + b^2 \Rightarrow \alpha = \min(a, \frac{b}{a^2+b^2}).$$

The equation has a unique solution over $[-\alpha, \alpha]$. We would like to maximize α. Given a, we let $f(b) = \dfrac{b}{a^2+b^2}$. We have $f'(b) = \dfrac{a^2-b^2}{(a^2+b^2)}$. Therefore, the maximum for f is obtained when $b = a$. This means the largest α can be is $\min(a, \frac{1}{2a})$. Setting $a = \frac{1}{\sqrt{2}}$ we conclude the equation has a solution with $|t| \leq 1/\sqrt{2}$.

(b) Let $z(t) = -y(-t)$. By the Chain Rule we have $z'(t) = y'(-t)$. Since y is a solution to the given IVP we have

$$y'(-t) = (-t)^2 + (y(-t))^2 \Rightarrow z'(t) = t^2 + (z(t))^2.$$

This means z satisfies $z' = t^2 + z^2$. On the other hand $z(0) = -y(-0) = -y(0) = 0$. Thus, z also satisfies the given IVP. The uniqueness implies $z = y$. Therefore, $-y(-t) = y(t)$, i.e. y is an odd function. $\square$

Example 5.16. Prove that the equation $y' = \dfrac{\sin(t+ty^2)}{1-t^2}, y(0) = 1$ has a unique solution over $(-1, 1)$.

Solution. Let $a \in (0,1)$ and $b > 0$. Note that $f(t,y) = \dfrac{\sin(t+ty^2)}{1-t^2}$ and its partial $f_y = \dfrac{2ty\cos(t+ty^2)}{1-t^2}$ are continuous over the rectangle $[-a,a] \times [-b,b]$. Furthermore, $|f(t,y)| \leq 1/(1-a^2) = M$. Thus, if we take $\alpha = \min(a, b/M) = \min(a, b(1-a^2))$, then the equation has a unique solution over $[-\alpha, \alpha]$. Taking $b = a/(1-a^2)$, we obtain $\alpha = a$. Thus, for every $a \in (0,1)$ the given initial value problem has a unique solution over $[-a,a]$. Assume y_n is the solution over $[-1+1/n, 1-1/n]$ for $n = 2, 3, \ldots$. By uniqueness $y_n(t) = y_{n+1}(t)$ for every $t \in [-1+1/n, 1-1/n]$. So, if we define $y(t) = y_n(t)$ for every $t \in [-1+1/n, 1-1/n]$ we can show this y is the unique solution over $(-1,1)$. The process is the same as the one in Example 5.10. $\square$

Example 5.17. Suppose $y = y_0$ is a stationary solution to the autonomous equation $y' = g(y)$, where $g : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function. Prove that for every $t_0 \in \mathbb{R}$, the solution $y = y_0$ is the unique solution to the initial value problem

$$y' = g(y), \; y(t_0) = y_0, \; t \in \mathbb{R} \qquad (*)$$

Solution. We will show this IVP has a unique solution over any interval $[t_0, t_0+\alpha]$ for every $\alpha > 0$. Let $a > 0$ and consider the rectangle R given by $t_0 \leq t \leq t_0 + a$, $|y - y_0| \leq 1$. Note that since y_0 is a stationary solution, $g(y_0) = 0$. Therefore, by the Mean-Value Theorem there is c between y and y_0 for which

$$|g(y)| = |g(y) - g(y_0)| = |g'(c)(y - y_0)| \leq L,$$

where L is the maximum of g' over $|y - y_0| \leq 1$, which is guaranteed to exist, by the Extreme Value Theorem, since g' is continuous. This yields $M = L$, which means $\alpha = \min(a, 1/L)$. Letting $a = 1/L$ we obtain $\alpha = 1/L$. Therefore, $y = y_0$ is the unique solution over $[t_0, t_0 + 1/L]$, so every solution to $(*)$ must be constant over $[t_0, t_0 + 1/L]$. Now, we will consider the initial value problem $y' = g(y), \; y(t_0 + 1/L) = y_0$. Similar to above this initial value problem has the unique solution $y = y_0$ over $[t_0 + 1/L, t_0 + 2/L]$. Therefore, every solution to $(*)$ must be constant over $[t_0, t_0 + 2/L]$. Repeating this, we see that each solution to $(*)$ is constant over

$[t_0, t_0 + n/L]$. Thus the only solution to $(*)$ over $[t_0, \infty)$ must be $y = y_0$. A similar argument shows $y = y_0$ is the unique solution to $(*)$ over $\mathbb{R}$. $\square$

A more general version of the above example is left as an exercise. See Exercise 5.25.

5.4 Exercises

Exercise 5.1. *Find the largest interval that the Existence and Uniqueness Theorem for Linear Equations guarantees the existence of a unique solution to each IVP:*

(a) $ty'' + (t^2 - \sin t)y' + \sqrt[3]{t^2 - 1}\,y = \sin t,\ y(2) = 5.$

(b) $y''' + p(t)y'' - q(t)y' + \tan t = \csc t,\ y(\pi/4) = 2\pi/3,$ *where* $p(t), q(t)$ *are polynomials.*

(c) $y'' - \ln(2 - \sqrt{t-1})y = e^t,\ y(0) = 6.$

Exercise 5.2. *In this chapter we proved that* $\sin(t^2)$ *is not a solution to any second order homogeneous linear differential equation. (See Example 5.2.) Let* n *be a positive integer. Prove that the function* $\sin(t^n)$ *is not a solution to any homogeneous linear differential equation in standard form of order not exceeding* n*, for which its coefficients are all continuous over* $(-1, 1)$*.*

Hint: Taylor series for $\sin t$ might help.

Exercise 5.3. *Consider the following IVP:*

$$y' \cos t - y \sin t = \cos t,\ y(0) = 1.$$

(a) Using the Existence Uniqueness Theorem for Linear Equations prove that the largest interval for which a unique solution is guaranteed to exist is $(-\pi/2, \pi/2)$*.*

(b) Find the solution.

(c) Prove a solution exists over the larger interval $(-3\pi/2, \pi/2)$*. Hint: Some trigonometric identities would help change the format of the solution.*

Exercise 5.4. *Consider the IVP* $y' = e^y,\ y(0) = 0.$

(a) Prove that the Existence and Uniqueness Theorem guarantees the existence of a solution over $(-1/e, 1/e)$*.*

(b) By solving the equation show that a solution exists over the larger interval $(-\infty, 1)$*.*

Exercise 5.5. *Prove that each IVP has a unique solution over the given interval:*

(a) $y' = e^{-t^2} + y^4,\ y(0) = 0,$ *with* $0 \le t \le 0.5.$

(b) $y' = t + y^2,\ y(0) = 0,$ *with* $0 \le t \le (1/2)^{2/3}.$

(c) $y' = \sin t + \cos(ty),\ y(1) = 5,$ *with* $t \in \mathbb{R}.$

(d) $y' = \dfrac{\sin(t+y)}{1+t^2+y^2},\ y(0) = 0,$ *with* $t \in \mathbb{R}$.

(e) $y' = \sin(ty^2 + y) + \cos t,\ y(0) = -1,$ *with* $t \in \mathbb{R}$.

(f) $y' = \dfrac{e^{-y^2}}{t^2+4t+5},\ y(0) = 2,$ *with* $t \in \mathbb{R}$.

(g) $y' = t(y+1),\ y(0) = -1,$ *with* $t \in \mathbb{R}$. *Hint: This one is different from all the others!*

Exercise 5.6. *Consider the following initial value problems.*

1. $ty' = 2y - 2,\ y(0) = -1.$

2. $ty' = 2y - 2,\ y(0) = 1.$

Prove that the first IVP has no solutions, while the second one has multiple solutions. How do you reconcile these with the Existence and Uniqueness Theorem, i.e. Theorem 5.2?

Exercise 5.7. *Prove that the IVP has infinitely many solutions:* $y' = -2t\sqrt{1-y^2},\ y(0) = 1$. *How do you reconcile this with the Existence and Uniqueness Theorem?*

Exercise 5.8. *Consider the initial value problem* $y' = 3y + 2,\ y(0) = -0.5$.

(a) Solve the IVP.

(b) Find the Picard iterates $y_n(t)$ *of this equation.*

(c) Does $y_n(t)$ *approach the solution found in part (a)?*

Exercise 5.9. *Let* t_0, y_0 *be real numbers. Suppose for every real number* $r > t_0$ *the initial value problem* $y' = f(t,y),\ y(t_0) = y_0$ *has a unique solution over* $[t_0, r]$. *Prove that there is a unique solution to the initial value problem* $y' = f(t,y),\ y(t_0) = y_0$ *over* $[t_0, \infty)$.

Exercise 5.10. *Suppose* $f(t,y)$ *is continuous and bounded on* $\mathbb{R}^2$. *Assume also that* f_y *is continuous over* $\mathbb{R}^2$. *Prove that for every* $t_0, y_0 \in \mathbb{R}$ *the IVP* $y' = f(t,y),\ y(t_0) = y_0$ *has a unique solution defined over* $\mathbb{R}$.

Exercise 5.11. *For each initial value problem find* $y_n(t)$, *the* n*-th Picard iterate. Show that this sequence approaches the solution to the equation.*

(a) $y' = y + 1 - t,\ y(0) = 1.$

(b) $y' = ty + 2t - t^3,\ y(0) = 0.$

(c) $y' = 2y,\ y(0) = 0.$

Exercise 5.12. *Prove that the initial value problem* $\dfrac{dy}{dt} = \sin t + \sqrt[3]{1+y^2}, y(0) = 0$ *has a unique solution defined over* $\mathbb{R}$.

Exercise 5.13. *Prove* $y = \sin t$ *is the only solution to the following IVP*

$$y' = \frac{2\cos t}{y^2 + \cos^2 t + 1},\ y(0) = 0,\ t \in \mathbb{R}.$$

Exercise 5.14. *Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, and $p, q : \mathbb{R} \to \mathbb{R}$ are continuously differentiable functions. Prove that for every $t_0, y_0 \in \mathbb{R}$ the following IVP has a unique solution defined over $\mathbb{R}$.*

$$y' = f(t)p(\cos y) + g(t)q(\sin y),\ \ y(t_0) = y_0.$$

Hint: M does not depend on b.

Exercise 5.15 (Existence and Uniqueness Theorem for Separable Equations)**.** *Suppose $f(t)$ is continuous over an open interval $(t_0 - a, t_0 + a)$ and $g(y)$ is differentiable over an interval $(y_0 - b, y_0 + b)$. Prove that, the IVP*

$$y' = f(t)g(y),\ \ y(t_0) = y_0$$

has a unique solution defined over some open interval $(t_0 - \epsilon, t_0 + \epsilon)$.

Exercise 5.16. *Suppose $y = y_0$ is a stationary solution for the first order autonomous equation*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y).$$

Assume $f : \mathbb{R} \to \mathbb{R}$ is C^1. Prove that if y is a nonstationary solution to $y' = f(y)$ defined over an open interval I, then either $y(t) > y_0$ for all $t \in I$ or $y(t) < y_0$ for all $t \in I$.

Hint: Use Example 5.17.

Exercise 5.17 (Gronwall's Inequality)**.** *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and nonnegative. Assume A, B, C are three positive constants for which*

$$f(t) \leq A + B \int_0^t f(s)\ \mathrm{d}s \text{ for all } t \in [0, C].$$

Prove $f(t) \leq Ae^{tB}$ for every $t \in [0, C]$.

Exercise 5.18. *Consider the linear IVP of order n:*

$$L[y] = f(t), y(t_0) = y_0, \ldots, y^{(n-1)}(t_0) = y_{n-1}.$$

Suppose this IVP has two distinct solutions. Prove that it must have infinitely many solutions.

Hint: Assume y_1, y_2 are two distinct solutions. Use $y = cy_1 + (1 - c)y_2$.

Exercise 5.19. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a C^1 even function. Assume $y_0(t)$ is a solution to the IVP*

$$y' = f(y),\ \ y(0) = 0.$$

Prove that y_0 is an odd function defined over some open interval $(-\epsilon, \epsilon)$.

Hint: First, prove that the given IVP has a unique solution over some interval centered at the origin. Then, show that $-y_0(-t)$ is a solution to the same IVP.

Exercise 5.20. *Consider the initial value problem*

$$y' = |t| + e^{-y^2}, y(0) = 0.$$

(a) Prove that this equation has a unique solution on $\mathbb{R}$*.*

(b) Prove the solution to this IVP is an odd function.

Hint: Let y be the unique solution to this IVP. Prove that $-y(-t)$ is also another solution to this IVP.

Exercise 5.21. *Consider the initial value problem*

$$y' = y(1-y),\ y(0) = y_0.$$

(a) Show that if $y_0 \in [0,1]$*, then there is a unique solution to this IVP that is defined over* $\mathbb{R}$*.*

(b) Show that if $y_0 > 1$*, then there is a unique solution to this IVP that is defined over* (c, ∞) *for some constant* c*. Find the smallest such* c *in terms of* y_0*.*

(c) Show that if $y_0 < 0$*, then there is a unique solution to this IVP that is defined over* $(-\infty, c)$ *for some constant* c*. Find the largest such* c *in terms of* y_0*.*

Hint: Use Example 5.17.

Exercise 5.22. *Consider the initial value problem*

$$y' = (y-1)^2,\ y(0) = 0.$$

Show that both of the following functions are solutions to this IVP.

$$(1)\ y = \frac{t}{1+t} \text{ if } t \neq -1, \text{ and } (2)\ y = \begin{cases} t/(1+t) & \text{if } t > -1 \\ (1+t)/(2+t) & \text{if } t < -1 \end{cases}$$

How do you reconcile this with the Existence and Uniqueness Theorem for first order equations?

Exercise 5.23. *Show that for every real numbers* t_0, t_1, y_0, y_1 *the following has a unique solution defined over* $\mathbb{R}$*:*

$$y'' = \sin(t + y') + t^2,\ y(t_0) = y_0,\ y'(t_1) = y_1.$$

Exercise 5.24. *Consider the differential equation* $\dfrac{dy}{dt} = \sin^2 y - 2y$*.*

(a) Find all stationary solutions of this differential equation.

(b) Suppose y *is a nonstationary solution to this differential equation with* $y(0) = y_0$*, for some* $y_0 \in \mathbb{R}$*. Determine whether* y *is concave up or concave down. Your answer may depend on* y_0*.*

Exercise 5.25. *Suppose* $y = y_1$ *is a solution to the IVP*

$$y' = f(t,y),\ y(t_0) = y_0,\ t \in (a,b) \qquad (*)$$

Assume $f(t,y)$ *and* $f_y(t,y)$ *are continuous over an open subset containing the graph of* $y_1(t)$ *in the ty-plane. Prove that* y_1 *is the unique solution to* $(*)$*.*

Hint: Assume y_1 and z are two distinct solutions to this IVP, and assume $y(t_1) \neq z(t_1)$, where $t_0 < t_1$. From now on, restrict the domain of both y and z to $[t_0, t_1]$. Note that $(y-z)(t_0) = 0$, and since $y - z$ is continuous, $(y-z)^{-1}(0)$ is a closed nonempty subset of $[t_0, t_1]$. For simplicity let $K = (y-z)^{-1}(0)$. Show K is compact. Then, use the Extreme Value Theorem to show K has a maximum element c. Now, use the fact that $y(c) = z(c)$ and the Existence and Uniqueness Theorem to show $y = z$ on an open interval containing c. Use this to obtain a contradiction.

5.5 Challenge Problems

Exercise 5.26. *Prove that the following initial value problem has a unique solution defined over* $[0, \infty)$*:*

$$\frac{dy}{dt} = y + e^{-y} + e^{-t}, y(0) = 0.$$

Exercise 5.27. *Suppose y is a solution to the IVP below defined over* $[0, \alpha]$*:*

$$y' = yf(t, y), \ y(0) = 1,$$

where $f(t, y)$ is a bounded, and f, f_y are both continuous on $\mathbb{R}^2$. Prove that there is a constant C for which $|y(t)| \leq e^{Ct}$ for all $t \in [0, \alpha]$.

Exercise 5.28. *Find all solutions to the initial value problem*

$$y' = 3y^{2/3}, \ y(0) = 0.$$

Exercise 5.29. *Are there any solutions to the differential equation $y' = t^2 + y^2$ that is defined over $\mathbb{R}$?*

5.6 Summary

- In order to make sure a linear IVP has a unique solution:
  - Write down the linear equation in normal form.
  - Find all points of discontinuity for all coefficients and forcing and place them on a number line.
  - Find the largest interval that contains t_0 and does not contain any of the points of discontinuity.
- Picard iterates can be found using $y_{n+1} = y_0 + \displaystyle\int_{t_0}^{t} f(s, y_n(s))ds$.
- To find an interval in which the Existence and Uniqueness Theorem for $y' = f(t, y)$ is valid:
  - Start with two positive constants a, b and assume $t_0 \leq t \leq t_0 + a$ and $|y - y_0| \leq b$.
  - Find M, the maximum of $|f(t, y)|$ for t and y given above. If the exact value of M is difficult to find, find some upper bound for $|f(t, y)|$ and call that M.
  - Evaluate $\alpha = \min(a, b/M)$.
  - Find a and b for which α is the number given in the problem.

- If that seems difficult, in terms of a, find b that maximized b/M. Then find out when $a \leq b/M$ and when $a > b/M$.
- A unique solution exists over $[t_0, t_0 + \alpha]$.
- For intervals of the form $[t_0 - \alpha, t_0]$ or $[t_0 - \alpha, t_0 + \alpha]$ use the same process.

Chapter 6

Numerical Methods

6.1 Numerical Approximations; Euler's Method

Throughout this chapter, we assume a unique solution to the initial value problem $\frac{dy}{dt} = f(t, y),\ y(t_0) = y_0$ exists over $[t_0, t_0 + a]$, the function $f(t, y)$ and its partials f_t and f_y are all continuous over a rectangle R given by $t_0 \le t \le t_0 + a$, $|y - y_0| \le b$. The objective is to approximate $y(t_0 + a)$. We divide $[t_0, t_0 + a]$ into N subintervals of equal width. Each subinterval has width $h = a/N$. We let $t_n = t_0 + nh$, and note that the solution has slope $f(t_0, y_0)$ at point (t_0, y_0). This means $y(t_1)$ can be approximated by $y_1 = y_0 + hf(t_0, y_0)$. Repeating this, we get the following sequence of approximations:

$$
\begin{aligned}
y_0 &= y(t_0)\\
y_1 &= y_0 + hf(t_0, y_0)\\
y_2 &= y_1 + hf(t_1, y_1)\\
&\vdots\\
y_N &= y_{N-1} + hf(t_{N-1}, y_{N-1})
\end{aligned}
$$

Example 6.1. Approximate $y(0.2)$ using the Euler's method once with 1 and once with 2 steps:

$$y' = y^2 + t^2,\ y(0) = 1.$$

Compare these with the value $y(0.2) \approx 1.25302$

As usual, after any approximation we need to understand the error. We will do so by the so called Lagrange Remainder Theorem stated below:

Theorem 6.1 (Lagrange Remainder Theorem)**.** *Suppose $f : [a, a + h] \to \mathbb{R}$ is continuous and f has n derivatives over the open interval $(a, a + h)$. Then, there exists $c \in (a, a + h)$ for which*

$$f(a + h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(c).$$

Applying the above theorem to $y(t)$ with $n = 2$ we will estimate the error. Note that the second derivative of $y(t)$ is the first derivative of $f(t, y(t))$ which is $f_t + f_y\frac{dy}{dt} = f_t + f_y f$.

$$y(t_{n+1}) = y(t_n + h) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(c_n)$$

for some c_n. Using the fact that $y' = f$ and that $y'' = f_t + f_y f$ we obtain:

$$y(t_{n+1}) = y(t_n) + hf(t_n, y(t_n)) + \frac{h^2}{2}\left(f_t + f_y f\right)(c_n, y(c_n))$$

Subtracting the recursion $y_{n+1} = y_n + hf(t_n, y_n)$ we obtain the following:

$$y(t_{n+1}) - y_{n+1} = y(t_n) - y_n + h[f(t_n, y(t_n)) - f(t_n, y_n)] + \frac{h^2}{2}\left(f_t + f_y f\right)(c_n, y(c_n)).$$

Denote by E_n the error $|y(t_n) - y_n|$, by L the maximum of $|f_y|$, and by D the maximum of $|f_t + f_y f|$. By the Mean Value Theorem we can see that $f(t_n, y(t_n)) - f(t_n, y_n) = (y(t_n) - y_n)f_y(t_n, d_n)$ for some d_n. Therefore, $|f(t_n, y(t_n)) - f(t_n, y_n)| \le LE_n$.

Using the triangle inequality we obtain the following:

$$E_{n+1} \le E_n + hLE_n + \frac{h^2 D}{2} = \underbrace{(1 + hL)}_{A} E_n + \underbrace{\frac{h^2 D}{2}}_{B}.$$

Combining these we get:

$$E_N \le AE_{N-1} + B \le A^2 E_{N-2} + AB + B \le \cdots \le A^N E_0 + A^{N-1}B + \cdots AB + B = \frac{(A^N - 1)B}{A - 1}.$$

Substituting back, we obtain $E_N \le \dfrac{((1 + hL)^N - 1)Dh}{2L}$. Note that $1 + x \le e^x$ for every positive real number x and thus, $E_N \le \dfrac{(e^{hLN} - 1)Dh}{2L}$. Since $hN = a$ we obtain the following error bound:

$$\boxed{E_N \le \frac{(e^{aL} - 1)Dh}{2L}.}$$

We say this error is of order h, and denote it by $O(h)$.

Example 6.2. Suppose the error in approximating the value of a solution to a first-order IVP using Euler's method is apprximated to be no more than 0.1. What changes should we make in order to guarantee the error does not exceed 0.01?

6.2 Other Numerical Methods

Similar to above, we are trying to approximate $y(t_0 + a)$, where y is the solution to the IVP $y' = f(t, y),\ y(t_0) = y_0$. For that we divide $[t_0, t_0 + a]$ into N subintervals of equal width $h = a/N$. So, $t_n = t_0 + nh$. We know $y(t) = y_0 + \displaystyle\int_{t_0}^{t} f(s, y(s))\ ds$. The integral on the right can be approximated using any of the methods of approximating integrals. As a result we obtain other numerical methods for approximation $y(t_0 + a)$.

Using the left-endpoint Riemann sum we have $y(t + h) = y(t) + \displaystyle\int_{t}^{t+h} f(s, y(s))\ ds \approx y(t) + hf(t, y(t))$. So we can approximate $y(t_n)$ by a sequence y_n defined by $y_{n+1} = y_n + hf(t_n, y_n)$. This is clearly the same as the Euler's method.

Similar to above we use the fact that $y(t_n+h) = y(t_n) + \displaystyle\int_{t_n}^{t_n+h} f(s,y(s))\ ds$. Then we approximate this integral using different methods of approximating integrals. We will get the following:

Runge-Midpoint: $\displaystyle\int_t^{t+h} f(s,y(s))\ ds \approx hf(t+\tfrac{h}{2}, y(t+\tfrac{h}{2}))$. We then apply the Euler's approximation to estimate $y(t+\frac{h}{2}) \approx y(t)+\frac{h}{2}f(t,y(t)))$. Thus we obtain that $\displaystyle\int_t^{t+h} f(s,y(s))\ ds \approx hf(t+\tfrac{h}{2}, y(t)+\tfrac{h}{2}f(t,y(t)))$. This yields the recurrence: $y_{n+1} = y_n + hf(t_n+\frac{h}{2}, y_n+\frac{h}{2}f(t_n,y_n))$. This can be memorized more easily if we write it as follows:

$$\begin{aligned} f_n &= f(t_n,y_n), & t_{n+\frac{1}{2}} &= t_n+\tfrac{h}{2} \\ y_{n+\frac{1}{2}} &= y_n+\tfrac{h}{2}f_n, & f_{n+\frac{1}{2}} &= f(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) \\ y_{n+1} &= y_n+hf_{n+\frac{1}{2}} \end{aligned}$$

Essentially the idea is to use the "slope at the midpoint" instead of the left endpoint slope.

Runge-Trapezoidal: $\displaystyle\int_t^{t+h} f(s,y(s))\ ds \approx \tfrac{h}{2}[f(t,y(t))+f(t+h,y(t+h))]$. Similar to above, using the Euler's approximation we can substitute $y(t+h)$ by $y(t)+hf(t,y(t))$. Thus, we obtain the following recursion:

$$\begin{aligned} f_n &= f(t_n,y_n), & \widetilde{y}_{n+1} &= y_n+hf_n \\ \widetilde{f}_{n+1} &= f(t_{n+1},\widetilde{y}_{n+1}), & y_{n+1} &= y_n+\tfrac{h}{2}(f_n+\widetilde{f}_{n+1}) \end{aligned}$$

Runge-Kutta: In this method we approximate the integral $\displaystyle\int_t^{t+h} f(s,y(s))\ ds$ using Simpson's rule. The computations for this method are too complicated to do manually.

Example 6.3. Approximate $y(0.2)$ using Midpoint, and Trapezoidal methods, where y is the solution to $\dfrac{dy}{dt} = t+y^2,\ y(0)=1$. Use 1 step for each.

It turns out that the above methods are all better than the Euler's method. The error for each of these methods is listed in the following table.

Method	Error
Euler	$O(h)$
Midpoint	$O(h^2)$
Trapezoidal	$O(h^2)$
Runge-Kutta	$O(h^4)$

Example 6.4. In estimating the value of a solution to a first order differential equation, the error is estimated to be less than 0.1. Given that we have used 10 steps, how many steps do we need in order to guarantee the error does not exceed 10^{-5} if each of the following methods is used?

(a) Euler's.

(b) Runge-Midpoint.

(c) Runge-Trapezoidal.

(d) Runge-Kutta.

6.3 More Examples

Example 6.5. Given the initial value problem

$$y' = t + y, \; y(0) = 2,$$

approximate $y(1)$ using the Runge-Trapezoidal method with 1 step.

Solution. The given information yields:

$$h = \frac{1-0}{1} = 1, t_0 = 0, t_1 = 1, y_0 = 2$$

$$\begin{aligned} f_0 &= f(0,2) = 0 + 2 = 2, & \tilde{y}_1 &= y_0 + hf_0 = 2 + 1 \cdot 2 = 4, \\ \tilde{f}_1 &= f(t_1, \tilde{y}_1) = 1 + 4 = 5, & y_1 &= y_0 + \frac{f_0 + \tilde{f}_1}{2} h = 2 + \frac{2+5}{2} = 5.5 \end{aligned}$$

□

6.4 Exercises

Exercise 6.1. *Consider the IVP*

$$y' = y + t, \; y(0) = -1.$$

(a) Approximate $y(1)$ using Euler's method once with step size $h = 1$ and once with $h = 0.1$.

(b) Solve the IVP.

(c) What is the error in each case? Explain your observation.

Exercise 6.2. *For each IVP approximate $y(0.5)$ in six different ways:*

1. *Euler's method with 1 step,*

2. *Euler's method with 10 steps,*

3. *Runge-Midpoint with 1 step,*

4. *Runge-Midpoint with 10 steps,*

5. *Runge-Trapezoidal with 1 step, and*

6. *Runge-Trapezoidal with 10 steps.*

Solve the equation using MATLAB or some other software, and compare the errors.

(a) $y' = y^2 + t^2,\ y(0) = 0.$

(b) $y' = \sin t + ty^2,\ y(0) = 1.$

Exercise 6.3. *Consider the IVP*

$$y' = t^2 + \tan^2 y,\ y(0) = 0.$$

(a) *Prove that there is a unique solution to this IVP with* $|t| \leq 0.5$ *and the solution satisfies* $|y| \leq \pi/4$.

(b) *We would like to approximate* $y(0.5)$ *using Euler's method. Find a reasonable number of steps that we need use in order to guarantee the value of* $y(0.5)$ *is accurate to* 2 *decimal places.*

Exercise 6.4. *The value of the solution at a specific point to a first-order differential equation has been approximated using the Runge-Kutta method with 10 steps. We would like to decrease the error by a factor of* $\frac{1}{16}$ *(i.e. multiply the error by* $\frac{1}{16}$*). What change should we make in the number of steps? What if we use the Midpoint or Trapezoidal methods?*

The following exercise shows that when using Euler's method, the points (t_n, y_n) will remain inside the original rectangle.

Exercise 6.5. *Suppose* $h, a,$ *and* b *are positive real numbers,* t_0, y_0 *are real numbers, and* $f(t, y)$ *is a function. Let* t_n, y_n *satisfy the recursions*

$$t_{n+1} = t_n + h, \text{ and } y_{n+1} = y_n + hf(t_n, y_n).$$

Let R *be the rectangle given by* $t_0 \leq t \leq t_0 + a,\ y_0 - b \leq y \leq y_0 + b,$ *and assume that* $|f(t, y)| \leq M$ *for all* (t, y) *in* R. *Finally, let* $\alpha = \min(a, b/M)$.

(a) *Prove that* $|y_j - y_0| \leq jhM,$ *as long as* $jh \leq \alpha$. *Hint: Use induction.*

(b) *Conclude from (a) that the points* (t_j, y_j) *all lie in* R *as long as* $j \leq \alpha/h$.

Exercise 6.6. *For each IVP approximate* $y(1)$ *in three different ways: Using Euler's Method, Runge Trapezoidal, and Runge Midpoint Methods, once with 1 step, and once with* 5 *steps. Evaluate the error in each case, by finding the solution, and determine if the change in error matches the expected error estimates.*

(a) $y' = ty,\ y(0) = 1.$

(b) $y' = y^2 + 1,\ y(0) = -1.$

(c) $y' + y = 0,\ y(0) = 2.$

(d) $y' = y \cos t,\ y(0) = 1.$

Exercise 6.7. *Suppose $y(t)$ is the solution to the IVP*

$$\frac{dy}{dt} = f(t, y), \quad y(0) = 0.$$

Suppose f, f_t, and f_y are continuous and $|f|, |f_t|, |f_y| \leq 1$, in rectangle $[0,1]\times[-1,1]$. When the Euler method with 10 steps is used to approximate $y(1)$, we obtain the following values

$$y_5 = -0.15[(1.1)^5 - 1], \text{ and } y_6 = 0.12[(1.1)^6 - 1].$$

Prove that $y(t) = 0$ for some $t \in (0.5, 0.6)$.

6.5 Summary

- Euler's method approximates the value of a solution to a first-order equation using the recursion: $y_{n+1} = y_n + hf(t_n, y_n)$.
- Runge-Midpoint formulas are

$$\begin{aligned} f_n &= f(t_n, y_n), & t_{n+\frac{1}{2}} &= t_n + \tfrac{h}{2} \\ y_{n+\frac{1}{2}} &= y_n + \tfrac{h}{2} f_n, & f_{n+\frac{1}{2}} &= f(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) \\ y_{n+1} &= y_n + hf_{n+\frac{1}{2}} \end{aligned}$$

- Runge-Trapezoidal formulas are

$$\begin{aligned} f_n &= f(t_n, y_n), & \widetilde{y}_{n+1} &= y_n + hf_n \\ \widetilde{f}_{n+1} &= f(t_{n+1}, \widetilde{y}_{n+1}), & y_{n+1} &= y_n + \tfrac{h}{2}(f_n + \widetilde{f}_{n+1}) \end{aligned}$$

- Errors for different numerical methods are listed below:

Method	Error
Euler	$O(h)$
Midpoint	$O(h^2)$
Trapezoidal	$O(h^2)$
Runge-Kutta	$O(h^4)$

Chapter 7

Higher Order Linear Equations

So far, most of our discussion has been around equations of first order. We will now focus on equations of order 2 or more, i.e. higher order equations.

7.1 General Strategy

A linear differential equation is one of the form

$$\frac{d^n y}{dt^n} + a_n(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_2(t)\frac{dy}{dt} + a_1(t)y = f(t).$$

For simplicity we write this equation as

$$D^n(y) + a_n(t)D^{n-1}(y) + \cdots + a_2(t)D(y) + a_1(t)y = f(t),$$

where $D = \dfrac{d}{dt}$ is the differentiation operator. Note that D is linear. In other words, for every two scalars c_1, c_2 and every two differentiable functions y_1, y_2 we have $D(c_1y_1 + c_2y_2) = c_1D(y_1) + c_2D(y_2)$. Similarly D^n is also linear for every n. We often write the above equation in a more compact form $L[y] = f(t)$, where L indicates the differential operator $D^n + a_n(t)D^{n-1} + \cdots + a_2(t)D + a_1(t)$. Note that since D is linear, L is also linear. In other words, $L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2]$.

Recall that by the Existence and Uniqueness Theorem for Linear Equations, any initial value problem of the form

$$L[y] = f(t), y(t_0) = y_0, \ldots, y^{(n-1)}(t_0) = y_{n-1}$$

has a unique solution provided all coefficients, and the forcing are continuous over an interval (a, b) containing t_0.

Definition 7.1. The **general solution** to a differential equation is a solution depending on some constants, where changing the constants gives us all solutions of the differential equation. A **particular** solution to a differential equation is some solution to that equation.

Example 7.1. Show that $y = c_1e^t + c_2e^{-t}$ with $c_1, c_2 \in \mathbb{R}$ is the general solution to the equation $y'' - y = 0$.

Theorem 7.1. *Let $Y_H(t)$ be the general solution to a homogeneous linear equation $L[y] = 0$, and let $f(t)$ be a function. Suppose $Y_P(t)$ is a particular solution to $L[y] = f(t)$, then the general solution to the differential equation $L[y] = f(t)$ is given by $y(t) = Y_H(t) + Y_P(t)$.*

The above theorem shows in order to solve a linear equation $L[y] = f(t)$ we need to

- Find the general solution $Y_H(t)$ to the associated homogeneous equation $L[y] = 0$, and
- Find a particular solution $Y_P(t)$ to the equation $L[y] = f(t)$.
- The result must be added up in order to find the general solution $y(t) = Y_H(t) + Y_P(t)$ to $L[y] = f(t)$.

7.2 Homogeneous Linear Equations

Example 7.2. Find the general solution of the equation $y'' - 5y' + 4y = 0$.

It appears that since the order of the above equation is 2, every solution can be written in terms of two solutions. This can be seen in the following theorem.

Theorem 7.2. *Suppose $L[y] = 0$ is an n-th order linear equation whose coefficients are continuous over an interval (a, b). Then, the set of all solutions to this equation is an n-dimensional vector space.*

Definition 7.2. Let $L[y] = 0$ be an n-th order linear equation whose coefficients are continuous over an open interval I. Solutions $Y_1, \ldots, Y_n$ are said to form a **Fundamental Set of Solutions (FSoS)** if $Y_1, \ldots, Y_n$ is a basis for the solution set of $L[y] = 0$ over I.

Definition 7.3. The set $\{N_0(t), N_1(t), \ldots, N_{n-1}(t)\}$ consisting of solutions to $L[y] = 0$ for which each $N_k(t)$ satisfies the initial conditions:

$$y(t_0) = y'(t_0) = \cdots = y^{(k-1)}(t_0) = y^{(k+1)}(t_0) = \cdots = y^{(n-1)}(t_0) = 0, \text{ and } y^{(k)}(t_0) = 1,$$

is called the **Natural Fundamental Set of Solutions (or NFSoS) of $L[y] = 0$ at t_0.**

It is now natural to ask: How do we know solutions $Y_1, \ldots, Y_n$ form a basis for the set of solutions of $L[y] = 0$?

This means given $t_0 \in (a, b)$ the following system must have a solution for $c_1, \ldots, c_n$, for all $y_0, y_1, \ldots, y_{n-1}$.

$$\begin{cases} c_1Y_1(t_0) + \cdots + c_nY_n(t_0) = y_0 \\ c_1Y_1'(t_0) + \cdots + c_nY_n'(t_0) = y_1 \\ \vdots \\ c_1Y_1^{(n-1)}(t_0) + \cdots + c_nY_n^{(n-1)}(t_0) = y_{n-1} \end{cases}$$

For this to have a solution for every $y_0, y_1, \dots, y_{n-1}$ we need

$$\det \begin{pmatrix} Y_1(t_0) & Y_2(t_0) & \cdots & Y_{n-1}(t_0) & Y_n(t_0) \\ Y_1'(t_0) & Y_2'(t_0) & \cdots & Y_{n-1}'(t_0) & Y_n'(t_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_1^{(n-2)}(t_0) & Y_2^{(n-2)}(t_0) & \cdots & Y_{n-1}^{(n-2)}(t_0) & Y_n^{(n-2)}(t_0) \\ Y_1^{(n-1)}(t_0) & Y_2^{(n-1)}(t_0) & \cdots & Y_{n-1}^{(n-1)}(t_0) & Y_n^{(n-1)}(t_0) \end{pmatrix} \neq 0$$

The above determinant is denoted by $W[Y_1, \dots, Y_n](t_0)$ and is called the **Wronskian** of $Y_1, \dots, Y_n$ at t_0. Similarly $W[Y_1, \dots, Y_n](t)$ is defined. So, Wronskian is a function of t defined over the entire open interval I, where all coefficients of $L[y] = 0$ are continuous. Similarly we can define the Wronskian of any n functions over an open interval for which they have $n-1$ derivatives.

The following theorem shows that it cannot be the case that $W[Y_1, \dots, Y_n](t)$ is zero for some values of t, but not for all values of t.

Theorem 7.3 (Abel's Theorem)**.** *Suppose W is the Wronskian of n solutions to an n-th order linear homogeneous equation $L[y] = 0$ whose coefficients are continuous over an open interval I. Then $W'(t)+a_n(t)W(t) = 0$, where $a_n(t)$ is the coefficient of $y^{(n-1)}$ in L. Furthermore, if $W(t)$ is zero at one point inside I, then it is zero everywhere on I.*

What we proved above can be stated in the following theorem:

Theorem 7.4. *If $Y_1, \dots, Y_n$ is a FSoS to an n-th order linear equation $L[y] = 0$, then its general solution is given by $y = c_1Y_1 + \cdots + c_nY_n$.*

Example 7.3. Suppose the Wronskian W of 3 solutions to the equation $y''' + 2ty'' - y = 0$ satisfies $W(0) = 1$. Find $W(t)$.

Theorem 7.5. *Assuming $N_0, \dots, N_{n-1}$ form a NFSoS at t_0 for $L[y] = 0$, the solution to the initial value problem*

$$L[y] = 0, y(t_0) = y_0, \dots, y^{(n-1)}(t_0) = y_{n-1}$$

is given by $y = y_0N_0 + \cdots y_{n-1}N_{n-1}$.

Example 7.4. Given that e^t, e^{2t} are solutions to $y'' - 3y' + 2y = 0$, find the NFSoS to this equation at 0.

Example 7.5. We know $t^2 - 1$ and t are solutions to the equation $(1 + t^2)y'' - 2ty' + 2y = 0$. Find the general solution to this equation. Use that to find a solution that satisfies $y(0) = 2, y'(0) = 3$.

Example 7.6. Prove that if the Wronskian of one FSoS to $L[y] = 0$ is constant, then the Wronskian of every FSoS is constant.

The next theorem shows that for solutions to linear homogeneous equations, linear independence and Wronskian being nonzero are equivalent.

Theorem 7.6. *Suppose $Y_1, \ldots, Y_n$ are solutions to an n-th order linear homogeneous equation. Then, $W[Y_1, \ldots, Y_n] \neq 0$ if and only if $Y_1, \ldots, Y_n$ are linearly independent.*

For functions that are not solutions of linear differential equations one direction of the above theorem is valid.

Theorem 7.7. *Suppose $f_1, \ldots, f_n : (a, b) \to \mathbb{R}$ are $(n-1)$ times differentiable and linearly dependent over an open interval I. Then, $W[f_1, \ldots, f_n](t) = 0$ for all $t \in I$.*

7.3 More Examples

Example 7.7. Suppose the Wronskian of two solutions Y_1, Y_2 of a second order equation is zero everywhere. Prove that one of the two solutions must be scalar multiple of the other.

Solution. Since $W[Y_1, Y_2] = 0$, by Theorem 7.6, the functions Y_1 and Y_2 are linearly dependent. Therefore, one must be a multiple of the other. □

Example 7.8. Suppose Y_1, Y_2, Y_3 are three solutions to a third order linear equation

$$y''' + a_3(t)y'' + a_2(t)y' + a_1(t)y = 0.$$

Assume c is a constant in such a way that all coefficients $a_1(t), a_2(t), a_3(t)$ are continuous over (c, ∞). Assume further that $W[Y_1, Y_2, Y_3](t) = t$ for all $t > c$. Prove that $c \geq 0$.

Solution. We will prove this by contradiction. Assume $c < 0$. This implies $0 \in (c, \infty)$. By assumption $W[Y_1, Y_2, Y_3](0) = 0$. By Theorem 7.3 the Wronskian must be identically zero over (c, ∞), which is a contradiction. □

Example 7.9. Prove that $W[\sin^2 t, 7, \cos(2t)] = 0$ without evaluating the Wronskian.

Solution. Note that $\cos(2t) = 1 - 2\sin^2 t$, which implies $\cos(2t) - \frac{1}{7}7 + 2\sin^2 t = 0$, and thus $\cos(2t), 7, \sin^2 t$ are linearly dependent. Therefore, $W[\sin^2 t, 7, \cos(2t)] = 0$, as desired. □

Example 7.10. Suppose Y_1, Y_2 are two solutions to a linear n-th order homogeneous differential equation $L[y] = 0$, where all coefficients are continuous over an open interval (a, b). Suppose

$$Y_1(c) = Y_1'(c) = \cdots = Y_1^{(n-2)}(c) = 0, \text{ and } Y_2(c) = Y_2'(c) = \cdots = Y_2^{(n-2)}(c) = 0,$$

for some $c \in (a, b)$. Prove that Y_1 and Y_2 are linearly dependent over (a, b).

Sketch. We need to show $c_1 Y_1 + c_2 Y_2 = 0$. In order to do that we will find an IVP that both $y = c_1 Y_1 + c_2 Y_2$ and 0 satisfy. By linearity, y satisfies $L[y] = 0$. By assumption $y(c) = \cdots = y^{(n-2)}(c) = 0$, so we need to select c_1, c_2 so that $c_1 Y_1^{(n-1)}(c) + c_2 Y_2^{(n-1)}(c) = 0$. This is possible, because $Y_1^{(n-1)}(c)$, and $Y_2^{(n-1)}(c)$ are two real numbers.

Solution. Since $Y_1^{(n-1)}(c)$, and $Y_2^{(n-1)}(c)$ are two real numbers, they are linearly dependent. Thus,

$$c_1 Y_1^{(n-1)}(c) + c_2 Y_2^{(n-1)}(c) = 0 \qquad (*)$$

for some constants c_1, c_2, not both zero. Now, consider the function $y = c_1 Y_1 + c_2 Y_2$. By assumption

$$y(c) = y'(c) = \cdots = y^{(n-2)}(c) = 0.$$

By $(*)$ we know $y^{(n-1)}(c) = 0$, and thus by the Existence and Uniqueness Theorem for Linear Equations, $y = 0$, which implies $c_1 Y_1 + c_2 Y_2 = 0$, as desired. □

Example 7.11. Consider the equation

$$L[y] = (t^2 - 2t)y'' + 2(1-t)y' + 2y = 0.$$

(a) Prove that $t^2, t-1$ form a FSoS for this equation.

(b) Find a NFSoS at $t = 1$.

(c) Solve the initial value problem $L[y] = 0,\ y(1) = 2,\ y'(1) = -1$.

(d) Show that $y = 0$ and $y = t^2$ are both solutions to the IVP $L[y] = 0,\ y(0) = y'(0) = 0$. How do you reconcile this with the Existence and Uniqueness Theorem for Linear Equations.

(e) Find all constants a for which
$$L[y] = 0, y(-1) = \alpha, y(a) = \beta$$
has a solution for all $\alpha, \beta \in \mathbb{R}$.

Solution. (a) First, note that both t^2 and $t-1$ are solutions to this equation. Next,

$$W[t^2, t-1] = \det \begin{pmatrix} t^2 & t-1 \\ 2t & 1 \end{pmatrix} = t^2 - 2t^2 + 2t = -t^2 + 2t,$$

which is not identically zero. Therefore, $t^2, t-1$ form a FSoS.

(b) We will have to solve the IVP

$$\begin{cases} y(1) = y_0 \\ y'(1) = y_1 \end{cases}$$

After substituting $y = c_1 t^2 + c_2(t-1)$ we obtain the following system:

$$\begin{cases} c_1 = y_0 \\ 2c_1 + c_2 = y_1 \end{cases}.$$

Therefore, $c_2 = y_1 - 2y_0$. Therefore, the general solution is

$$y = y_0 t^2 + (y_1 - 2y_0)(t-1) = y_0(t^2 - 2t + 2) + y_1(t-1).$$

This yields $t^2-2t+2, t-1$ are a NFSoS.

(c) Using what we found above, the solution is $y = 2(t^2-2t+2)-(t-1) = 2t^2-5t+5$.

(d) The fact that both $y=0$ and $y=t^2$ are solutions can be manually verified. When the equation is written in normal form, the coefficient $2(1-t)/(t^2-2t)$ is not continuous at $t=0$. Therefore, the Existence Uniqueness Theorem does not apply for the initial value $t_0=0$.

(e) We know $y = c_1t^2+c_2(t-1)$. We would like the following system to have a solution for all $\alpha, \beta \in \mathbb{R}$:

$$\begin{cases} c_1 - 2c_2 = \alpha \\ c_1a^2 + c_2(a-1) = \beta \end{cases}$$

This is equivalent to

$$\begin{pmatrix} 1 & -2 \\ a^2 & a-1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

For this to have a solution for every α, β, we need the rank of the above 2×2 matrix to be 2 and thus its determinant must be nonzero. Therefore, $a-1+2a^2 \neq 0$, or $a \neq -1, 1/2$. □

Example 7.12. Create a second order linear homogeneous differential equation, where t, t^2 form a FSoS.

Solution. The functions t, t^2 are linearly independent over any interval. So, if they are solutions of a second order linear homogeneous equation, they must form a FSoS. The equation $y'' + p(t)y' + q(t)y = 0$ is such a differential equation if and only if $y=t$ and $y=t^2$ satisfy this equation. This yields the following system:

$$\begin{cases} p(t) + q(t)t = 0 \\ 2 + p(t)2t + q(t)t^2 = 0 \end{cases}$$

This can be written as

$$\begin{pmatrix} 1 & t \\ 2t & t^2 \end{pmatrix} \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

This yields

$$\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} 1 & t \\ 2t & t^2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \frac{-1}{t^2} \begin{pmatrix} t^2 & -t \\ -2t & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

Therefore, $p(t) = -2/t$ and $q(t) = 2/t^2$ yields the only such differential equation. □

Example 7.13. Suppose W is the Wronskian of a FSoS to a n-th order linear homogeneous equation $L[y]=0$ whose coefficients are continuous over $\mathbb{R}$, and whose coefficient of $y^{(n-1)}$ is $a_n(t)$. Suppose $a_n(t) > 0$ over $\mathbb{R}$. Prove that $W(t)$ is a strictly decreasing function if and only if $W(0) > 0$.

Solution. By Abel's Theorem $W' + a_n(t)W = 0$. Suppose $W(0) > 0$. Since W is not zero, and $W(0)$ is positive, W is always positive. On the other hand, we know $a_n(t)W$ is always positive, which means $W' = -a_n(t)W$ is always negative. Therefore, W is strictly decreasing. The converse can be similarly done by assuming $W(0) < 0$. $\square$

Example 7.14. Prove that the equation $y'' + (t^4 + 3t^2 + 1)y = 0$ has a nontrivial even solution and a nontrivial odd solution.

Sketch. In order to find an even solution, we need a solution y satisfying $y(-t) = y(t)$. We will find an IVP for which both $y(t)$ and $y(-t)$ satisfy. $y(-0) = y(0)$ is immediate. $-y'(-0) = y'(0)$ yields $y'(0) = 0$. Thus, we will take y to be a solution satisfying $y(0) = 1$ and $y'(0) = 0$.

In order to find an odd solution, we need a solution y satisfying $-y(-t) = y(t)$. We will find an IVP for which both $y(t)$ and $-y(-t)$ satisfy. $-y(-0) = y(0)$ yields $y(0) = 0$. By taking the derivative we have $y'(-0) = y'(0)$, which is trivial. Thus, we will take y to be a solution satisfying $y(0) = 0$ and $y'(0) = 1$.

Solution. Let N_0, N_1 be the NFSoS at $t_0 = 0$ to this equation, and let $y(t) = N_0(-t)$, and $z(t) = -N_1(-t)$. We have $y(0) = N_0(0) = 1$, and $y'(0) = -N_0'(0) = 0$. Furthermore, $z(0) = -N_1(0) = 0$ and $z'(0) = N_1'(0) = 1$. We have the following:

$$y''(t) + (t^4 + 3t^2 + 1)y(t) = N_0''(-t) + (t^4 + 3t^2 + 1)N_0(-t) = N_0''(-t) + ((-t)^4 + 3(-t)^2 + 1)N_0(-t) = 0.$$

The last equality is obtained by substituting t by $-t$ and using the fact that N_0 is a solution to the given differential equation. Therefore, by uniqueness we have $y(t) = N_0(t)$, i.e. $N_0(-t) = N_0(t)$, and hence N_0 is even. Similarly we can prove N_1 is odd. $\square$

7.4 Exercises

Exercise 7.1. *Given $L = D^3 + t^2D^2 - \sin(t)$. Evaluate $L[y]$ for each of the following functions.*

(a) $y = \sin t$.

(b) $y = e^t$.

(c) $y = t^2 - t$.

Exercise 7.2. *Compute the Wronskian of each set of functions.*

(a) $\sin(at), \cos(bt)$, *where a, b are constants.*

(b) e^{at}, e^{bt}, *where a, b are constants.*

(c) $\sin(2t), \sin t, \cos t, \sin t \cos t$.

Exercise 7.3. *For each equation, some information about the Wronskian $W(t)$ of a FSoS is given. Find $W(t)$.*

(a) $y''' - 2t\ y'' + 7\sin(t)\ y = 0$, *with* $W(0) = -2$.

(b) $ty^{(4)} - t^2y'' + \cos(t)y' = 0$, *with* $W(1) = 5$.

(c) $\sin(t)y''' - \cos(t)y'' + ty = 0$, *with* $W(\pi/2) = 1$.

Exercise 7.4. *Given that* $1, \sin t, \cos t$ *are solutions to* $y''' + y' = 0$*:*

(a) Find a NFSoS at $t_0 = 0$.

(b) Solve the initial value problem $y''' + y' = 0, y(0) = 1, y'(0) = 2, y''(0) = -1$.

Exercise 7.5. *Consider the differential equation*

$$y'' - 4y' + 4y = 0.$$

(a) Find a solution of the form e^{ct}.

(b) Find a solution of the form te^{ct}.

(c) Find a NFSoS at $t_0 = 1$.

(d) Solve the IVP $y'' - 2y' + y = 0, y(0) = 2, y'(0) = 1$.

Exercise 7.6. *Given that* $\sin t$ *and* $\cos t$ *are solutions of* $y'' + y = 0$ *prove that there are no solutions to*

$$y'' + y = 0,\ y(0) = 0,\ y(\pi) = 1.$$

How do you reconcile this with the Existence and Uniqueness Theorem for Linear Equations?

Exercise 7.7. *Suppose the coefficients of the linear equation* $L[y] = 0$ *are continuous over* (a, b)*. Prove that for every* $t_0 \in (a, b)$ *and every nonzero real number* r*, there is a FSoS,* $Y_1, \ldots, Y_n$ *for which*

$$W[Y_1, \ldots, Y_n](t_0) = r.$$

Exercise 7.8. *Suppose* $Y_1, \ldots, Y_n$ *are* C^{n-1} *functions over an open interval* I*. Let* $A \in M_n(\mathbb{R})$ *be an invertible matrix, and assume* $Z_1, \ldots, Z_n$ *satisfy*

$$(Z_1 \cdots Z_n) = (Y_1 \cdots Y_n)A.$$

Prove that $W[Z_1, \ldots, Z_n] = (\det A)W[Y_1, \ldots, Y_n]$.

Exercise 7.9. *Suppose* $Y_1, \ldots, Y_n$ *are* C^n *over an open interval* I*, and that* $W[Y_1, \ldots, Y_n](t) \neq 0$ *for all* $t \in I$*. Prove that there is a linear homogeneous equation* $L[y] = 0$ *for which* $\{Y_1, \ldots, Y_n\}$ *is a FSoS over* I*.*

Exercise 7.10. *Let y_1 and y_2 form a fundamental set of solutions for a standard homogeneous linear equation*

$$y'' + p(t)y' + q(t)y = 0,$$

whose coefficients are continuous over $\mathbb{R}$. Prove that if there is some $t_0 \in \mathbb{R}$ for which $y_1''(t_0) = y_2''(t_0) = 0$, then $p(t_0) = q(t_0) = 0$.

Exercise 7.11. *Let*

$$y'' + p(t)y' + q(t)y = 0 \qquad (*)$$

be a second order linear equation, where $p(t)$ and $q(t)$ are continuous over (a, b). Suppose y_1, y_2 form a FSoS for $(*)$ over (a, b).

(a) Prove that y_1 and y_2 cannot have a common root inside (a, b).

(b) Prove that y_1 and y_2 cannot achieve a local maximum or a local minimum at the same point inside (a, b).

(c) Prove that between every two consecutive roots of y_1 there is precisely one root of y_2. Hint: Use contradiction and apply the Mean Value Theorem to y_1/y_2.

(d) Suppose y_1, y_2 have an inflection point at some $t_0 \in (a, b)$. Prove that $p(t_0) = q(t_0) = 0$. Hint: Write down what it means for y_1 and y_2 to be solutions and substitute $t = t_0$.

Exercise 7.12. *Let I be an open interval, $f_1, \ldots, f_{n+1} : I \to \mathbb{R}$ be $n - 1$ times differentiable, and a, b be two constants. Prove the following:*

(a) $W[f_1, \ldots, f_{n-1}, af_n + bf_{n+1}] = aW[f_1, \ldots, f_{n-1}, f_n] + bW[f_1, \ldots, f_{n-1}, f_{n+1}]$.

(b) $W[af_1, f_2, \ldots, f_n] = aW[f_1, \ldots, f_n]$.

Exercise 7.13. *Let $p(t)$ and $q(t)$ be continuous functions on an interval $(-c, c)$. Assume q is an even function and the equation $y'' + p(t)y' + q(t)y = 0$ has an even solution Y over $(-c, c)$. Suppose Y has no roots in $(-c, c)$. Prove that $p(t) = 0$ for all $t \in (-c, c)$.*

Hint: See part (b) of Example 5.15.

Exercise 7.14. *Consider the functions t^2 and $t|t|$ on $\mathbb{R}$.*

(a) Prove that $W[t^2, t|t|] = 0$ for every $t \in \mathbb{R}$.

(b) Prove t^2 and $t|t|$ are linearly independent.

Exercise 7.15. *Consider the differential equation $y''' + y = 0$. Suppose y_1, y_2, y_3 form the NFSoS of this equation at $t_0 = 0$. Without finding y_i's prove that $y_1' = -y_3$ and $y_1'' = -y_2$.*

Hint: Create IVP's and use uniqueness.

Exercise 7.16. *Suppose $Y_1, \ldots, Y_n$ are n times differentiable over an open interval I for which their Wronskian $W[Y_1, \ldots, Y_n]$ is never zero over I. Prove that there is a unique n-th order linear homogeneous differential equation $L[y] = 0$ for which $Y_1, \ldots, Y_n$ is a FSoS.*

Hint: See Example 7.12

Exercise 7.17. *Suppose $L[y] = f(t)$ is a linear differential equation all of whose coefficients and forcing are C^∞ over an open interval I. Prove that every solution to this equation is C^∞ over I.*

Exercise 7.18. *Prove that it is not possible for all solutions of a linear homogeneous differential equation defined over $(-1, 1)$ to be odd.*

Exercise 7.19. *Prove that if all solutions of a linear homogeneous differential equation defined over $(-1, 1)$ are even, then the equation is first order.*

7.5 Challenge Problems

Exercise 7.20. *Suppose $p(t), q(t)$ are continuous functions over $\mathbb{R}$, and $q(t) < 0$ for all $t \in \mathbb{R}$. Let y be a solution to the differential equation*

$$y'' + p(t)y' + q(t)y = 0$$

that has at least two distinct roots. Prove that y is the zero function.

Exercise 7.21. *Let $p(t)$ and $q(t)$ be continuous functions on an interval $(-c, c)$. Suppose $p(t)$ is odd and $q(t)$ is even. Prove that there is a FSoS Y_1, Y_2 for which Y_1 is odd and Y_2 is even.*

Exercise 7.22. *Suppose $Y_1(t), \dots, Y_n(t), v(t)$ are $n-1$ times differentiable functions over the interval (a, b). Prove that*

$$W[vY_1, vY_2, \dots, vY_n](t) = (v(t))^n W[Y_1, Y_2, \dots, Y_n](t), \text{ for all } t \in (a, b).$$

Exercise 7.23. *Prove that every n-th order linear differential equation with constant coefficients $L[y] = 0$ has a FSoS of the form*

$$Y, Y', \dots, Y^{(n-1)}.$$

By an example show this result does not hold if the coefficients are not constant.

Exercise 7.24. *Suppose $y_1, \dots, y_n : (a, b) \to \mathbb{R}$ are all $n-1$ times differentiable. Assume $W[y_1, \dots, y_n](t) = 0$ for every $t \in (a, b)$ and $W[y_1, \dots, y_{n-1}](t) \neq 0$ for every $t \in (a, b)$. Prove that $y_1, \dots, y_n$ are linearly dependent.*

7.6 Summary

- In order to find the general solution to a linear equation $L[y] = f(t)$:
  - Find the general solution Y_H to $L[y] = 0$.
  - Find a particular solution Y_P to $L[y] = f(t)$.
  - $y = Y_H + Y_P$ is the general solution to $L[y] = f(t)$.

- $Y_1, \dots, Y_n$ form a fundamental set of solutions for $L[y] = 0$ if $W[Y_1, \dots, Y_n] \neq 0$, in which case a general solution is given by $y = c_1 Y_1 + \cdots + c_n Y_n$.

- $N_0, \dots, N_{n-1}$ form a natural fundamental set of solutions at a point t_0 if the matrix of Wronskian evaluated at t_0 is the identity matrix.

- To find a NFSoS we need to solve the initial value problem $L[y] = 0, y(t_0) = y_0, \dots, y^{(n-1)}(t_0) = y_{n-1}$. The solution is $y = y_0 N_0 + \cdots + y_{n-1} N_{n-1}$.

- If $N_0, \dots, N_{n-1}$ form a NFSoS at point t_0, then $y = y_0 N_0 + \cdots + y_{n-1} N_{n-1}$ is the solution to the IVP

$$L[y] = 0, \quad y(t_0) = y_0, y'(t_0) = y_1, \dots, y^{(n-1)}(t_0) = y_{n-1}.$$

Chapter 8

Linear Equations with Constant Coefficients

In this chapter, we will consider linear equations of the form $L[y] = f(t)$, where

$$L = D^n + a_n D^{n-1} + \cdots + a_2 D + a_1$$

is a linear differential operator, with constant coefficients $a_1, \ldots, a_n$, and forcing $f(t)$ that is continuous over an open interval I.

We have seen in the previous chapter that solving these equations requires:

- Finding the general solution to the corresponding homogeneous equation, and
- Finding a particular solution to the nonhomogeneous equation.

We will explore these separately.

8.1 Homogeneous Linear Equations with Constant Coefficients

Example 8.1. Find the general solution to $y'' + 7y' + 12y = 0$.

The polynomial $p(z) = z^n + a_n z^{n-1} + \cdots + a_2 z + a_1$ is called the **characteristic polynomial** corresponding to L. We know $D[e^{zt}] = ze^{zt}$. Repeatedly applying D we obtain $D^m[e^{zt}] = z^m e^{zt}$. This implies $L[e^{zt}] = p(z)e^{zt}$, which means $L[e^{zt}] = 0$ if $p(z) = 0$. This is the main idea that we used in solving the previous problem.

Example 8.2. Find the general solution to $y''' + 2y'' - y' - 2y = 0$.

Theorem 8.1. *Suppose $z_1, \ldots, z_n$ are distinct complex numbers. Then $e^{z_1 t}, \ldots, e^{z_n t}$ are linearly independent over $\mathbb{C}$.*

Example 8.3. Solve the equation $y'' + 2y' + 2y = 0$.

When the characteristic polynomial has a nonreal root $a+ib$, since $L[e^{(a+ib)t}] = p(a+ib)e^{(a+ib)t} = 0$, we obtain $L[e^{at}\cos(bt) + ie^{at}\sin(bt)] = 0$. Using linearity we conclude $L[e^{at}\cos(bt)] + iL[e^{at}\sin(bt)] = 0$. Taking the real and imaginary parts we get

$$L[e^{at}\cos(bt)] = L[e^{at}\sin(bt)] = 0,$$

and thus $e^{at}\cos(bt), e^{at}\sin(bt)$ are solutions of the linear homogeneous equation $L[y] = 0$.

Example 8.4. Solve the equation $y''' + 2y'' + y' = 0$.

When the characteristic polynomial has repeated roots, we need a different strategy. We repeatedly differentiate $L[e^{zt}] = p(z)e^{zt}$ with respect to z. Note that $\dfrac{\partial^2}{\partial z\partial t} = \dfrac{\partial^2}{\partial t\partial z}$ and thus we can move the differentiation with respect to z inside L to obtain $L\left[\frac{\partial^k}{\partial z^k}(e^{zt})\right] = \frac{\partial^k}{\partial z^k}(p(z)e^{zt})$. This gives us the following equalities called the **Key Identities**:

$$\begin{aligned}
&L[e^{zt}] = p(z)e^{zt}\\
&L[te^{zt}] = p'(z)e^{zt} + p(z)te^{zt}\\
&L[t^2e^{zt}] = p''(z)e^{zt} + 2p'(z)te^{zt} + p(z)t^2e^{zt}\\
&L[t^3e^{zt}] = p'''(z)e^{zt} + 3p''(z)te^{zt} + 3p'(z)t^2e^{zt} + p(z)t^3e^{zt}\\
&\vdots
\end{aligned}.$$

The coefficients above are those in the Pascal's triangle. In other words, the general form of a key identity is as follows:

$$L[t^n e^{zt}] = \sum_{j=0}^{n} \binom{n}{j} p^{(n-j)}(z)\ t^j e^{zt}.$$

Theorem 8.2. *The multiplicity of a complex root c of a polynomial $p(z)$ is m if and only if*

$$p(c) = p'(c) = \cdots = p^{(m-1)}(c) = 0, \ \text{and } p^{(m)}(c) \neq 0.$$

Theorem 8.3. *Let L be a linear differential operator with constant coefficients, and $p(z)$ be its characteristic polynomial. Suppose c is a root of $p(z)$ with multiplicity m. Then*

$$L[e^{ct}] = L[te^{ct}] = \cdots = L[t^{m-1}e^{ct}] = 0.$$

Furthermore, if $c = a+ib$, with a, b real, then $L[t^j\ e^{at}\cos(bt)] = L[t^j\ e^{at}\sin(bt)] = 0$ for $j = 0, 1, \ldots, m-1$.

Recall that since $p(z)$ has real coefficients, its nonreal roots come in complex conjugate pairs. Therefore, for every pair of nonreal roots $a \pm ib$ ($a, b \in \mathbb{R}$) with multiplicity m we obtain $2m$ solutions $t^j\ e^{at}\cos(bt), t^j\ e^{at}\sin(bt)$. This means the above method yields n solutions for $L[y] = 0$, where n is the order of this differential equation. We now need to prove these solutions do in fact form a FSoS. This is the subject of the next theorem.

Theorem 8.4. *Let L be a linear differential operator with constant coefficients, and $p(z)$ be its characteristic polynomial. For each real root r of $p(z)$ with multiplicity m consider m functions listed below:*

$$e^{rt}, te^{rt} \ldots, t^{m-1}e^{rt}.$$

For each pair of nonreal root $a \pm ib$ of $p(z)$, each with multiplicity k, consider $2k$ functions listed below:

$$e^{at}\cos(bt), e^{at}\sin(bt), te^{at}\cos(bt), te^{at}\sin(bt), \ldots, t^{k-1}e^{at}\cos(bt), t^{k-1}e^{at}\sin(bt).$$

These functions form a FSoS for $L[y] = 0$.

Note that we have already shown that the above functions are solutions to $L[y] = 0$, and there are n of them, where n is the degree of $p(z)$ and thus the order of $L[y] = 0$. Therefore, to prove the above theorem we need to show they are linearly independent. This is done in Exercise 8.5.

Example 8.5. Solve each of the following equations:

(a) $y^{(4)} + 6y'' + 9y = 0$.

(b) $(D^2 + 1)^2(D - 1)^3 Dy = 0$.

8.2 Reduction of Order

When we have repeated roots for the characteristic polynomial, we can also employ the method of Reduction of Order to find another solution. For example, to solve $y'' - 2y' + y = 0$, we see $p(z) = z^2 - 2z + 1 = (z-1)^2$, which means $y = e^t$ is a solution. We know ce^t is also a solution for every constant c. This motivates us to assume a second solution of the form $y = ve^t$ for a function v. We then substitute this into the equation and solve for v.

$$y' = v'e^t + ve^t,\ y'' = v''e^t + 2v'e^t + ve^t \Rightarrow y'' - 2y' + y = e^t(v'' + 2v' + v - 2v' - 2v + v) = e^t v''$$

For this to be zero we need $v'' = 0$. Thus, $v = t$ is one such solution. Therefore, te^t is a second solution to the above equation, which is what we obtained by the Key Identities as well.

In general, suppose a nonzero solution y_1 to a homogeneous linear differential equation is known. We are often able to find another solution by writing $y = v(t)y_1$, substituting into $L[y] = 0$ and solving for v. This is especially useful when the equation is of second order. This strategy works for linear equations, regardless of whether or not the coefficients are constant.

Example 8.6. Given $y = t$ is a solution to $t^2y'' - t(t+2)y' + (t+2)y = 0$ with $t > 0$. Find the general solution to this equation.

8.3 Nonhomogeneous Linear Equations with Constant Coefficients

Note that by Theorem 7.1, in order to solve $L[y] = f(t)$ we need to find Y_H, the general homogeneous solution, and Y_P, a particular nonhomogeneous solution. In the previous section we discussed how to find Y_H. Now, we will discuss several methods for finding Y_P.

8.3.1 Key Identities

Example 8.7. Find a particular solution to the equation $y'' + 2y = e^{5t}$.

The idea is to write down the Key Identities, substitute appropriate z values and use them to obtain the desired forcing. Recall the Key Identities are the ones listed below:

$$\begin{aligned}
&L[e^{zt}] = p(z)e^{zt} \\
&L[te^{zt}] = p'(z)e^{zt} + p(z)te^{zt} \\
&L[t^2e^{zt}] = p''(z)e^{zt} + 2p'(z)te^{zt} + p(z)t^2e^{zt} \\
&L[t^3e^{zt}] = p'''(z)e^{zt} + 3p''(z)te^{zt} + 3p'(z)t^2e^{zt} + p(z)t^3e^{zt} \\
&\vdots
\end{aligned}.$$

Example 8.8. Find a particular solution for $y'' - 6y' + 9y = 4e^{3t}$.

Example 8.9. Find a particular solution for $y'' + 2y' + 10y = \cos(2t)$.

In general, this method works well if the forcing has the following form:

$$f(t) = (\text{polynomial}) \cdot e^{at}\cos(bt) + (\text{polynomial}) \cdot e^{at}\sin(bt).$$

In which case, we write down the Key Identities and substitute $z = a + ib$. Then we take appropriate linear combinations to obtain the forcing.

Example 8.10. Find a particular solution for $y'' + 2y' + 10y = 4te^{2t}$.

Example 8.11. Find a particular solution for $y'' + y = \sin t + t$.

8.3.2 Undetermined Coefficients

Theorem 8.5. *Suppose L is a linear differential operator with constant coefficients with $p(z)$ as its characteristic polynomial. Suppose the multiplicity of $a+ib$ (with $a, b \in \mathbb{R}$) as a root of $p(z)$ is m (So, if $p(a+ib) \neq 0$, then $m = 0$). Assume $f(t) = g(t)e^{at}\cos(bt) + h(t)e^{at}\sin(bt)$, where $g(t), h(t)$ are polynomials and d is the maximum degree of g and h. Then, the equation $L[y] = f(t)$ has a particular solution of the form*

$$Y_P = t^m(A_0 + A_1t + \cdots + A_dt^d)e^{at}\cos(bt) + t^m(B_0 + B_1t + \cdots + B_dt^d)e^{at}\sin(bt),$$

where A_j, B_j are constants.

Example 8.12. Using the method of undetermined coefficients, find a particular solution for each of the following:

(a) $y'' + 4y = t\cos(2t)$.

(b) $y'' - 6y' + 9y = 4e^{3t}$.

(c) $y'' + 2y' + 10y = 5e^{-t}\ \sin(3t)$.

(d) $y'' + 3y' - 4y = 2\sin t\ \cos(3t)$.

8.3.3 Variation of Parameters

This method is used to find particular solutions to nonhomogeneous equations. It can be applied regardless of whether or not the coefficients are constant.

We assume $Y_1, \ldots, Y_n$ form a FSoS for $L[y] = 0$. We are looking for a particular solution to $L[y] = f(t)$. We find functions $u_1(t), \ldots, u_n(t)$ for which $y = u_1(t)Y_1(t) + \cdots + u_n(t)Y_n(t)$ is a solution to $L[y] = f(t)$. This gives us only one equation for $u_1, \ldots, u_n$. We will choose other equations to make evaluation of derivatives of y simpler. We start from

$$y = u_1Y_1 + \cdots + u_nY_n$$

Differentiating we obtain

$$y' = (u_1Y_1' + \cdots + u_nY_n') + (u_1'Y_1 + \cdots + u_n'Y_n).$$

Setting $u_1'Y_1 + \cdots + u_n'Y_n = 0$ simplifies the above equality to

$$y' = u_1Y_1' + \cdots + u_nY_n'.$$

Differentiating this again we obtain

$$y'' = (u_1Y_1'' + \cdots + u_nY_n'') + (u_1'Y_1' + \cdots + u_n'Y_n').$$

Again we simplify this by setting $u_1'Y_1' + \cdots + u_n'Y_n' = 0$. Repeating this we get the following:

$$\begin{aligned} u_1'Y_1^{(k)} + \cdots + u_n'Y_n^{(k)} &= 0, \text{ for } k = 0, \ldots, n-2. \\ y^{(k)} &= u_1Y_1^{(k)} + \cdots + u_nY_n^{(k)}, \text{ for } k = 0, \ldots, n-1. \\ y^{(n)} &= (u_1Y_1^{(n)} + \cdots + u_nY_n^{(n)}) + (u_1'Y_1^{(n-1)} + \cdots + u_n'Y_n^{(n-1)}) \end{aligned}$$

Substituting these into $L[y] = f(t)$ we obtain the following:

$$u_1L[Y_1] + \cdots + u_nL[Y_n] + u_1'Y_1^{(n-1)} + \cdots + u_n'Y_n^{(n-1)} = f(t).$$

Since $Y_1, \ldots, Y_n$ are solutions to the homogeneous equation $L[y] = 0$, we obtain the last equation that we need: $u_1'Y_1^{(n-1)} + \cdots + u_n'Y_n^{(n-1)} = f(t)$. Therefore, we need to solve the followings system:

$$\begin{cases} u_1'Y_1 + u_2'Y_2 + \cdots + u_n'Y_n = 0 \\ u_1'Y_1' + u_2'Y_2' + \cdots + u_n'Y_n' = 0 \\ \vdots \\ u_1'Y_1^{(n-2)} + u_2'Y_2^{(n-2)} + \cdots + u_n'Y_n^{(n-2)} = 0 \\ u_1'Y_1^{(n-1)} + u_2'Y_2^{(n-1)} + \cdots + u_n'Y_n^{(n-1)} = f(t) \end{cases}$$

This equation must be solved in terms of $u_1', \ldots, u_n'$. Note that the determinant of the ceofficient matrix is $W[Y_1, \ldots, Y_n] \neq 0$ and thus a solution always exists.

Example 8.13. Find a particular solution for $y'' + y = \tan t$.

8.4 More Examples

Example 8.14. Find the general solution of each equation.

(a) $D^4[y] + 4D^2[y] + 4y = 0$.

(b) $y''' + y'' - 4y' - 4y = 0$.

(c) $y^{(7)} - 6y^{(5)} + 9y''' = 0$.

(d) $y''' + 2y'' + 2y' = 0$.

(e) $y'' + 3y' + 2y = \dfrac{1}{e^t + 1}$.

Solution. (a) $p(z) = z^4 + 4z^2 + 4 = (z^2 + 2)^2$. The roots are $z = \pm\sqrt{2}i$. Therefore, the general solution is

$$y = c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t) + c_3 t \cos(\sqrt{2}t) + c_4 t \sin(\sqrt{2}t).$$

(b) $p(z) = z^3 + z^2 - 4z - 4$. We can either factor this by grouping or guess a root. A root is $z = -1$ and thus $p(z)$ must have a factor $z + 1$. This yields $p(z) = (z + 1)(z^2 - 4) = (z + 1)(z - 2)(z + 2)$. The roots are $-1, 2, -2$. Therefore, the general solution is

$$y = c_1 e^{-t} + c_2 e^{2t} + c_3 e^{-2t}.$$

(c) $p(z) = z^7 - 6z^5 + 9z^3 = z^3(z^4 - 6z^2 + 9) = z^3(z^2 - 3)^2 = z^3(z - \sqrt{3})^2(z + \sqrt{3})^2$. The general solution is

$$y = c_1 + c_2 t + c_3 t^2 + c_4 e^{\sqrt{3}t} + c_5 t e^{\sqrt{3}t} + c_6 e^{-\sqrt{3}t} + c_7 t e^{-\sqrt{3}t}.$$

(d) $p(z) = z^3 + 2z^2 + 2z = z(z^2 + 2z + 2) = z((z + 1)^2 + 1)$. The roots are $z = 0, -1 \pm i$. The general solution is

$$y = c_1 + c_2 e^{-t} \cos t + c_3 e^{-t} \sin t.$$

(e) $p(z) = z^2 + 3z + 2 = (z + 1)(z + 2)$. The general homogeneous solution is

$$Y_H = c_1 e^{-t} + c_2 e^{-2t}.$$

To find a particular solution we will use Variation of Parameters. Note that because of the format of the forcing, Key Identities and Undetermined Coefficients would not work. Set $Y_p = u_1 e^{-t} + u_2 e^{-2t}$. We need to solve the following system:

$$\begin{cases} u_1' e^{-t} + u_2' e^{-2t} = 0 \\ -u_1' e^{-t} - 2u_2' e^{-2t} = \dfrac{1}{e^t + 1} \end{cases}$$

Adding the two equations we obtain

$$-u_2' e^{-2t} = \frac{1}{e^t + 1} \Rightarrow u_2' = \frac{-e^{2t}}{e^t + 1} \Rightarrow u_2 = -\int \frac{e^{2t}}{e^t + 1}\, \mathrm{d}t = -\int \frac{x\, \mathrm{d}x}{x + 1}, \text{ where } x = e^t.$$

$$\int \frac{x}{x + 1}\, \mathrm{d}x = \int 1 - \frac{1}{x + 1}\, \mathrm{d}x = x - \ln|x + 1| + C.$$

Therefore, one such u_2 is $-e^t + \ln(e^t + 1)$.

Substituting u_2' into the first equation we obtain

$$u_1' = -u_2' e^{-t} = \frac{e^t}{e^t + 1} \Rightarrow u_1 = \int \frac{e^t}{e^t + 1}\,\mathrm{d}t = \int \frac{1}{x+1}\,\mathrm{d}x, \text{ where } x = e^t.$$

One such u_1 is $\ln(e^t + 1)$. Therefore,

$$Y_P = \ln(e^t + 1)e^{-t} + (-e^t + \ln(e^t + 1))e^{-2t} = \ln(e^t + 1)(e^{-t} + e^{-2t}) - e^{-t}.$$

$-e^{-t}$ can be absorbed by Y_H. Thus, the general solution to the given equation is

$$y = c_1 e^{-t} + c_2 e^{-2t} + \ln(e^t + 1)(e^{-t} + e^{-2t}).$$

□

Example 8.15. Find a particular solution to the following equation using each of the following methods:

$$y'' - y' = te^t$$

(a) Undetermined Coefficients.

(b) Key Identities.

(c) Variation of Parameters.

Solution. (a) $p(z) = z^2 - z$. Roots are $0, 1$, and $d = m = 1$. A particular solution is of the form $Y_P = t(A_0 + A_1 t)e^t$. We have the following:

$$Y_P' = (A_0 + 2A_1 t)e^t + (A_0 t + A_1 t^2)e^t = (A_0 + 2A_1 t + A_0 t + A_1 t^2)e^t$$

$$Y_P'' = (2A_1 + A_0 + 2A_1 t + A_0 + 2A_1 t + A_0 t + A_1 t^2)e^t = (2A_0 + 2A_1 + A_0 t + 4A_1 t + A_1 t^2)e^t.$$

$$Y_P'' - Y_P' = (2A_1 t + 2A_1 + A_0)e^t$$

In order for Y_P to satisfy the equation we need to have

$$(2A_1 t + 2A_1 + A_0)e^t = te^t \Rightarrow 2A_1 = 1, 2A_1 + A_0 = 0 \Rightarrow A_0 = -1, A_1 = 1/2.$$

Therefore, a particular solution is $Y_P = -te^t + t^2 e^t/2$.

(b) Similar to the previous case $d = m = 1$. We need to write down the Key Identities for the 1st and 2nd derivatives and substitute $z = 1$. We have $p'(z) = 2z - 1, p''(z) = 2$. Therefore,

$$p(1) = 0, p'(1) = 1, p''(1) = 2.$$

The Key Identities yield:

$$L[te^t] = p'(1)e^t + p(1)te^t = e^t, \text{ and } L[t^2e^t] = p''(1)e^t + 2p'(1)te^t + p(1)t^2e^t = 2e^t + 2te^t.$$

Subtracting twice the first one from the second we obtain

$$L[t^2e^t - 2te^t] = 2e^t + 2te^t - 2e^t = 2te^t \Rightarrow L[t^2e^t/2 - te^t] = te^t \Rightarrow Y_P = t^2e^t/2 - te^t.$$

(c) Roots of the characteristic polynomial are $0, 1$. A particular solution is of the form $Y_P = u_1 + u_2e^t$, where u_1, u_2 satisfy the following system:

$$\begin{cases} u_1' + u_2'e^t = 0 \\ u_1'0 + u_2'e^t = te^t \Rightarrow u_2' = t \Rightarrow u_2 = t^2/2 \text{ works.} \end{cases}$$

Substituting into the first equation we obtain $u_1' = -te^t$. Integrating yields $u_1 = -te^t + e^t$. Therefore, a particular solution is

$$Y_P = -te^t + e^t + t^2e^t/2.$$

This solution is different from the one we got in parts (a) and (b), however it differs by those by a homogeneous solution e^t. □

Example 8.16. Given the solution y_1 to each equation, find the general solution.

(a) $ty'' - ty' + y = 0,\ y_1 = t.$

(b) $y'' + e^ty' + (e^t - 1)y = 0,\ y_1 = e^{-t}.$

(c) $ty'' - (t+3)y' + 2y = 0,\ y_1 = at^2 + bt + c$ for some constants a, b, c.

Solution. We will use the method of Reduction of Order for all parts.

(a) Let $y = vt$. We have $y' = v't + v$, and $y'' = v''t + 2v'$. Substituting we obtain $t^2v'' + 2tv' - t^2v' = 0$. Note that when using Reduction of Order, the term involving v will vanish so we do not need to compute that term. Setting $w = v'$ we obtain the following equation $tw' + (2-t)w = 0$. The integrating factor is $e^{2\ln t - t} = t^2e^{-t}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^2e^{-t}w\right) = 0 \Rightarrow w = e^t/t^2 \text{ is one solution.}$$

Since $w = v'$ is nonzero, v is not constant and thus the solutions $y_2 = t\int_0^t e^s/s^2\ \mathrm{d}s$ and y_1 are not scalar multiples and thus, they are linearly independent. The general solution is thus

$$y = c_1t + c_2t\int_0^t e^s/s^2\ \mathrm{d}s.$$

(b) Let $y = ve^{-t}$. We have $y' = v'e^{-t} - ve^{-t}$ and $y'' = v''e^{-t} - 2v'e^{-t} + ve^{-t}$. Substituting we obtain

$$v''e^{-t} - 2v'e^{-t} + e^tv'e^{-t} = 0 \Rightarrow v''e^{-t} + v'(-2e^{-t} + 1) = 0 \Rightarrow v'' + (-2 + e^t)v' = 0.$$

An integrating factor for this linear equation is e^{-2t+e^t}. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{2t-e^t}v'\right) = 0 \Rightarrow v' = e^{e^t-2t} \text{ works.}$$

Integrating and using substituting $u = e^t$ we obtain

$$v = \int_1^{e^t} e^u u^{-2}\frac{\mathrm{d}u}{u} = \int_1^{e^t} e^u u^{-3}\,\mathrm{d}u.$$

The general solution is

$$y = c_1 e^{-t} + c_2 e^{-t}\int_1^{e^t} e^u u^{-3}\,\mathrm{d}u.$$

(c) First, we will find a, b, c by substituting $at^2 + bt + c$ into the equation:

$$y' = 2at + b, y'' = 2a \Rightarrow ty'' - (t+3)y' + 2y = 2at - (t+3)(2at+b) + 2(at^2+bt+c) = 0.$$

Simplifying we obtain:

$$(2a - b - 6a + 2b)t - 3b + 2c = 0 \Rightarrow b - 4a = -3b + 2c = 0 \Rightarrow b = 4a, c = 6a.$$

Setting $a = 1$ we obtain a solution $y_1 = t^2 + 4t + 6$.

We will now use Reduction of Order to find a second solution.

$$y = v(t^2+4t+6), y' = v'(t^2+4t+6) + v(2t+4) \Rightarrow y'' = v''(t^2+4t+6) + 2v'(2t+4) + 2v.$$

Substituting into the equation we obtain

$$v''(t^3+4t^2+6t) + 2v'(2t^2+4t) - (t+3)v'(t^2+4t+6) = 0 \Rightarrow \frac{v''}{v'} = \frac{t^3+3t^2+10t+18}{t^3+4t^2+6t}.$$

The right hand side can be written as

$$1 + \frac{-t^2+4t+18}{t^3+4t^2+6t} = 1 + \frac{3}{t} - \frac{4t+8}{t^2+4t+6},$$

where the latter equality is obtained using partial fractions method. We will now integrate the last fraction:

$$\int \frac{4t+8}{t^2+4t+6}\,\mathrm{d}t = \int \frac{4(t+2)}{(t+2)^2+2}\,\mathrm{d}t = \int \frac{4u}{u^2+2}\,\mathrm{d}u = 2\ln(u^2+2) + C, \text{ where } u = t+2.$$

Therefore,

$$\ln(v') = t + 3\ln t - 2\ln(u^2+2) \Rightarrow v' = \frac{e^t t^3}{(t^2+4t+6)^2}.$$

This yields $v = \dfrac{e^t(t-3)}{t^2+4t+6}$. (It is not immediately clear to me how to do this integral manually!) The general solution is

$$y = c_1(t^2+4t+6) + c_2 e^t(t-3).$$

□

Example 8.17. Prove that if $y = t^3$ is a solution to a linear homogeneous equation with constant coefficients $L[y] = 0$, then so are $1, t$, and t^2.

Solution. Let $p(z)$ be the characteristic polynomial of L. By the third Key Identity with $z = 0$ we have

$$L[t^3] = p'''(0) + 3p''(0)t + 3p'(0)t^2 + p(0)t^3.$$

Since $L[t^3] = 0$ we must have $p'''(0) + 3p''(0)t + 3p'(0)t^2 + p(0)t^3 = 0$ for all $t \in \mathbb{R}$, which implies $p(0) = p'(0) = p''(0) = p'''(0) = 0$. Therefore, by Theorem 8.2, zero as a root of $p(z)$ has multiplicity at least 4. Combining this with Theorem 8.4 we conclude that functions $1, t, t^2$ are all solutions of $L[y] = 0$. □

Example 8.18. Find a linear differential equation with constant coefficients in normal form $L[y] = f(t)$ whose general solution is given by

$$y = c_1 + c_2 t + c_3 \cos t + c_4 \sin t.$$

How many such equations are there?

Solution. We need to make sure $1, t, \sin t, \cos t$ form a FSoS. So, the characteristic polynomial should have roots $0, 0, \pm i$, and thus it should be $p(z) = z^2(z^2 + 1) = z^4 + z^2$. So, one such equation is $y^{(4)} + y'' = 0$.

Now, we will see if there are other such equations. Note that by setting $c_j = 0$ for all j we conclude that $y = 0$ must be a solution, and hence $L[0] = f(t)$. This means $f(t) = 0$. On the other hand we know $1, t, \sin t, \cos t$ are linearly independent by proof of Theorem 8.4. Thus, the solution set to $L[y] = 0$ must be of dimension 4. Furthermore, we know $L[1] = 0$, which by the first Key Identity we conclude $p(1) = 0$. We also know $L[t] = p'(1) + p(1)t$, which means $p'(1) = 0$. In addition to that $L[e^{it}] = p(i)e^{it}$. On the other hand

$$L[e^{\pm it}] = L[\cos t \pm i \sin t] = L[\cos t] \pm iL[\sin t] = 0 \pm i0 = 0.$$

Therefore, $p(\pm i) = 0$. So far we have shown $p(z)$ has four roots $0, 0, \pm i$, and has degree 4. Since we are assuming the equation is in normal form $p(z) = z^2(z + i)(z - i)$ and thus, $L = D^4 + D^2$ is the only such linear operator with constant coefficients. □

Example 8.19. Find all values of constant α where all solutions of the equation $y'' - 5\alpha y' + 4\alpha^2 y = 0$ tend to zero as $t \to \infty$. Also, find all values of α for which all solutions are bounded for $t > 0$.

Solution. The roots of the characteristic polynomial are $r_1, r_2 = \alpha, 4\alpha$. We will take three cases:
Case I. $\alpha < 0$. In this case, since the roots are distinct and real, the general solution is $y = c_1 e^{\alpha t} + c_2 e^{4\alpha t}$, which tends to zero as $t \to \infty$. Furthermore, for $t > 0$ we have $|y| \leq |c_1| + |c_2|$, which means all solutions are bounded.

Case II. $\alpha = 0$. In this case the roots are both 0 and thus the general solution is $c_1 + c_2 t$ which does not always approach zero as $t \to \infty$, e.g. when $c_2 = 1$, and at least some solutions, e.g. $y = t$, are unbounded.

Case III. $\alpha > 0$. In this case the roots are both real and distinct. Thus, the general solution $c_1 e^{\alpha t} + c_2 e^{4\alpha t}$ does not always tends to zero as $t \to \infty$., e.g. when $c_1 = c_2 = 1$. The same solution is unbounded.

To summarize: All solutions approach zero as $t \to \infty$ if and only if $\alpha < 0$ if and only if all solutions are bounded. □

Example 8.20. Suppose $y = \sin t + t \cos t$ is a solution to a homogeneous linear equation with constant coefficients $L[y] = 0$. Prove that for every $c_1, c_2, c_3, c_4 \in \mathbb{R}$, the function

$$y = c_1 \sin t + c_2 \cos t + c_3 t \sin t + c_4 t \cos t$$

is also a solution to $L[y] = 0$.

Solution. Let $z_1, \ldots, z_m$ be all distinct roots of the characteristic polynomial $p(z)$ of L. By Theorem 8.4 every solution to $L[y] = 0$ is a linear combination of $t^k e^{z_j t}$'s with complex coefficients. On the other hand

$$\sin t + t \cos t = \frac{e^{it} - e^{-it}}{2i} + t\frac{e^{it} + e^{-it}}{2}.$$

Since $t^k e^{at}$'s are linearly independent, $i, -i$ must be roots of $p(z)$. Therefore, $\sin t$ is a solution to $L[y] = 0$. By linearity $t \cos t$ is also a solution to $L[y] = 0$. By an argument similar to Example 8.17 we conclude that $p(z)$ has i as a root with multiplicity at least 2. Thus, $\sin t, \cos t, t \sin t, t \cos t$ are all solutions to $L[y] = 0$. By linearity we obtain the result. □

Example 8.21. Find an analytic function that is not a solution to any linear equation with constant coefficients.

Sketch. We need to find a function that is not a linear combination of functions of the form

$$t^j e^{at} \cos(bt), t^j e^{at} \sin(bt)$$

Note that these functions grow at most exponentially, so we will choose a function that grows faster.

Solution. One such function is e^{t^2}. First, note that since $e^t = \sum_{n=0}^{\infty} t^n/n!$ we have $e^{t^2} = \sum_{n=0}^{\infty} t^{2n}/n!$ and thus e^{t^2} is analytic.

Suppose on the contrary e^{t^2} is a solution to a homogeneous linear equation with constant coefficients. By Theorem 8.4 it can be written as a linear combination of functions of the following form

$$t^j e^{at} \cos(bt), t^j e^{at} \sin(bt) \qquad (*).$$

Note that

$$\frac{t^j e^{at}\cos(bt)}{e^{t^2}} = t^j e^{(a-t)t}\cos(bt).$$

When $t > a+1$ we have $a - t < -1$ and thus $t^j e^{(a-t)t} < t^j e^{-t}$, which approaches zero as $t \to \infty$. Thus the entire ratio above approaches zero. Similarly the ratio of $t^j e^{at}\sin(bt)$ by e^{t^2} approaches zero. Therefore , e^{t^2} approaches infinity faster than any solution to a homogeneous linear equation with constant coefficients. $\square$

Example 8.22. Evaluate

$$\underbrace{\int\int\cdots\int\int}_{8 \text{ times}} te^t dt\cdots dt$$

Solution. We are looking for every function whose 8-th derivative is te^t. In other words we are trying to solve $y^{(8)} = te^t$. The characteristic polynomial is $p(z) = z^8$. This yields the characteristic equation $z^8 = 0$, and thus the homogeneous solution is

$$Y_H = c_0 + c_1 t + \cdots + c_7 t^7.$$

Using the usual notations, we know $d = 1$ and $m = 0$. Therefore, we get the following identities:

$$L[e^t] = p(1)e^t = e^t, \text{ and } L[te^t] = p'(1)e^t + p(1)te^t = 8e^t + te^t.$$

This implies $L[te^t - 8e^t] = te^t$, and hence $Y_P = te^t - 8e^t$. Thus, the answer is

$$c_0 + c_1 t + \cdots + c_7 t^7 + te^t - 8e^t.$$

$\square$

Example 8.23. Given that $y = t$ is a solution to the following equation, find the general solution to this equation:

$$t^3 y''' - 3t^2 y'' + (6t - 4t^3)y' + (-6 + 4t^2)y = 0, \ t > 0.$$

Solution. We will use Reduction of Order to reduce the order of the equation. Set $y = vt$. We will then obtain:

$$y = vt, \ y' = v + v't, \ y'' = 2v' + v''t, \text{ and } y''' = 3v'' + v'''t.$$

Substituting this into the equation we obtain the following:

$$t^4 v''' + (3t^3 - 3t^3)v'' + (-6t^2 + 6t^2 - 4t^4)v' + (6t - 4t^3 - 6t + 4t^3)v = 0.$$

This simplifies to $v''' - 4v' = 0$. This is an equation with constant coefficients whose characteristic polynomial is $z^3 - 4z$ that has roots $0, \pm 2$. Thus, $v = 1, e^{\pm 2t}$ are three solutions. This means t, te^t and te^{-t} are three solutions. By Exercise 8.5 these functions are linearly independent. Since the given equation is of order 3, its solution set must have dimension 3. Thus, the general solution to the given equation is $y = c_1 t + c_2 te^t + c_3 te^{-t}$. $\square$

Example 8.24. Prove that all periodic solutions of the following equation form a two dimensional vector space.

$$y^{(4)} - y = 0.$$

Solution. The characteristic polynomial is $p(z) = z^4 - 1$. Its roots are $z = \pm 1, \pm i$. Therefore, the general solution is of the form

$$y = c_1 \cos t + c_2 \sin t + c_3 e^t + c_4 e^{-t}.$$

Note that if y is periodic, since it is continuous, by the Extreme Value Theorem, it must be bounded. If $c_3 > 0$, then as $t \to \infty$, $c_3 e^t \to \infty$. Since $e^{-t} \to 0$ and $c_1 \cos t + c_2 \sin t$ are bounded, y would be unbounded. Therefore, y would not be periodic. Similarly if $c_3 < 0$, the solution y would not be periodic. Therefore, if y is periodic $c_3 = 0$. Similarly, we must have $c_4 = 0$. Thus, the only solutions that may be periodic are of the form $c_1 \cos t + c_2 \sin t$. These solutions are all periodic with period 2π. These solutions form a vector space of dimension 2 with basis $\{\cos t, \sin t\}$. □

Example 8.25. Given a positive integer n, solve the following differential equation.

$$y^{(n)} + y^{(n-1)} + \cdots + y' + y = 0.$$

Solution. The characteristic polynomial is $p(z) = z^n + \cdots + z + 1 = \dfrac{z^{n+1} - 1}{z - 1}$. Its roots satisfy $z^{n+1} = 1$. Taking the absolute value of both sides we conclude that $|z|^{n+1} = 1$, i.e. $|z| = 1$. Using the polar form of z we have $z = e^{i\theta}$ for some angle $\theta \in [0, 2\pi)$. Since $z^{n+1} = 1$, we have $\cos((n+1)\theta) = 1$ and $\sin((n+1)\theta) = 0$. Therefore, $\theta = \dfrac{2\pi k}{n+1}$, with $k = 0, 1, \ldots, n$. Note that $z = 1$ is not a root of $p(z)$, thus, we need to eliminate $\theta = 0$. Therefore, roots of the characteristic polynomial of this equation are

$$\cos(2k\pi/(n+1)) + i \sin(2k\pi/(n+1)), \; k = 1, \ldots, n.$$

If $n+1$ is odd, then $\sin(2k\pi/(n+1))$ is not zero and thus all of these roots are nonreal. If $n+1$ is even, there is one root of -1 corresponding to $k = (n+1)/2$ and the rest of the roots are nonreal.

The conjugate of $\cos(2k\pi/(n+1)) + i \sin(2k\pi/(n+1))$ is

$$\cos(2k\pi/(n+1)) - i \sin(2k\pi/(n+1)) = \cos(2(n+1-k)\pi/(n+1)) + \sin(2(n+1-k)\pi/(n+1)).$$

If $n+1$ is odd, then the general solution to this equation is

$$y = \sum_{k=1}^{\frac{n}{2}} c_k e^{t\cos(2k\pi/(n+1))} \cos(\sin(2k\pi/(n+1))) + \sum_{k=1}^{\frac{n}{2}} d_k e^{t\cos(2k\pi/(n+1))} \sin(\sin(2k\pi/(n+1))).$$

If $n+1$ is even, then the general solution to this equation is

$$y = \sum_{k=1}^{\frac{n+1}{2}} c_k e^{t\cos(2k\pi/(n+1))} \cos(\sin(2k\pi/(n+1))) + \sum_{k=1}^{\frac{n-1}{2}} d_k e^{t\cos(2k\pi/(n+1))} \sin(\sin(2k\pi/(n+1))).$$

□

8.5 Exercises

Exercise 8.1. *Find the general solution for each equation*

(a) $y''' - 4y'' + 4y' - y = 0.$

(b) $4y'' - 12y' + 9y = t.$

(c) $y^{(4)} + y = 0.$

(d) $y'' + 4y = \sin(3t)\cos t + e^t.$

(e) $y'' - 6y' + 9y = t^{3/2}e^{3t},\ t > 0.$

Exercise 8.2. *Given the solution y_1 to each equation, find the general solution.*

(a) $(1-t^2)y'' - 2ty' + 2y = 0,\ y_1 = t.$

(b) $ty'' - y' - 4t^3y = 0,\ y_1 = e^{t^2}.$

(c) $(e^t+1)y'' - 2y' - e^ty = 0,\ y_1 = e^t - 1.$

Exercise 8.3. *Find a particular solution for $y'' + 4y' + 4y = te^{-2t} + t$ using*

(a) *Key Identities.*

(b) *Undetermined Coefficients.*

(c) *Variation of Parameters.*

Exercise 8.4. *Find a particular solution to each equation:*

(a) $y^{(4)} + 4y''' + 6y'' + 4y' + y = e^{-t}.$

(b) $y^{(4)} + y'' = t.$

In the following exercise you will prove Theorem 8.4.

Exercise 8.5. *Let $z_1, \dots, z_n$ be distinct complex numbers, $z = a + ib$ with $a, b \in \mathbb{R}$, be a nonreal complex number, and m be a nonnegative integer.*

(a) *Prove that the span of $\{t^m\ e^{zt}, t^m\ e^{\overline{z}t}\}$ over $\mathbb{C}$ is the same as the span of $\{t^m\ e^{at}\cos(bt), t^m\ e^{at}\sin(bt)\}$ over $\mathbb{C}$.*

(b) *Prove that if polynomials $p_1(t), \dots, p_n(t)$ with complex coefficients satisfy $p_1(t)e^{z_1t} + \cdots + p_n(t)e^{z_nt} = 0$ for all $t \in \mathbb{R}$, then all of the polynomials $p_1(t), \dots, p_n(t)$ must be identically zero.*

(c) *Use the previous part to prove that the set of functions*

$$\{t^k e^{z_jt} \mid k = 0, 1, \dots, m,\ \text{and}\ j = 1, 2, \dots, n\}$$

is linearly independent over $\mathbb{C}$.

(d) Use part (a) to show if in the set in part (c) we replace each $t^k e^{(a\pm ib)t}$ by $t^k e^{at}\cos(bt)$ and $t^k e^{at}\sin(bt)$ we obtain another set of functions that are also linearly independent over $\mathbb{C}$. (Here $a, b \in \mathbb{R}$ and $b \neq 0$.)

Hint: To prove part (b), use induction on n. To prove the inductive step, start with $\sum_{j=1}^{n} p_j(t)e^{z_j t} = 0$ and assume $p_j(t) \neq 0$ for all j. Divide both sides by $e^{z_1 t}$. Prove that all derivatives of $p_j(t)e^{(z_j - z_1)t}$, with $j > 1$, are of the form $q(t)e^{(z_j - z_1)t}$, where the degree of $q(t)$ is the same as the degree of $p_j(t)$. To prove this, you need to use yet another induction. This means by differentiating enough times you would be able to eliminate $p_1(t)$ altogether, without changing the degrees of $p_2, \dots, p_n$. Then apply inductive hypothesis to yield a contradiction,

Exercise 8.6. *Suppose two different linear homogeneous differential equations with constant coefficients $L_1[y] = 0$ and $L_2[y] = 0$ have a common nonzero solution $y = y(t)$, defined over $\mathbb{R}$. Prove that the characteristic polynomials $p_1(z)$ and $p_2(z)$ of L_1 and L_2 have at least one common root.*

Exercise 8.7. *Using induction on n, prove the Key Identity $L[t^n e^{zt}] = \sum_{j=0}^{n} \binom{n}{j} p^{(n-j)}(z)\ t^j e^{zt}$.*

Exercise 8.8. *Find a general solution for the equation $y'' - 2y' + y = t^2 e^t$.*

Exercise 8.9. *Suppose $y = \sin t$ is a solution to a homogeneous linear equation with constant coefficients. Prove that $y = \cos t$ must also be a solution to this equation.*

Hint: See Example 8.17.

Exercise 8.10. *For each equation do all of the following:*

1. *Determine all values of α for which all solutions tend to zero as $t \to \infty$.*

2. *Determine all values of α for which all solutions are periodic.*

3. *Determine all values of α for which all nonzero solutions are unbounded over $[0, \infty)$.*

(a) $y'' - 2\alpha y' + (\alpha^2 - 1)y = 0$.

(b) $y'' + \alpha y = 0$.

Exercise 8.11. *Let a, b, c be three positive constants. Prove that all solutions of $ay'' + by' + cy = 0$ approach zero as $t \to \infty$.*

Exercise 8.12. *Let α and β be two constants. Show that if $y'' + 4y = 0, y(\alpha) = y_0, y'(\beta) = y_1$ has more than one solution for some y_0, y_1, then $4\alpha - 4\beta = (2k+1)\pi$ for some integer k.*

Exercise 8.13. *Suppose $\phi_1(t) = e^t + e^{t^2}, \phi_2(t) = 2e^t + e^{t^2}$, and $\phi_3(t) = e^{-t} + e^{t^2}$ are solutions to the differential equation $y'' + p(t)y' + q(t)y = f(t)$. Solve the initial value problem:*

$$y'' + p(t)y' + q(t)y = f(t), y(0) = 0, y'(0) = 2.$$

Exercise 8.14. *Let L be a linear differential operator with constant coefficients. Suppose all roots of the characteristic polynomial associated to $L[y] = 0$ have negative real parts. Prove that every solution to $L[y] = 0$ tends to zero at $t \to \infty$.*

Exercise 8.15. *Let n be a positive integer. Consider the differential equation*

$$ty'' - (t+n)y' + ny = 0, \ t > 0.$$

(a) Prove that $y = e^t$ is a solution to this equation.

(b) Find the general solution. Can you find a simple form for the general solution?

Exercise 8.16. *Find the general solution of each equation, where one solution is given:*

(a) $ty'' - y' + 4t^3 y = 0, \ t > 0, \ y_1 = \sin(t^2)$.

(b) $(t-1)y'' - ty' + y = 0, \ t > 1, \ y_1 = e^t$.

Exercise 8.17. *Suppose $p(t)$ is a continuous over $\mathbb{R}$. Find the general solution of*

$$y'' + p(t)y' - (p(t)+1)y = 0.$$

Hint: $y = e^t$ is a solution.

Exercise 8.18. *Suppose the equation $y'' + p(t)y' + q(t)y = 0$ has two solutions defined over $(0, \infty)$ whose Wronskian is t. Assume also that this differential equation has $y = t^2$, with $t > 0$, as a solution. Find a general solution for the differential equation $y'' + p(t)y' + q(t)y = t^2 \sin t$ with $t > 0$. As usual show your work completely.*

Exercise 8.19. *Find a particular solution for $y'' + y = 2 \sin t \cos(2t)$.*

Exercise 8.20. *Create a second order linear differential equation $L[y] = f(t)$ whose general solution is $y = \sin(t^2) + c_1 \cos(t) + c_2 \sin(t)$. How many such equations are there?*

Exercise 8.21. *Prove that the equation $y'' + y = 2 \sin t$ has no periodic solutions.*

Exercise 8.22. *Consider the differential equation $y'' + y = 0$, and two given constants α, c. Determine how many solutions y of this equation satisfy $y(0) = 0$ and $y'(\alpha) = c$. Your answer could depend on α and c.*

Exercise 8.23. *Suppose y is a solution to an n-th order linear homogeneous equation with constant coefficients, and T is a positive real number for which*

$$y(0) = y(T), y'(0) = y'(T), \ldots, y^{(n-1)}(0) = y^{(n-1)}(T).$$

Prove that y is periodic. By an example show that this result does not hold if the coefficients are not constant.

Exercise 8.24. *Given that $y = t^2$ is a solution to*

$$t^3 y''' + (t^3 - 6t^2)y'' + (t^3 - 4t^2 + 18t)y' - (2t^2 - 6t + 24)y = 0, \ t > 0$$

find the general solution to this equation.

Exercise 8.25. *Suppose $L = D^n + a_n D^{n-1} + \cdots + a_2 D + a_1$ is a differential operator with constant coefficients $a_1, \ldots, a_n$ for which $a_1 \neq 0$. Prove that for every $t_0 \in \mathbb{R}$ and every $y_1, \ldots, y_n \in \mathbb{R}$ the following problem has a unique solution.*

$$L[y] = 0, y'(t_0) = y_1, y''(t_0) = y_2, \ldots, y^{(n)}(t_0) = y_n.$$

(Note that this is NOT an IVP.)

Exercise 8.26. *Prove that the equation $y'' \cos t + y' \sin t + y \cos t = t^3$ has an odd solution defined over $(-\frac{\pi}{2}, \frac{\pi}{2})$.*

Exercise 8.27. *Assume $L[y] = 0$ is an n-th order linear homogeneous equation with constant coefficients all of whose solutions are bounded.*

(a) Prove that if n is odd, then $L[1] = 0$.

(b) Prove that if $n = 3$, then all solution of this equation are periodic.

(c) Prove that the characteristic polynomial of L is of the form $p(z) = z^k(z^2 + a_1^2) \cdots (z^2 + a_m^2)$, where $k = 0$ or 1, and $a_1, \ldots, a_m$ are distinct positive real numbers.

Exercise 8.28. *For every two n times differentiable functions $u(t), v(t)$, and any linear differential operator with constant coefficients*

$$p(D) = D^n + a_n D^{n-1} + \cdots + a_1,$$

prove that

$$p(D)[uv] = \frac{1}{n!} \sum_{k=0}^{n} p^{(k)}(D)[u] p^{(n-k)}(D)[v].$$

8.6 Challenge Problems

Exercise 8.29. *Suppose L is a linear differential operator with constant coefficients. Assume the order of the equation $L[y] = 0$ is odd and it has no nonzero constant solutions. Prove that $L[y] = 0$ has a solution that is strictly increasing over $\mathbb{R}$.*

Exercise 8.30. *Suppose $p : \mathbb{R} \to \mathbb{R}$ is a periodic continuous function with period T. Assume $\{N_0, N_1\}$ is a NFSoS of $y'' + p(t)y = 0 \quad (*)$ at time $t_0 = 0$. Prove that $(*)$ has a nontrivial solution with period T if and only if $N_0(T) + N_1'(T) = 2$.*

Exercise 8.31. *Prove that the method of Reduction of Order always reduces the order of the equation. In other words, assume y_1 is a solution to an n-th order linear equation $L[y] = 0$, and let $y = vy_1$. Prove that $L[vy_1]$ is an $(n-1)$-th order equation in terms of $w = v'$.*

8.7 Summary

- To solve a homogeneous linear differential equation with constant coefficients:

  - Write down the characteristic polynomial $p(z)$ of the linear operator.
  - Find all roots of $p(z)$.
  - For every real root r with multiplicity m, include $e^{rt}, \dots, t^{m-1}e^{rt}$ in a fundamental set of solutions.
  - For a nonreal root $a + bi$ include $e^{at}\cos(bt)$ and $e^{at}\sin(bt)$ in the fundamental set of solutions.
  - If a nonreal root has multiplicity, similar to above multiply $e^{at}\cos(bt)$ and $e^{at}\sin(bt)$ by powers of t to get the appropriate number of solutions.
  - Steps above create a fundamental set of solutions. Take a linear combination to get the general solution.

- To find a particular solution to $L[y] = f(t)$ using the method of Key Identities:
  - Find $p(z)$, the characteristic polynomial of L.
  - Break $f(t)$ into functions of the form

    $$(\text{polynomial}) \cdot e^{at}\cos(bt) + (\text{polynomial}) \cdot e^{at}\sin(bt).$$

    Follow the steps below for each one of them separately.
  - Find d, the larger degree of the polynomials in the forcing.
  - Find m, the multiplicity of $a + ib$ as a root of $p(z)$. If $a + ib$ is not a root of $p(z)$, then $m = 0$.
  - Write down the Key Identities from the m-th to the $(m + d)$-the derivatives.
  - Take a linear combination to obtain the desired forcing.
  - If needed, add up the particular solutions obtained.

- To find a particular solution to $L[y] = f(t)$ using the method of Undetermined Coefficients:
  - Break $f(t)$ into functions of the form

    $$(\text{polynomial})e^{at}\cos(bt) + (\text{polynomial})e^{at}\sin(bt).$$

    Follow the steps below for each one of them separately.
  - Find d and m as above.
  - A particular solution has the form

    $$Y_P = t^m(A_0 + \cdots + A_d t^d)e^{at}\cos(bt) + t^m(B_0 + \cdots + B_d t^d)e^{at}\sin(bt).$$

  - Substitute Y_P into the equation and find all constants A_j, B_j.

- To find Y_P using Variation of Parameters:
  - We need a FSoS $\{Y_1, \dots, Y_n\}$ for the homogeneous equation.
  - Make sure the equation is in normal form, and let $f(t)$ be the forcing.

– Solve the system:

$$\begin{cases} u_1'Y_1 + u_2'Y_2 + \cdots + u_n'Y_n = 0 \\ u_1'Y_1' + u_2'Y_2' + \cdots + u_n'Y_n' = 0 \\ \vdots \\ u_1'Y_1^{(n-2)} + u_2'Y_2^{(n-2)} + \cdots + u_n'Y_n^{(n-2)} = 0 \\ u_1'Y_1^{(n-1)} + u_2'Y_2^{(n-1)} + \cdots + u_n'Y_n^{(n-1)} = f(t) \end{cases}$$

– This gives $Y_P = u_1Y_1 + \cdots + u_nY_n$. Remember that you only need *one* $u_1, \ldots, u_n$ and not all of them.

– Variation of Parameters can be used for constant or nonconstant linear equations.

- Reudction of order allows us to find a second solution to $y'' + p(t)y' + q(t)y = 0$ given one solution y_1. To use Reduction of Order we let $y = vy_1$, substitute into the equation and obtain a first order equation in terms of $w = v'$.

- Reduction of order can be used for equations of higher order as well. Though, solving the resulting equation for $w = v'$ may prove to be difficult.

Chapter 9

Power Series Solutions

9.1 Series Solutions Near an Ordinary Point

In this section we will focus on finding solutions to second order linear equations of the form

$$y'' + p(t)y' + q(t)y = 0.$$

Example 9.1. Solve $y'' + y = 0$.

The general strategy is to replace y by a power series, and find a relation between the coefficients of both sides. This yields a recurrence relation for the coefficients which in turns gives us a solution. This all works out, if either we know analytic solutions exist or if the resulting power series converges.

Definition 9.1. A function $f(t)$ is said to be **analytic at** t_0 if both of the following hold:

- The radius of convergence R of the Taylor series of $f(t)$ at t_0 is positive, and
- $f(t)$ equals its Taylor series centered at t_0 for all $t \in (t_0 - R, t_0 + R)$.

Theorem 9.1. *Suppose $f(t)$, and $g(t)$ are analytic at $t = t_0$. Then, the functions $f(t) + g(t)$, and $f(t)g(t)$ are also analytic at $t = t_0$. Furthermore, if $g(t_0) \neq 0$, then $f(t)/g(t)$ is analytic at $t = t_0$.*

The following theorem which guarantees the existence of an analytic solution allows us to find solutions using the above method.

Definition 9.2. A point t_0 is said to be an **ordinary point** for the equation

$$y'' + p(t)y' + q(t)y = 0,$$

if $p(t)$ and $q(t)$ are analytic functions at t_0. In other words,

$$p(t) = \sum_{n=0}^{\infty} p_n(t - t_0)^n, \text{ and } q(t) = \sum_{n=0}^{\infty} q_n(t - t_0)^n \text{ for all } t \text{ with } 0 < |t - t_0| < \epsilon.$$

If the equation is not in normal form, but it can be written in normal form in such a way that the coefficients are analytic at $t = t_0$ we still call t_0 an ordinary point of the equation.

Theorem 9.2. *Suppose $p(t)$ and $q(t)$ are analytic at t_0. Then, every solution to the equation*

$$y'' + p(t)y' + q(t)y = 0$$

is analytic at t_0. Furthermore, the radius of convergence of the Taylor series of each solution centered at t_0 is at least the minimum of the radii of convergence of Taylor series of $p(t)$ and $q(t)$ centered at t_0.

The above theorem allows us to find solutions using their Taylor series expansions, and by finding a recursion for the coefficients.

Example 9.2. Solve $y'' - ty = 0$.

Remark. If the coefficients a_n for $y = \sum_{n=0}^{\infty} a_n(t - t_0)^n$ $(*)$ are determined in a such a way that the power series for y satisfies $y'' + p(t)y' + q(t)y = 0$, then y is a solution over the interval of convergence of $(*)$.

Example 9.3. Solve the initial value problem: $y'' + t^2y' + 2ty = 0,\ y(0) = 1,\ y'(0) = 0$.

Example 9.4. Solve the initial value problem: $y'' + (t^2 + 2t + 1)y' - (4t + 4)y = 0,\ y(-1) = 0,\ y'(-1) = 1$.

Often times, solutions obtained using the Power Series Method cannot be written in terms of common functions: polynomials, rational functions, radicals, trigonometric functions and their inverses, exponential functions, logarithms, etc. If a function can be written in terms of one of these common functions we say it is in **closed form**.

9.2 Series Solutions Near a Regular Singular Point

Definition 9.3. We say the differential equation $y'' + p(t)y' + q(t)y = 0$ is **singular** (or has a singularity) at t_0 if $p(t)$ or $q(t)$ are unbounded near $t = t_0$, and $p(t), q(t)$ are both continuous over $0 < |t - t_0| < \epsilon$, for some positive ϵ.

Solutions near singularities may not even be continuous, which means we will not be able to find them using the Power Series Method. We will start by looking at a particular type of singularity.

Definition 9.4. Any differential equation of the form $t^2y'' + \alpha t\, y' + \beta y = 0$, where $\alpha, \beta \in \mathbb{R}$ are real constants, is said to be an **Euler's equation**.

Note that the above Euler's equation is singular at zero, which means we may not be able to find a solution by the Power Series Method. We do notice that if $y = t^r$, then both t^2y'' and ty' will be multiples of t^r. This suggests $y = t^r$ may be a good candidate for a solution.

Example 9.5. Solve each of the following equations given $t > 0$.

(a) $t^2\, y'' + 6t\, y' + 4\, y = 0$.

(b) $t^2\, y'' + 3t\, y' + y = 0$.

(c) $4t^2\, y'' + 20t\, y' + 25y = 0$.

Theorem 9.3. *Let α, β be constants. Consider the differential equation*

$$t^2\ y'' + \alpha t\ y' + \beta y = 0,\ \ t > 0 \qquad (*)$$

Let r_1, r_2 be the roots of equation $r(r-1) + \alpha r + \beta = 0$.

- *If r_1, r_2 are distinct and real, then t^{r_1}, t^{r_2} form a FSoS for $(*)$.*
- *If $r_1 = r_2$, then $t^{r_1}, t^{r_1}\ln t$ form a FSoS for $(*)$.*
- *If $r_1, r_2 = a \pm ib$ are not real, with $a, b \in \mathbb{R}$, then $t^a \cos(b\ln t), t^a \sin(b\ln t)$ form a FSoS for $(*)$.*

When $t < 0$ the result remains the same when all t's are replaced by $-t$.

Remark. Equations of the form $(t-t_0)^2 y'' + \alpha(t-t_0)y' + \beta y = 0$ can also be solved using a similar method to what we did above by letting $y = (t-t_0)^r$. In which case the previous theorem is still valid when all t's are replaced by $t - t_0$.

The above equation can be written as

$$y'' + \frac{\alpha}{t-t_0}y' + \frac{\beta}{(t-t_0)^2}y = 0.$$

In this example, even though there is a singularity at t_0, the singularity is still "managable". This motivates the following definition:

Definition 9.5. Suppose the equation $y'' + p(t)y' + q(t)y = 0$ is singular at t_0. We say t_0 is a **regular singular point** for this equation if $(t-t_0)p(t)$ and $(t-t_0)^2 q(t)$ are analytic near $t = t_0$. In other words, there is $\epsilon > 0$ and two sequences p_n, q_n of real numbers for which, for all t with $0 < |t - t_0| < \epsilon$, we have

$$(t-t_0)p(t) = p_0 + p_1(t-t_0) + p_2(t-t_0)^2 + \cdots,\ \text{and}\ (t-t_0)^2 q(t) = q_0 + q_1(t-t_0) + q_2(t-t_0)^2 + \cdots.$$

Otherwise, we say t_0 is an **irregular singular point**.

Example 9.6. Find all regular and irregular singular points of the equation

$$(t^4 - 2t^2 + 1)y'' + (t-1)y' + 3y = 0.$$

It is often easier to do a change of variables $t = s + t_0$, $z(s) = y(s + t_0)$, to move the singularity to the origin and work with power series centered at $s = 0$. In which case the equation becomes

$$z''(s) + p(s+t_0)z'(s) + q(s+t_0)z(s) = 0.$$

So, we will only look at equations with zero as a regular singular point.

From Euler's equation we know a solution may not be analytic near zero. The idea of solving such equations is to write $y(t) = t^r \sum_{n=0}^{\infty} a_n t^n$, and find a recurrence relation for a_n.

Example 9.7. Find a FSoS for

$$t^2y'' + ty' + (t^2 - 1/4)y = 0, \ t > 0.$$

Theorem 9.4. *Suppose $t = 0$ is a regular singular point for the equation*

$$y'' + p(t)y' + q(t)y = 0 \qquad (*)$$

Suppose

$$tp(t) = p_0 + p_1t + p_2t^2 + \cdots, \text{ and } t^2q(t) = q_0 + q_1t + q_2t^2 + \cdots \text{ for all } t \in (0, R).$$

Let r_1, r_2 be roots of the equation $r(r-1) + p_0r + q_0 = 0$ with $r_1 \geq r_2$ if r_1, r_2 are both real. Then, the equation $(*)$ has a FSoS over $(0, R)$ of the following form:

$$y_1(t) = t^{r_1}\sum_{n=0}^{\infty} a_nt^n, \text{ and } y_2(t) = c\ y_1(t)\ln t + t^{r_2}\sum_{n=0}^{\infty} b_nt^n,$$

where c is a real constant. Furthermore,

- *If $r_1 - r_2$ is not an integer, then $c = 0$.*
- *If $r_1 = r_2$, then $c = 1$.*
- *If $r_1 - r_2$ is a positive integer, then c could be any real constant.*

When $r_1, r_2 = a \pm bi$ are nonreal, we replace t^{r_1} by its real and imaginary parts: $t^a\cos(b\ln t)$ and $t^a\sin(b\ln t)$ to obtain the solutions:

$$y_1(t) = t^a\cos(b\ln t)\sum_{n=0}^{\infty} a_nt^n, \text{ and } y_2(t) = t^a\sin(b\ln t)\sum_{n=0}^{\infty} a_nt^n.$$

Furthermore, a similar result holds for when $t \in (-R, 0)$, if we replace each t^r by $(-t)^r$, and $\ln t$ is replaced by $\ln(-t)$.

Remark. The equation $r(r-1) + p_0r + q_0 = 0$ is called the **indicial equation** and the solutions r_1, r_2 are called **exponents at the singularity** for the given second order equation.

Example 9.8. Write down the form of a FSoS to each equation near $t_0 = 0$.

(a) $t^2y'' + (\sin t + t)y' + y = 0, \ t > 0.$

(b) $t^2y'' + (e^t - 1)y' - (t+1)y = 0, \ t > 0.$

(c) $t^2y'' + (3t + t^4)y' + y = 0, \ t > 0.$

9.3 More Examples

Example 9.9. Find the general solution to each equation.

(a) $t^2y'' + 3ty' + 2y = 0, \ t < 0.$

(b) $t^2y'' - 5ty' + 9y = 0,\ t > 0.$

(c) $(t-1)^2y'' - 5(t-1)y' + 8y = 0,\ t < 1.$

Solution. These are all Euler's equations.

(a) The indicial equation is $r(r-1) + 3r + 2 = 0$. This yields $r = -1 \pm i$. The general solution is thus

$$y = c_1t^{-1}\cos(\ln(-t)) + c_2t^{-1}\sin(\ln(-t)).$$

(b) The indicial equation is $r(r-1) - 5r + 9 = 0$. Its roots are $r = 3, 3$. Thus, the general solution is

$$y = c_1t^3 + c_2t^3\ln t.$$

(c) The indicial equation is $r(r-1) - 5r + 8 = 0$. The roots are $r = 2, 4$. The general solution is thus

$$y = c_1(1-t)^2 + c_2(1-t)^4.$$

□

Example 9.10. Solve the equation: $t^2y'' - 2y = t^3$, with $t > 0$.

Solution. First we will solve the corresponding homogeneous equation, which is an Euler's equation. Letting $y = t^r$, we obtain the equation $r(r-1) - 2 = 0$. This gives $r = -1, 2$, which means t^{-1}, t^2 form a FSoS, and thus $Y_H = c_1/t + c_2t^2$ is the general homogeneous solution. To find a particular solution we will use Variation of Parameters. Let $Y_P = u_1/t + u_2t^2$. We have the following system:

$$\begin{cases} u_1'/t + u_2't^2 = 0 \\ -u_1'/t^2 + 2u_2't = t \end{cases}$$

Note that the forcing is $t^3/t^2 = t$. Solving, we obtain $u_2' = 1/3$ and $u_1' = -t^3/3$. Integrating we will find $u_1 = -t^4/12$ and $u_2 = t/3$ are two such solutions. Thus, $Y_P = -t^3/12 + t^3/3 = t^3/4$. Therefore, the general solution is

$$y = \frac{c_1}{t} + c_2t^2 + \frac{t^3}{4}.$$

□

Example 9.11. Find all constants α for which all solutions to the equation $t^2y'' + ty' + \alpha y = 0,\ t > 0$ are bounded near $t = 0$.

Solution. This is an Euler's equation. The indicial equation is $r(r-1) + r + \alpha = 0$. The roots are $\pm\sqrt{\alpha}$. We will take three cases.

Case I. If $\alpha = 0$, then the general solution is $y = c_1 + c_2\ln t$. Thus, in that case $y = \ln t$ is a solution which is not bounded near $t = 0$.

Case II. If $\alpha > 0$, then the general solution is $y = c_1t^{\sqrt{\alpha}} + c_1t^{-\sqrt{\alpha}}$. The solution $t^{-\sqrt{\alpha}}$ is unbounded near

$t = 0$.

Case III. If $\alpha < 0$, then the general solution is $y = c_1 \cos(\sqrt{-\alpha}\ln t) + c_2 \sin(\sqrt{-\alpha}\ln t)$, which is bounded since $\sin x$ and $\cos x$ are both bounded functions.

The answer is $\alpha < 0$. $\square$

Example 9.12. For each of the following find an Euler's equation whose general solution is given.

(a) $c_1 t^4 + c_2 t^5$.

(b) $c_1 t^2 + c_2 t^2 \ln t$.

(c) $c_1 t^5 \cos(3\ln t) + c_2 t^5 \sin(3\ln t)$.

Solution. (a) We will make sure that the indicial equation in Theorem 9.3 has roots $r = 4, 5$. Thus, we need

$$r(r-1) + \alpha r + \beta = (r-4)(r-5) \Rightarrow r^2 + (\alpha-1)r + \beta = r^2 - 9r + 20 \Rightarrow \alpha - 1 = -9, \beta = 20.$$

This yields the equation $t^2 y'' - 8ty' + 20y = 0$.

(b) Similar to above, but there must be a repeated root of 2. This means $r(r-1) + \alpha r + \beta = (r-2)^2$, i.e. $\alpha - 1 = -4$ and $\beta = 4$. Thus, we obtain the equation $t^2 y'' - 3ty' + 4y = 0$.

(c) The roots must be $5 \pm 3i$, and thus $r(r-1) + \alpha r + \beta = (r-5)^2 + 9$. This implies $\alpha - 1 = -10$ and $\beta = 34$. This yields the equation $t^2 y'' - 9ty' + 34y = 0$. $\square$

Example 9.13. Find all singularities of the following equations. For each singularity, determine if it is regular or irregular.

(a) $(t^2 \sin t)y'' + t\, y' - (\cos t)y = 0$.

(b) $(t^2 - 1)^3 y'' + (1-t)^2 y' + \sin(1-t)\; y = 0$.

Solution. (a) The equation, written in normal form, is

$$y'' + \frac{y'}{t \sin t} - \frac{\cos t}{t^2 \sin t} y = 0.$$

We see that $p(t) = \dfrac{1}{t \sin t}$ and $q(t) = \dfrac{-\cos t}{t^2 \sin t}$ are continuous everywhere, except when $t^2 \sin t = 0$. At these points $p(t)$ is unbounded. Thus, the singularities are when $t = k\pi$ with $k \in \mathbb{Z}$.

First, we will look at $t = 0$. The function $tp(t) = \dfrac{1}{\sin t}$ is not analytic at zero, since it is unbounded near zero. Therefore, zero is an irregular singularity. Assume $t_0 = k\pi$ with $0 \neq k \in \mathbb{Z}$. Let $s = t - k\pi$. We have $t = s + k\pi$. This yields:

$$(t-k\pi)p(t) = \frac{s}{(s+k\pi)\sin(s+k\pi)} = \frac{s}{\pm(s+k\pi)\sin s}.$$

The function

$$\frac{\sin s}{s} = \sum_{n=0}^{\infty}(-1)^n \frac{s^{2n}}{(2n+1)!}$$

is analytic at zero, and is not zero at zero. Therefore, $\dfrac{s}{\pm(s+k\pi)\sin s}$ is analytic at zero. Similarly, $\dfrac{(t-k\pi)^2\cos t}{t^2\sin t}$ is also analytic at $k\pi$. Therefore, points of the form $k\pi$ with k a nonzero integer are all regular singular points.

(b) The coefficients are all analytic and the roots of $(t^2-1)^3$ are $t=\pm 1$. The equation can be written as

$$y'' + \frac{(1-t)^2}{(t^2-1)^3}y' + \frac{\sin(1-t)}{(t^2-1)^3}\,y = 0.$$

The coefficients are then

$$\frac{1}{(t-1)(t+1)^3}, \text{ and } \frac{\sin(1-t)/(t-1)}{(t-1)^2(1+t)^3}.$$

Note that $1/(t+1)^3$ is analytic near $t=1$, and $\dfrac{\sin(1-t)/(t-1)}{(1+t)^3}$ is analytic near $t=1$, since similar to above $\sin(1-t)/(t-1)$ is analytic. Therefore, $t=1$ is a regular singularity.

Near $t=-1$, the function $\dfrac{1}{(t-1)(t+1)^2}$ is not analytic, as it is unbounded. Thus $t=-1$ is an irregular singularity. □

Example 9.14. Find the first six coefficients of the Taylor series of the solution centered at $t_0=0$ to each IVP.

(a) $y'' + (\sin t)y' + y = 0,\ y(0)=1, y'(0)=0.$

(b) $y'' + (t^3-1)y' + e^t y = 0,\ y(0)=0,\ y'(0)=2.$

Solution. First note that since all coefficients are analytic the solutions to both equations are analytic.
(a) Let $y = \sum\limits_{n=0}^{\infty} a_n t^n$. We have

$$y' = \sum_{n=1}^{\infty} n a_n t^{n-1},\ y'' = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}.$$

Substituting y, y', y'' and $\sin t$ by their power series we obtain:

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \left(\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{(2n+1)!}\right)\sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n = 0.$$

Comparing the coefficients we obtain the following:

$$2a_2 + a_0 = 6a_3 + a_1 + a_1 = 12a_4 + 2a_2 + a_2 = 20a_5 + 3a_3 + \frac{-1}{3!}a_1 + a_3 = 0.$$

Note that $y(0) = 1$ implies $a_0 = 1$, and $y'(0) = 0$ implies $a_1 = 0$. Substituting and solving the above system we obtain $a_2 = -1/2, a_3 = 0, a_4 = 1/8, a_5 = 0$.

(b) Similar to above we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + (t^3 - 1)\sum_{n=1}^{\infty} n a_n t^{n-1} + \left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\right)\sum_{n=0}^{\infty} a_n t^n = 0.$$

Comparing the coefficients we obtain:

$$2a_2 - a_1 + a_0 = 6a_3 - 2a_2 + a_1 + \frac{a_0}{1!} = 12a_4 - 3a_3 + a_2 + \frac{a_1}{1!} + \frac{a_0}{2!} = 20a_5 + a_1 - 4a_4 + a_3 + \frac{a_2}{1!} + \frac{a_1}{2!} + \frac{a_0}{3!} = 0.$$

The given assumption yileds $a_0 = 0, a_1 = 2$. Substituting and solving the system we obtain $a_2 = 1, a_3 = 0, a_4 = -1/4, a_5 = -1/4$. □

Example 9.15. Solve each equation. Write down each solution in closed form.

(a) $(1 + t^2)y'' - 6ty' + 6y = 0$ with $t \in \mathbb{R}$.

(b) $(1 - t^2)y'' - 8ty' - 12y = 0$ near $t = 0$.

Solution. (a) Note that the cofficients $-6t/(1 + t^2)$ and $6/(1 + t^2)$ are analytic near $t = 0$. Thus, any solution near $t_0 = 0$ is analytic. Let $y = \sum_{n=0}^{\infty} a_n t^n$. This yields $y' = \sum_{n=1}^{\infty} n a_n t^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$. Substituting into the equation and distributing $1 + t^2$ we obtain the following:

$$\begin{aligned} &\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n t^n - 6\sum_{n=1}^{\infty} n a_n t^n + 6\sum_{n=0}^{\infty} a_n t^n \\ &= 2a_2 + 6a_3 t - 6a_1 t + 6a_0 + 6a_1 t + \sum_{n=2}^{\infty}((n+2)(n+1)a_{n+2} + n(n-1)a_n - 6na_n + 6a_n)t^n = 0 \end{aligned}$$

This yields the following system:

$$\begin{cases} 2a_2 + 6a_0 = 0 \Rightarrow a_2 = -3a_0 \\ a_3 = 0 \\ (n+2)(n+1)a_{n+2} + (n^2 - 7n + 6)a_n = 0 \Rightarrow a_{n+2} = -\dfrac{(n-6)(n-1)}{(n+2)(n+1)}a_n, \text{ if } n \geq 2 \end{cases}$$

Substituting $n = 2, 4, 6$ in the last equality we obtain

$$a_4 = \frac{4}{12}a_2 = -a_0, \; a_6 = \frac{6}{30}a_4 = -\frac{a_0}{5}, \; a_8 = 0.$$

Since a_{n+2} is a multiple of a_n we obtain $a_{2n} = 0$ for all $n \geq 4$. Similarly, since $a_3 = 0$ we have $a_{2n+1} = 0$ for all $n \geq 1$. Therefore, the general solution is given by

$$y = a_0 + a_1 t - 3a_0 t^2 - a_0 t^4 - \frac{a_0}{5}t^6 = a_0(1 - 3t^2 - t^4 - \frac{1}{5}t^6) + a_1 t.$$

Note that since the above solution is a polynomial, substituting into the equation we get a solution that is valid over $\mathbb{R}$. On the other hand, since the coefficients $-6t/(1 + t^2)$ and $6/(1 + t^2)$ are continuous over $\mathbb{R}$, by the Existence and Uniqueness Theorem for Linear Equations, the above solution is the unique solution satisfying $y(0) = a_0, y'(0) = a_1$. Thus, the general solution over $\mathbb{R}$ is given by $y = a_0(1-3t^2-t^4-\frac{1}{5}t^6)+a_1t$.

(b) Similar to above, both coefficients $-8t/(1-t^2)$ and $-12/(1-t^2)$ are analytic near $t_0 = 0$. Thus, all solutions of the equation are analytic near $t_0 = 0$. Substituting $y = \sum_{n=0}^{\infty} a_n t^n$ we obtain:

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n t^n - 8\sum_{n=1}^{\infty} n a_n t^n - 12\sum_{n=0}^{\infty} a_n t^n \\
&= 2a_2 + 6a_3t - 8a_1t - 12a_0 - 12a_1t + \sum_{n=2}^{\infty}((n+2)(n+1)a_{n+2} - n(n-1)a_n - 8na_n - 12a_n)t^n = 0
\end{aligned}$$

This yields the following system:

$$\begin{cases}
-12a_0 + 2a_2 = 0 \Rightarrow a_2 = 6a_0 \\
6a_3 - 20a_1 = 0 \Rightarrow a_3 = \dfrac{20}{6}a_1 \\
(n+2)(n+1)a_{n+2} - (n^2+7n+12)a_n = 0 \Rightarrow a_{n+2} = \dfrac{(n+3)(n+4)}{(n+1)(n+2)}a_n, \text{ for all } n \geq 2
\end{cases} \qquad (*)$$

Substituting $n = 2, 4, 6$ in the last equation we obtain

$$a_4 = \frac{5\cdot 6}{3\cdot 4}a_2 = \frac{5\cdot 6}{2}a_0, a_6 = \frac{7\cdot 8}{5\cdot 6}a_4 = \frac{7\cdot 8}{2}a_0, a_8 = \frac{9\cdot 10}{7\cdot 8}a_6 = \frac{9\cdot 10}{2}a_0.$$

Similar equalities hold for when n is odd.

$$a_5 = \frac{6\cdot 7}{4\cdot 5}a_3 = \frac{6\cdot 7}{4\cdot 5}\frac{20}{6}a_1 = \frac{6\cdot 7}{6}a_1, a_7 = \frac{8\cdot 9}{6\cdot 7}a_5 = \frac{8\cdot 9}{6\cdot 7}\frac{6\cdot 7}{6}a_1 = \frac{8\cdot 9}{6}a_1.$$

We will now prove by induction that for every $n \geq 0$ we have

$$\begin{cases}
a_{2n} = \dfrac{(2n+1)(2n+2)}{2}a_0 \\
a_{2n+1} = \dfrac{(2n+2)(2n+3)}{6}a_1
\end{cases} \qquad (**)$$

The base case is $a_0 = \frac{1\cdot 2}{2}a_0$ and $a_1 = \frac{2\cdot 3}{6}a_1$, which are both clear.

For the inductive step, assume the statement $(**)$ is true for n. Using $(*)$ and the indutive hypotheses we obtain:

$$\begin{cases}
a_{2n+2} = \dfrac{(2n+3)(2n+4)}{(2n+1)(2n+2)}a_{2n} = \dfrac{(2n+3)(2n+4)}{(2n+1)(2n+2)}\dfrac{(2n+1)(2n+2)}{2}a_0 \\
\\
a_{2n+3} = \dfrac{(2n+4)(2n+5)}{(2n+2)(2n+3)}a_{2n+1} = \dfrac{(2n+4)(2n+5)}{(2n+2)(2n+3)}\dfrac{(2n+2)(2n+3)}{6}a_1 = \dfrac{(2n+4)(2n+5)}{6}a_1
\end{cases}$$

This completes the proof of $(**)$. Therefore, the general solution is given by

$$y = a_0\sum_{n=0}^{\infty}\frac{(2n+1)(2n+2)}{2}t^{2n} + a_1\sum_{n=0}^{\infty}\frac{(2n+2)(2n+3)}{6}t^{2n+1}.$$

The first sums can be written in closed form, by differentiating the following power series twice:

$$\frac{1}{1-t^2}=\sum_{n=0}^{\infty}t^{2n}\Rightarrow\frac{2t}{(1-t^2)^2}=\sum_{n=1}^{\infty}2nt^{2n-1}\Rightarrow\frac{2+6t^2}{(1-t^2)^3}=\sum_{n=1}^{\infty}2n(2n-1)t^{2n-2}.$$

Therefore,

$$\sum_{n=0}^{\infty}\frac{(2n+2)(2n+1)}{2}t^{2n}=\frac{1+3t^2}{(1-t^2)^3}.$$

Similarly by differentiating the power series for $t/(1-t^2)$ twice we obtain

$$\sum_{n=0}^{\infty}\frac{(2n+2)(2n+3)}{6}t^{2n+1}=\frac{3t+t^3}{3(1-t^2)^3}.$$

Therefore, the general solution to this equation is

$$y=\frac{3a_0(1+3t^2)+a_1(3t+t^3)}{3(1-t^2)^3}.$$

□

Example 9.16. Find the general solution to $y''+t^5y'+6t^4y=0$ with $t\in\mathbb{R}$ as a power series. Use that to solve the IVP.

$$y''+t^5y'+6t^4y=0,\ y(0)=0,\ y'(0)=1.$$

Your solution to this IVP must be in closed form.

Solution. Since coefficients t^5 and $6t^4$ are analytic over $\mathbb{R}$, all solutions are analytic over $\mathbb{R}$. Substituting $y=\sum\limits_{n=0}^{\infty}a_nt^n$ we obtain

$$\sum_{n=2}^{\infty}n(n-1)a_nt^{n-2}+t^5\sum_{n=1}^{\infty}na_nt^{n-1}+6t^4\sum_{n=0}^{\infty}a_nt^n=0$$
$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n+\sum_{n=5}^{\infty}(n-4)a_{n-4}t^n+\sum_{n=4}^{\infty}6a_{n-4}t^n=0$$
$$2a_2+6a_3t+12a_4t^2+20a_5t^3+30a_6t^4+6a_0t^4+\sum_{n=5}^{\infty}[(n+2)(n+1)a_{n+2}+(n-4)a_{n-4}+6a_{n-4}]t^n=0$$

Therefore, we obtain the following:

$$\begin{cases}30a_6+6a_0=0\Rightarrow a_6=\dfrac{-a_0}{5}\\ a_2=a_3=a_4=a_5=0.\\ a_{n+2}=-\dfrac{a_{n-4}}{n+1},\ \text{for all } n\geq 5 \qquad (*)\end{cases}$$

Since a_{n+4} is a multiple of a_{n-4}, and $a_2=a_3=a_4=a_5=0$, we have $a_8=a_9=a_{10}=a_{11}=0$, and similarly $a_{14}=a_{15}=a_{16}=a_{17}=0$. Therefore, the only coefficients that may be nonzero are those of the form a_{6n} and a_{6n+1}. We evaluate a few terms using $(*)$.

$$a_6=-\frac{1}{5}a_0, a_{12}=-\frac{a_6}{11}=\frac{a_0}{5\cdot 11}, a_{18}=-\frac{a_{12}}{17}=-\frac{a_0}{5\cdot 11\cdot 17}.$$

Similarly we have

$$a_7=-\frac{a_1}{6}, a_{13}=\frac{a_1}{6\cdot 12}, a_{19}=\frac{a_1}{6\cdot 12\cdot 18}.$$

We will prove by induction on n that

$$\begin{cases} a_{6n} = (-1)^n \dfrac{a_0}{5 \cdot 11 \cdots (6n-1)} \\ a_{6n+1} = (-1)^n \dfrac{a_1}{6 \cdot 12 \cdots (6n)} = (-1)^n \dfrac{a_1}{6^n n!} \end{cases}$$

The base case is clear. The inductive step follows from $(*)$. Therefore, the general solution is given by

$$y = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n t^{6n}}{5 \cdot 11 \cdots (6n-1)} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n t^{6n+1}}{6^n n!}.$$

When $y(0) = 0$ and $y'(0) = 1$ we have $a_0 = 0$ and $a_1 = 1$. Thus, we obtain the following

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n t^{6n+1}}{6^n n!} = t \sum_{n=0}^{\infty} \frac{(-t^6)^n}{6^n n!} = te^{-t^6/6}.$$

□

Example 9.17. Using power series, find the general solution of the equation

$$2ty'' + y' + 2ty = 0, \ t > 0.$$

Solution. (a) We see $p(t) = \frac{1}{2t}$, and $q(t) = 1$. This gives $tp(t) = 1/2$ and $t^2 q(t) = t^2$, which means $t_0 = 0$ is a regular singularity. The indicial equation is $r(r-1) + \frac{1}{2}r = r(r - 1/2)$. The roots are $r = 0, 1/2$. By Theorem 9.4, this equation has two solutions of the form

$$y_1 = \sum_{n=0}^{\infty} a_n t^n, \text{ and } y_2 = \sqrt{t} \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+1/2}.$$

The first one yields:

$$\begin{array}{cl} & 2ty'' + y' + 2ty \\ = & \sum\limits_{n=2}^{\infty} 2n(n-1)a_n t^{n-1} + \sum\limits_{n=1}^{\infty} n a_n t^{n-1} + \sum\limits_{n=0}^{\infty} 2a_n t^{n+1} \\ = & \sum\limits_{n=1}^{\infty} 2(n+1)n a_{n+1} t^n + \sum\limits_{n=0}^{\infty} (n+1)a_{n+1} t^n + \sum\limits_{n=1}^{\infty} 2a_{n-1} t^n \\ = & a_1 + \sum\limits_{n=1}^{\infty} [2(n+1)n a_{n+1} + (n+1)a_{n+1} + 2a_{n-1}] t^n \end{array}.$$

Setting this equal to zero we obtain the following system:

$$\begin{cases} a_1 = 0 \\ (n+1)(2n+1)a_{n+1} + 2a_{n-1} = 0 \Rightarrow a_{n+1} = -\dfrac{2}{(n+1)(2n+1)} a_{n-1} \text{ for all } n \geq 1 \end{cases} \qquad (*)$$

Since a_{n+1} is a multiple of a_{n-1} and $a_1 = 0$ we obtain $a_{2n-1} = 0$ for all $n \geq 1$.

Setting $n = 1, 3, 5$ into $(*)$ we obtain the following:

$$a_2 = -\frac{2}{2 \cdot 3} a_0, a_4 = -\frac{2}{4 \cdot 7} a_2 = \frac{2^2}{2 \cdot 3 \cdot 4 \cdot 7} a_0, a_6 = -\frac{2}{6 \cdot 11} a_4 = \frac{2^3}{2 \cdot 3 \cdot 4 \cdot 7 \cdot 6 \cdot 11} a_0.$$

By induction we can prove

$$a_{2n} = (-1)^n \frac{2^n}{2 \cdot 4 \cdot 6 \cdots (2n) \cdot 3 \cdot 7 \cdots (4n-1)} a_0.$$

(Prove this!) We can write $2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!$, so this can be written as

$$a_{2n} = (-1)^n \frac{a_0}{n! \cdot 3 \cdot 7 \cdots (4n-1)}.$$

Setting $a_0 = 1$ we obtain the solution:

$$y_1 = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n}}{n! \cdot 3 \cdot 7 \cdots (4n-1)}.$$

Substituting y_2 into the equation we obtain

$$\begin{aligned} & 2ty'' + y' + 2ty \\ = \; & 2\sum_{n=0}^{\infty}(n+1/2)(n-1/2)a_n t^{n-1/2} + \sum_{n=0}^{\infty}(n+1/2)a_n t^{n-1/2} + \sum_{n=0}^{\infty} 2a_n t^{n+3/2} \\ = \; & -\tfrac{1}{2}a_0 t^{-1/2} + \tfrac{3}{2}a_1 t^{1/2} + \tfrac{1}{2}a_0 t^{-1/2} + \tfrac{3}{2}a_1 t^{1/2} + \sum_{n=2}^{\infty}[\tfrac{(2n+1)(2n-1)}{2}a_n + \tfrac{2n+1}{2}a_n + 2a_{n-2}]t^{n-1/2} \end{aligned}$$

Setting this equal to zero, we obtain the following

$$\begin{cases} 3a_1 = 0 \Rightarrow a_1 = 0 \\ (2n+1)na_n + 2a_{n-2} = 0 \Rightarrow a_n = -\dfrac{2}{n(2n+1)}a_{n-2} \text{ for all } n \geq 2 \end{cases} \qquad (*)$$

Since a_n is a multiple of a_{n-2} and $a_1 = 0$ we obtain $a_{2n-1} = 0$ for all $n \geq 1$.

Setting $n = 2, 4, 6$ into $(*)$ we obtain the following:

$$a_2 = -\frac{2}{2\cdot 5}a_0,\, a_4 = -\frac{2}{4\cdot 9}a_2 = \frac{2^2}{2\cdot 5\cdot 4\cdot 9},\, a_6 = -\frac{2}{6\cdot 13}a_4 = -\frac{2^3}{2\cdot 5\cdot 4\cdot 9\cdot 6\cdot 13}a_0.$$

By induction we will show

$$a_{2n} = \frac{(-1)^n 2^n a_0}{2\cdot 4\cdots (2n)\cdot 5\cdot 9\cdots (4n+1)}.$$

(Prove this!) This simplifies to

$$a_{2n} = \frac{(-1)^n 2^n a_0}{2^n n! \cdot 5\cdot 9\cdots (4n+1)} = \frac{(-1)^n a_0}{n!\cdot 5\cdot 9\cdots (4n+1)}.$$

Setting $a_0 = 1$ we obtain the following solution:

$$y_2 = t^{1/2} + \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n+1/2}}{n!\cdot 5\cdot 9\cdots (4n+1)}.$$

The general solution is thus

$$y = c_1\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n t^{2n}}{n!\cdot 3\cdot 7\cdots (4n-1)}\right) + c_2\sqrt{t}\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n t^{2n}}{n!\cdot 5\cdot 9\cdots (4n+1)}\right).$$

$\square$

Example 9.18. Find one nontrivial power series solution to the equation:

$$t^2y'' - 2ty' + (t^2+2)y = 0,\; t > 0.$$

Your answer must be in closed form.

Solution. We see that $p(t) = -2/t$, $q(t) = (t^2+2)/t^2$ are continuous everywhere except at $t_0 = 0$, and they are both unbounded near zero. Thus there is a singularity at zero. This singularity is regular, since $tp(t) = -2$, $t^2 q(t) = t^2 + 2$ are both analytic.

The indicial equation is $r(r-1) - 2r + 2 = 0$. The roots are $r = 1, 2$. By Theorem 9.4 the solution corresponding to $r = 2$ is of the form $y = t^2 \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+2}$. Substituting we obtain

$$t^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_n t^n - 2t \sum_{n=0}^{\infty} (n+2) a_n t^{n+1} + t^2 \sum_{n=0}^{\infty} a_n t^{n+2} + 2 \sum_{n=0}^{\infty} a_n t^{n+2} = 0.$$

This yields the following:

$$2a_0 t^2 + 6a_1 t^3 - 4a_0 t^2 - 6a_1 t^3 + 2a_0 t^2 + 2a_1 t^3 + \sum_{n=4}^{\infty} [n(n-1)a_{n-2} - 2na_{n-2} + a_{n-4} + 2a_{n-2}] t^n = 0.$$

Therefore, we obtain the following system:

$$\begin{cases} a_1 = 0 \\ (n^2 - 3n + 2)a_{n-2} + a_{n-4} = 0 \Rightarrow a_{n-2} = -\dfrac{a_{n-4}}{(n-1)(n-2)} \text{ for all } n \geq 4 \end{cases}$$

Since a_{n-2} is a multiple of a_{n-4} for every $n \geq 4$, and $a_1 = 0$ we conclude that $a_{2n+1} = 0$ for all $n \geq 1$. Substituting $n = 4, 6, 8$ we obtain

$$a_2 = -\frac{a_0}{3 \cdot 2} = \frac{-a_0}{3!}, a_4 = -\frac{a_2}{5 \cdot 4} = \frac{a_0}{5!}, a_6 = -\frac{a_4}{7 \cdot 6} = -\frac{a_0}{7!}.$$

By induction we can show $a_{2n} = (-1)^n \dfrac{a_0}{(2n+1)!}$. (Show this!) Setting $a_0 = 1$ we obtain one nontrivial solution as

$$\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+2}}{(2n+1)!} = t \sin t.$$

□

Example 9.19. Solve the initial value problem

$$2y'' - (5t+2)y' + (t+9)y = t^3 + 2, \ y(0) = 0, \ y'(0) = 1.$$

Your solution must be in closed form.

Solution. First, note that since this equation is not homogeneous, we may not use any of the theorems in this section, however we know that the Existence and Uniqueness Theorem for Linear Equations guarantees the existence of a unique solution. We will see if we can find an analytic solution. Substituting $y = \sum_{n=0}^{\infty} a_n t^n$ we obtain

$$\begin{aligned} & 2y'' - (5t+2)y' + (t+9)y \\ = \ & 2\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - 5\sum_{n=1}^{\infty} na_n t^n - 2\sum_{n=1}^{\infty} na_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^{n+1} + 9\sum_{n=0}^{\infty} a_n t^n \\ = \ & 2\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n - 5\sum_{n=1}^{\infty} na_n t^n - 2\sum_{n=0}^{\infty} (n+1)a_{n+1} t^n + \sum_{n=1}^{\infty} a_{n-1} t^n + 9\sum_{n=0}^{\infty} a_n t^n \\ = \ & 4a_2 - 2a_1 + 9a_0 + \sum_{n=1}^{\infty} [2(n+2)(n+1)a_{n+2} - 5na_n - 2(n+1)a_{n+1} + a_{n-1} + 9a_n] t^n \end{aligned} \quad (*)$$

Setting this equal to $t^3 + 2$ we obtain the following system:

$$\begin{cases} 4a_2 - 2a_1 + 9a_0 = 2 \\ 2(n+2)(n+1)a_{n+2} - (5n-9)a_n - 2(n+1)a_{n+1} + a_{n-1} = 0 \text{ if } n \neq 0, 3 \\ 2(5)(4)a_5 - 5(3)a_3 - 2(4)a_4 + a_2 + 9a_3 = 1. \end{cases}$$

By assumption, $a_0 = 0, a_1 = 1$. Substituting into the first equation we obtain $a_2 = 1$. Letting $n = 1$ into the second equation we obtain

$$12a_3 + 4a_1 - 4a_2 + a_0 = 0 \Rightarrow a_3 = 0.$$

Substituting $n = 2$ into the second equation we obtain

$$24a_4 - a_2 - 6a_3 + a_1 = 0 \Rightarrow a_4 = 0.$$

The last equation in $(*)$ yields:

$$40a_5 - 6 \cdot 0 - 8 \cdot 0 + 1 = 1 \Rightarrow a_5 = 0.$$

So far we have shown $a_3 = a_4 = a_5 = 0$. The second equation in $(*)$ yields

$$a_{n+2} = \frac{(5n-9)a_n + 2(n+1)a_{n+1} - a_{n-1}}{2(n+2)(n+1)}.$$

A simple induction shows $a_n = 0$ for all $n \geq 3$. (Show this!) Therefore, $y = t + t^2$ is the unique solution to this IVP. $\square$

9.4 Exercises

Exercise 9.1. *Find the general solution to each equation:*

(a) $t^2y'' - 6y = 0$, *with* $t > 0$.

(b) $t^2y'' + 3ty' + 2y = 0$, *with* $t > 0$.

(c) $t^2y'' + 5ty' + 4y = 0$, *with* $t < 0$.

(d) $t^2y'' + \alpha ty' + y = 0$, *with* $t < 0$.

Exercise 9.2. *Find a linear equation whose general solution is each of the following:*

(a) $y = \dfrac{c_1}{t} + c_2t^7$.

(b) $y = c_1 t\cos(2\ln t) + c_2 t\sin(2\ln t)$.

(c) $y = c_1t^3 + c_2t^3\ln t$.

Exercise 9.3. *Find the general solution of the equation* $4t^2y'' - 4ty' + 3y = \sqrt{t}$, *where* $t > 0$.

Hint: First find the general solution to the homogeneous equation. Then use Variation of Parameters.

Exercise 9.4. *Find the general solution to the differential equation* $12t^3y''' + 32t^2y'' + 5ty' + y = 0$, *given* $t > 0$.

Hint: This is a higher order Euler's equation.

Exercise 9.5. *For each equation, find the largest interval for which all solutions are guaranteed to be equal to their Taylor series centered at* t_0.

(a) $t^2y'' + t(\sin t)y' + (e^{t^2} - 1)y = 0,\ t_0 = 0.$

(b) $(1 + t^2)y' + ty'' + y = 0,\ t_0 = 0.$

(c) $ty'' + y' + y = 0,\ t_0 = 1.$

Exercise 9.6. *Solve the initial value problem*

$$y'' + ty' + y = 0,\ y(0) = 1,\ y'(0) = 0.$$

Exercise 9.7. *Consider the differential equation*

$$t^3y'' + (t - t^2)y' - 2y = 0.$$

Prove that even though $t = 0$ *is an irregular singularity for this equation, this equation has a nontrivial solution that is analytic on* $\mathbb{R}$.

Exercise 9.8. *Find all singularities of each equation. Determine if each singularity is regular or irregular.*

(a) $(t\sin t)y'' + t^3y' + y = 0.$

(b) $(t^2 - t)^3y'' + (t - 1)^2y' + (t^3 - 1)y = 0.$

(c) $y'' + (t - 1)y' + (\cos t)y = 0.$

(d) $(\cos t)y'' + ty' + (\sin t)y = 0$

Exercise 9.9. *Solve each equation. Your solutions may be written as power series.*

(a) $(1 + t^2)y'' + 3ty' + y = 0.$

(b) $2ty'' + y' + ty = 0,\ t > 0.$

(c) $4ty'' + 3y' + 3y = 0,\ t > 0.$

(d) $(t^2 - 4)y'' + 3ty' + y = 0,\ t \in (-2, 2).$

Exercise 9.10. *Solve each initial value problem. Your solution must be in closed form.*

(a) $y'' - 2ty' - 2y = 0,\ y(0) = 1,\ y'(0) = 0.$

(b) $y'' + (t-4)y' + (3-2t)y = 0,\ y(0) = 0,\ y'(0) = 1.$

(c) $y'' + (t-3)y' - 3ty = 0,\ y(0) = 1,\ y'(0) = 3.$

(d) $y'' + t^2y' + 2ty = 0,\ y(0) = 1,\ y'(0) = 0.$

Exercise 9.11. *For a positive real number α, consider the Bessel equation of order α given by:*

$$t^2y'' + ty' + (t^2 - \alpha^2)y = 0,\ t > 0.$$

(a) Find a nontrivial solution for this equation.

(b) Find a second linearly independent solution if 2α is not an integer.

Exercise 9.12. *Let α be a real constant. Consider the Legendre's equation given by*

$$(1-t^2)y'' - 2ty' + (\alpha^2 + \alpha)y = 0.$$

(a) Find a FSoS for this equation.

(b) Show that if α is a positive integer, then this equation has a polynomial solution.

Exercise 9.13. *Find one solution to the equation:*

$$ty'' + y' + y = 0,\ t > 0.$$

Exercise 9.14. *For each equation find a FSoS.*

(a) $ty'' + ty' + 2y = 0,\ t > 0.$

(b) $ty'' + (1-t^2)y' + 4ty = 0,\ t > 0.$

(c) $t^2y'' + (t^2 - 3t)y' + 3y = 0,\ t > 0.$

Hint: Find the exponents at singularity $t_0 = 0$. Then find the solution corresponding to the larger exponent. Then apply the method of Reduction of order to find a second solution.

9.5 Challenge Problems

Exercise 9.15. *Solve the initial value problem*

$$\cos(t)y'' - \sin(t)y' + 2\cos(t)y = -2\sin(3t), y(0) = 0, y'(0) = 2.$$

Hint: First, prove that the solution is odd. This simplifies the form of y. Then use a power series centered at zero. Find the first few coefficients, and prove the pattern by induction.

Exercise 9.16. *Consider the differential equation*

$$t^4 y'' - 2ty' + y = 0, \ t \in \mathbb{R}.$$

Prove that this equation has a solution that is C^∞, but is not analytic.

Do the same for the equation

$$t^6 y'' + t^6 y' + (-4 + 6t^2 - 2t^3)y = 0, \ t \in \mathbb{R}.$$

Exercise 9.17. *Suppose $p(t), q(t), r(t)$ are C^∞ functions satisfying $p(0) = p'(0) = 0$, and $q(0) \neq 0$. Prove that the only solution to the following problem that is analytic over $\mathbb{R}$ is the trivial solution $y = 0$:*

$$p(t)y'' + q(t)y' + r(t)y = 0, \ y(0) = 0, \ t \in \mathbb{R}.$$

Exercise 9.18. *Solve the initial value problem.* $(1 - 2t^3)y'' - 10t^2 y' - 8ty = 0, \ y(0) = 1, \ y'(0) = 0.$

9.6 Summary

- To solve the Euler's equation $t^2 y'' + \alpha t y' + \beta y = 0$ we set $y = t^r$ and solve for r. Let r_1, r_2 be roots of the indicial equation $r(r-1) + \alpha r + \beta = 0$. (You do not need to memorize this. You can obtain it by substituting $y = t^r$.)
  - If $r_1 \neq r_2$ are real, then the general solution is $y = c_1 t^{r_1} + c_2 t^{r_2}$.
  - If $r_1 = r_2 = r$, then the general solution is $y = c_1 t^r + c_2 t^r \ln t$.
  - If $r_1, r_2 = a \pm ib$ are nonreal, then the general solution is $y = c_1 t^a \cos(b \ln t) + c_2 t^a \sin(b \ln t))$.
- If $p(t)$ and $q(t)$ are analytic near t_0, then we can solve $y'' + p(t)y' + q(t)y = 0$ by setting $y = \sum_{n=0}^{\infty} a_n (t - t_0)^n$, and finding a recursion for a_n. Note that by the Existence and Uniqueness Theorem, we must be able to find every a_n in terms of a_0 and a_1.
- When $p(t)$ or $q(t)$ are continuous but unbounded near t_0, we say t_0 is a singularity for the equation $y'' + p(t)y' + q(t)y = 0$.
- A singular point t_0 is called regular if $(t - t_0)p(t)$ and $(t - t_0)^2 q(t)$ are analytic at t_0.
- To solve an equation near a regular singularity we will use Theorem 9.4 to find r_1, r_2 and the format of each solution. Then find a recursion for a_n.
- If needed, we can always center everything at zero by using: $s = t - t_0$, $z(s) = y(s + t_0)$. This changes the initial values to $z(0) = y(t_0)$ and $z'(0) = y'(t_0)$.
- When solving recursions, find the first few terms of the sequence, guess the general term, and prove it by induction.

Chapter 10

Laplace Transform

Definition 10.1. Laplace transform, denoted by $\mathcal{L}$, assigns to any function $f(t)$ defined for all $t \geq 0$ the function

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\ dt,$$

for every s that the improper integral on the right converges. The Laplace of $f(t)$ at s is usually denoted by $F(s)$.

Note that by properties of integrals, $\mathcal{L}$ is linear.

Example 10.1. Find the Laplace transform of each of the following:

(a) e^{at}, where a is a real constant.

(b) $e^{at}\cos(bt)$, where a, b are real constants.

(c) $e^{at}\sin(bt)$, where a, b are real constants.

The improper integral in the definition of Laplace transform may not always converge. This typically happens in two instances:

- when $f(t)$ is discontinuous at too many points for $e^{-st}f(t)$ to be integrable, or
- when $e^{-st}f(t)$ is too large for the area underneath its graph to be finite. This typically (but not always, see Example 10.18) means $f(t)$ grows faster than e^{st}.

Definition 10.2. We say a function $f : [0, \infty) \to \mathbb{R}$ is **piecewise continuous** if for every $r > 0$ the function f is continuous over $[0, r]$ except possibly at finitely many points.

Definition 10.3. A function $f : [0, \infty) \to \mathbb{R}$ is said to be of **exponential order** if there are constants c, M for which $|f(t)| \leq Me^{ct}$ for all $t \geq 0$. In that case we say f is of exponential order not exceeding c.

Example 10.2. The following are some examples of functions that are of exponential order:

(a) $\sin t$ is of exponential order not exceeding 0.

(b) The function t is of exponential order not exceeding c for every positive real number c.

Theorem 10.1. *Suppose $f : [0, \infty) \to \mathbb{R}$ is piecewise continuous, and of exponential order. Then, its Laplace transform $\mathcal{L}\{f(t)\}(s)$ exists for sufficiently large s. Specifically, if f is piecewise continuous, and $|f(t)| \leq Me^{ct}$, for constants c, M, and for all $t \geq 0$, then $\mathcal{L}\{f(t)\}(s)$ exists for all $s > c$.*

It turns out that we are able to recover a function from its Laplace transform. The following theorem indicates that fact.

Theorem 10.2. *Suppose $f(t)$ and $g(t)$ are two functions continuous over $[0, \infty)$, both of which are of exponential order. Assume there is a real number A for which $\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{g(t)\}(s)$ for all $s > A$. Then $f(t) = g(t)$ for all $t \in [0, \infty)$.*

The above theorem shows that Laplace transform has an inverse. We denote this inverse by $\mathcal{L}^{-1}$.

Theorem 10.3. *Suppose $f : [0, \infty) \to \mathbb{R}$ is of exponential order. Let $L[y] = f(t)$ be a linear equation with constant coefficients. Then, every solution to this equation is of exponential order.*

Example 10.3. Solve $y' - 2y = e^{5t},\ y(0) = 3$.

Theorem 10.4. *Suppose $f : [0, \infty) \to \mathbb{R}$ is n times differentiable, $f^{(n)}(t)$ is piecewise continuous, and of exponential order not exceeding c. Let $F(s) = \mathcal{L}\{f(t)\}(s)$. Then,*

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0),$$

for every $s > c$.

Theorem 10.5. *Suppose $f : [0, \infty) \to \mathbb{R}$ is piecewise continuous and of exponential order not exceeding c. Then, its Laplace transform $F(s)$ is infinitely differentiable and for every positive integer n and every real number a we have:*

(a) $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$, for all $s > c$.

(b) $\mathcal{L}\{e^{at} f(t)\}(s) = F(s - a)$, for all $s > a + c$.

Example 10.4. Find the inverse Laplace of $\dfrac{1}{(s+1)^2}$.

Example 10.5. Using the method of Laplace Transform solve the initial value problem:

$$y''' + 2y'' + y' = 0, y(0) = 1, y'(0) = y''(0) = 0.$$

Example 10.6. Find the inverse Laplace of $\dfrac{5}{s^4 + 13s^2 + 36}$.

We often find Laplace of piecewise defined functions using the so-called Heaviside function defined below:

Definition 10.4. The **Heaviside** step function H is defined as

$$H(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Note that for every constant c we have

$$H(t-c) = \begin{cases} 1 & \text{if } t \geq c \\ 0 & \text{otherwise} \end{cases}$$

Example 10.7. Find $\mathcal{L}\{H(t-c)f(t-c)\}$ in terms of $F(s)$, the Laplace of f.

Example 10.8. Find the Laplace of $f(t)$, where $f(t) = \begin{cases} t^2 & \text{if } 0 \leq t < 2 \\ 1-t & \text{if } 2 \leq t < 3 \\ 1 & \text{if } t \geq 3 \end{cases}$

Definition 10.5. The **convolution** of two functions $f, g : [0, \infty) \to \mathbb{R}$ is defined by

$$(f \star g)(t) = \int_0^t f(x)g(t-x)\, dx.$$

Theorem 10.6. *Suppose $f, g : [0, \infty) \to \mathbb{R}$ are piecewise continuous and of exponential order not exceeding c. Then,*

$$\mathcal{L}\{f \star g\}(s) = \mathcal{L}\{f\}(s) \cdot \mathcal{L}\{g\}(s), \text{ for all } s > c.$$

Example 10.9. Find the inverse Laplace of $\dfrac{1}{s^4 + 2s^2 + 1}$

Table of Laplace Transform

$j(t) = \mathcal{L}^{-1}[J(s)]$	$J(s) = \mathcal{L}[j(t)]$
e^{at}, a is real	$\dfrac{1}{s-a}$
t^n, n is a nonnegative integer	$\dfrac{n!}{s^{n+1}}$
$e^{at}t^n, a$ is real and n is a nonnegative integer	$\dfrac{n!}{(s-a)^{n+1}}$
$e^{at}\sin(bt), a, b$ are real	$\dfrac{b}{(s-a)^2 + b^2}$
$e^{at}\cos(bt), a, b$ are real	$\dfrac{s-a}{(s-a)^2 + b^2}$
$j^{(n)}(t)$	$s^n J(s) - s^{n-1}j(0) - s^{n-2}j'(0) - \cdots - j^{(n-1)}(0)$
$H(t-c)j(t-c), c \geq 0$	$e^{-cs}J(s)$
$e^{at}j(t), a$ is real	$J(s-a)$
$j(t) \star k(t)$	$J(s)K(s)$
$t^n j(t), n$ is a nonnegative integer	$(-1)^n J^{(n)}(s)$

10.1 More Examples

Example 10.10. Find the Laplace of each function using the Table of Laplace Transform:

(a) $f(t) = \begin{cases} t^2 + t & \text{if } 0 \leq t < 1 \\ 2t^2 + 2t - 1 & \text{if } 1 \leq t \end{cases}$

(b) $\sin(2t) + e^t \cos(5t)$.

Solution. (a) $f(t) = (t^2+t)(H(t) - H(t-1)) + (2t^2+2t-1)H(t-1) = H(t)(t^2+t) + H(t-1)(t^2+t-1)$. From the table of Laplace Transform, we know $\mathcal{L}\{H(t-c)j(t-c)\}(s) = e^{-sc}J(s)$. So, $\mathcal{L}\{H(t)(t^2+t)\}(s) = \mathcal{L}\{t^2+t\} = \dfrac{2!}{s^3} + \dfrac{1!}{s^2}$. For the second part of $f(t)$ we need to first find $j(t)$ for which $j(t-1) = t^2+t-1$. Substituting $t+1$ for t we obtain $j(t) = (t+1)^2 + (t+1) - 1 = t^2 + 3t + 1$. Laplace of this function is $\dfrac{2!}{s^3} + \dfrac{3}{s^2} + \dfrac{0!}{s}$. Putting these together we obtain

$$\mathcal{L}\{f(t)\}(s) = \frac{2!}{s^3} + \frac{1!}{s^2} + e^{-s}\left(\frac{2!}{s^3} + \frac{3}{s^2} + \frac{0!}{s}\right).$$

(b) By linearity of Laplace and using the table we obtain $\mathcal{L}\{\sin(2t)+e^t\cos(5t)\} = \dfrac{2}{s^2+4} + \dfrac{s-1}{(s-1)^2+25}$. □

Example 10.11. Find the inverse Laplace of each of the following:

(a) $\dfrac{s}{s^2+2s+2}$.

(b) $\dfrac{s}{(s^2+1)^2}$.

(c) $\dfrac{e^{-2s}}{s^2+s}$.

Solution. (a) By completing the square we can write

$$\frac{s}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}.$$

From the table, the inverse Laplace of this function is $e^{-t}\cos t - e^{-t}\sin t$.
(b) We notice that the derivative of $\dfrac{1}{s^2+1}$ is $\dfrac{-2s}{(s^2+1)^2}$. Note that $\mathcal{L}^{-1}\{1/(s^2+1)\} = \cos t$. Therefore, from the table

$$\mathcal{L}^{-1}\{\frac{s}{(s^2+1)^2}\} = \frac{-1}{2}\mathcal{L}^{-1}\{\left(\frac{1}{s^2+1}\right)'\} = -\frac{1}{2}t\sin t.$$

(c) From the table $\mathcal{L}^{-1}\{\dfrac{e^{-2s}}{s^2+s}\} = H(t-2)\mathcal{L}^{-1}\{\dfrac{1}{s^2+s}\}(t-2)$. We need to find the inverse Laplace of $\dfrac{1}{s^2+s} = \dfrac{1}{s} - \dfrac{1}{s+1}$. From the table the Laplace inverse of the latter is $1 - e^{-t}$. Thus, the answer is $H(t-2)(1-e^{-t+2})$. □

Example 10.12. For every positive integer n evaluate $\mathcal{L}\{t^n \cos t\}$.

Solution. From the table we know $\mathcal{L}\{t^n\cos t\} = (-1)^n F^{(n)}(s)$, where $F(s) = \mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1}$. So, we need to find the n-th derivative of $s/(s^2+1)$. The first few derivatives of $F(s) = s/(s^2+1)$ are listed below:

$$\begin{aligned} F(s) &= \frac{s}{s^2+1} \\ F'(s) &= \frac{1-s^2}{(s^2+1)^2} \\ F''(s) &= \frac{2s(s^2-3)}{(s^2+1)^3} \\ F'''(s) &= \frac{-6(s^4-6s^2+1)}{(s^2+1)^4} \end{aligned}$$

As you can see it is difficult to see a pattern. Instead what we can do is to use partial fractions, but for that we need to use complex numbers:

$$F(s) = \frac{s}{s^2+1} = \frac{1/2}{s+i} + \frac{1/2}{s-i}$$

We will then prove by induction that for every constant c we have: $\dfrac{d^n}{ds^n}\left(\dfrac{1}{s+c}\right) = \dfrac{(-1)^n n!}{(s+c)^{n+1}}$. (Prove this!) Thus, the answer is

$$\mathcal{L}\{t^n \cos t\} = \frac{1}{2}\left(\frac{n!}{(s+i)^{n+1}} + \frac{n!}{(s-i)^{n+1}}\right).$$

□

Example 10.13. Let c be a positive constant and $f : [0,\infty) \to \mathbb{R}$ be piecewise continuous and of exponential order. Find a relation between $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{f(ct)\}$.

Solution. By definition of Laplace transform and substitution $x = ct$ we obtain the following:

$$\mathcal{L}\{f(ct)\}(s) = \int_0^\infty e^{-st} f(ct)dt = \int_0^\infty e^{-sx/c} f(x)\frac{dx}{c} = \frac{1}{c}\mathcal{L}\{f(t)\}(s/c).$$

□

Example 10.14. Solve the initial value problem:

$$y'(t) + \int_0^t (t-x)y(x)dx = t, y(0) = 0.$$

Solution. This equation can be written as $y'(t) + (t \star y)(t) = t$. Let $Y(s)$ be the Laplace of y. Taking Laplace of both sides we obtain:

$$sY(s) - y(0) + \mathcal{L}\{t \star y\}(s) = \frac{1}{s^2}.$$

This yields $sY(s) + \dfrac{1}{s^2}Y(s) = \dfrac{1}{s^2}$. Therefore, $Y(s) = \dfrac{1}{s^3+1}$. Then we will take the inverse Laplace using the method of partial fractions. □

Example 10.15. Prove that if $f : [0,\infty) \to \mathbb{R}$ is a differentiable function and $f'(t)$ is piecewise continuous and of exponential order not exceeding c, where c is a positive constant, then $f(t)$ is also of exponential order not exceeding c.

Solution. Suppose $|f'(t)| \le Me^{ct}$ for all $t \ge 0$ and for some constant $M \ge 0$. Thus,

$$-Me^{ct} \le f'(t) \le Me^{ct}.$$

By Theorem 5.3 we can integrate the above inequalities from 0 to t for every $t > 0$ to obtain the following:

$$-\frac{M}{c}(e^{ct}-1) \le f(t) - f(0) \le \frac{M}{c}(e^{ct}-1).$$

This implies

$$-\frac{M}{c}(e^{ct}-1) + f(0) \le f(t) \le \frac{M}{c}(e^{ct}-1) + f(0).$$

Note that $|f(0)| \le |f(0)|e^{ct}$ since $c, t \ge 0$. Therefore,

$$\frac{M}{c}(e^{ct}-1) + f(0) \le \frac{M}{c}e^{ct} + |f(0)| \le (M/c + |f(0)|)e^{ct}.$$

Furthermore,

$$-\frac{M}{c}(e^{ct}-1) + f(0) \ge -\frac{M}{c}e^{ct} - |f(0)| \ge -(M/c + |f(0)|)e^{ct}.$$

Therefore, f is of exponential order not exceeding c. □

Example 10.16. Consider the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(t) = (-1)^n$ if $n \le t < n+1$ for a nonnegative integer n.

(a) Prove that $F(s) = \mathcal{L}\{f\}(s)$ converges for all $s > 0$.

(b) Find $F(s)$.

Hint: Find $f(t) + f(t-1)$.

Solution. (a) Note that since $f(t) = (-1)^n$ is constant over $(n, n+1)$, it is continuous there. Therefore, the only points of discontinuity of f are integers. Thus, f is piecewise continuous. Furthermore, $|f(t)| = 1 \le e^{0t}$ for all $t \ge 0$. Thus, f is of exponential order not exceeding zero. Therefore, by Theorem 10.1, $F(s)$ exists for all $s > 0$.

(b) Note that if $t \in [n, n+1)$, then $t - 1 \in [n-1, n)$, and thus $f(t) + f(t-1) = (-1)^n + (-1)^{n-1} = 0$. Therefore, for all $t \ge 1$ we have $f(t) + f(t-1) = 0$. If $0 \le t < 1$, then $f(t) = (-1)^0 = 1$. These two can be combined as $f(t) = 1(1 - H(t-1)) - f(t-1)H(t-1) = 1 - (f(t-1)+1)H(t-1)$. Taking the Laplace of both sides we obtain

$$F(s) = \frac{1}{s} - e^{-s}F(s) - \frac{e^{-s}}{s}.$$

Therefore, $F(s) = \dfrac{1-e^{-s}}{s(1+e^{-s})}$ □

Example 10.17. Consider the function $\dfrac{\sin t}{t}$. Show this function is piecewise defined and of exponential order. Find its Laplace.

Solution. Note that the limit of this function as $t \to 0$ is 1. Thus, if we define this function to be 1 at $t = 0$, it becomes continuous over $[0, \infty)$. Thus, it attains a maximum value over $[0, 1]$. Let M be this maximum value. For all $t \geq 1$ we have

$$\left|\frac{\sin t}{t}\right| \leq \frac{1}{|t|} \leq 1.$$

Thus, for every $t \geq 0$ we have

$$\left|\frac{\sin t}{t}\right| \leq M + 1.$$

Therefore, the function is of exponential order not exceeding 0. The only point of discontinuity of this function is $t = 0$. Thus, it is piecewise continuous. Therefore, $F(s) = \mathcal{L}\{f(t)\}(s)$ is defined for all $s > 0$.

Let $F(s) = \mathcal{L}\{\frac{\sin t}{t}\}(s)$. We have $\mathcal{L}\{\sin t\} = -F'(s)$. From the table we obtain

$$F'(s) = -\frac{1}{s^2 + 1} \Rightarrow F(s) = -\tan^{-1}(s) + C$$

for all $s > 0$. By Exercise 10.10, we know $F(s) \to 0$ as $s \to \infty$. Therefore, $0 = -\pi/2 + C$, and thus $C = \pi/2$. Therefore, $\mathcal{L}\{(\sin t)/t)\} = \pi/2 - \tan^{-1}(s)$. □

Example 10.18. Show that the Laplace transform $F(s)$ of the function $f(t) = 1/t$ does not converge for any $s \in \mathbb{R}$.

Scratch. Near zero, $e^{-st}/t \approx 1/t$, and thus $\displaystyle\int e^{-st} f(t)\ \mathrm{d}t \approx \ln t$, which diverges near zero. We will now make this rigorous.

Solution. Let $s \in \mathbb{R}$. Note that $e^{-st} \to 1$ as $t \to 0$. By the definition of limit, there is some $\delta > 0$ that if $t \in (0, \delta)$ we have

$$\left|e^{-st} - 1\right| \leq \frac{1}{2} \Rightarrow \frac{1}{2} \leq e^{-st} \Rightarrow \frac{1}{2t} \leq \frac{e^{-st}}{t}.$$

We see $\displaystyle\int_0^\delta \frac{1}{2t}\ \mathrm{d}t = \lim_{c \to 0^+} \frac{1}{2}(\ln \delta - \ln c) = \infty$. Therefore, by the Comparison Theorem, $\displaystyle\int_0^\delta \frac{e^{-st}}{t}\ \mathrm{d}t$ diverges. Thus, $\mathcal{L}\{1/t\}(s)$ diverges for every $s \in \mathbb{R}$. □

Example 10.19. Show that the Laplace transform $F(s)$ of the function $f(t) = \dfrac{\sin t}{t^2}$ does not converge for any $s \in \mathbb{R}$.

Scratch. Near zero, we have $\dfrac{\sin t}{t^2} \approx \dfrac{1}{t}$ and $e^{-st} \approx 1$. Thus, the integral $\displaystyle\int e^{-st} f(t)\ \mathrm{d}t$ near zero is approximately $\displaystyle\int \mathrm{d}t/t$, which is $\ln t$ which diverges near zero. We will now make this rigorous.

Solution. Let $s \in \mathbb{R}$. Note that $\dfrac{\sin t}{t} e^{-st} \to 1$ as $t \to 0$. By the definition of limit, there is some $\delta > 0$ that if $t \in (0, \delta)$ we have

$$\left|\frac{\sin t}{t} e^{-st} - 1\right| \leq \frac{1}{2} \Rightarrow \frac{1}{2} \leq \frac{\sin t}{t} e^{-st} \Rightarrow \frac{1}{2t} \leq \frac{\sin t}{t^2} e^{-st}.$$

We see $\displaystyle\int_0^\delta \frac{1}{2t}\,\mathrm{d}t = \lim_{c\to 0^+} \frac{1}{2}(\ln\delta - \ln c) = \infty$. Therefore, by the Comparison Theorem, $\displaystyle\int_0^\delta \frac{\sin t}{t^2} e^{-st}\,\mathrm{d}t$ diverges. Thus, $\mathcal{L}\{\frac{\sin t}{t^2}\}(s)$ diverges for every $s \in \mathbb{R}$. $\square$

10.2 Exercises

Exercise 10.1. *Find the Laplace of each function using the definition.*

(a) $\sin^2 t$.

(b) t^2.

Exercise 10.2. *Find the Laplace of each function using the table of Laplace transform.*

(a) $\sin(4t) + \cos t$.

(b) $t \sin t \cos(3t)$.

(c) a^t*, where* $a > 0$ *is a constant.*

(d) $(t + \sin t)H(t - \pi)$.

(e) $f(t) = \begin{cases} 1 & \text{if } 2n \le t < 2n+1 \text{ for some } n \in \mathbb{Z} \\ 0 & \text{if } 2n+1 \le t < 2n+2 \text{ for some } n \in \mathbb{Z} \end{cases}$

Exercise 10.3. *Let* c *be a positive real number and* $p(t)$ *be a non-constant polynomial.*

(a) Prove that $p(t)$ *is of exponential order not exceeding* c*.*

(b) Prove that $p(t)$ *is not of exponential order not exceeding zero.*

Hint: Show $\frac{p(t)}{e^{ct}}$ is bounded for large t, by using the fact that is tends to zero as $t \to \infty$. For small values of t invoke the Extreme Value Theorem.

Exercise 10.4. *Prove that the function* $\frac{e^t - 1}{t}$ *is of exponential order and find its Laplace transform.*

Hint: See Example 10.17.

Exercise 10.5. *Show that the Laplace* $\mathcal{L}\{f(t)\}(s)$ *of the following function does not converge for any* $s \in \mathbb{R}$.

$$f(t) = \begin{cases} \dfrac{e^t - 1}{t^2} & \text{if } t \ne 0 \\ 1 & \text{if } t = 0 \end{cases}$$

Exercise 10.6. *Find the inverse Laplace of each function:*

(a) $\dfrac{1}{s^4 - 1}$.

(b) $\dfrac{2s + 4}{s^3 - s}$.

(c) $\dfrac{s+2}{s^2+4s+5}$.

(d) $\ln\left(\dfrac{s+1}{s-1}\right)$.

Exercise 10.7. *Solve each equation using the Laplace transform method.*

(a) $y' + 2y = \sin t,\ y(0) = 0$.

(b) $y'' - 4y = 2\cos t\cos(3t),\ y(0) = y'(0) = 0$.

(c) $y(t) = \sin t + 2\int_0^t \cos(t-u)y(u)\ du$.

(d) $y'' - y' = f(t),\ y(0) = 0, y'(0) = 1,$ *and* $f(t) = \begin{cases} t^2 - 1 & \text{if } t \in [0,1) \\ 1 & \text{if } t \geq 1 \end{cases}$

Exercise 10.8. *Find the Laplace of* $|\sin t|$.

Hint: Write $|\sin t|$ *as a piecewise defined function. Then, use the Heaviside function.*

Exercise 10.9. *Prove that* $\mathcal{L}\{e^{t^2}\}(s)$ *diverges for all* $s \in \mathbb{R}$.

Hint: Use the fact that $t^2 \geq st$ for large values of t.

Exercise 10.10. *Suppose* $f : [0,\infty) \to \mathbb{R}$ *is of exponential order and let* $F(s) = \mathcal{L}\{f\}(s)$. *Prove that* $\lim\limits_{s\to\infty} F(s) = 0$.

Exercise 10.11. *Prove Theorem 10.4 by induction.*

Exercise 10.12. *Let c be a real number, and suppose $f_1(t), f_2(t), \ldots$ is a sequence of piecewise continuous functions of exponential order not exceeding c. Let $F_1(s), F_2(s), \ldots$ be the sequence consisting of Laplace transforms of $f_1, f_2, \ldots$, respectively, for all $s > c$. Assume $f(t) = \sum\limits_{n=1}^{\infty} f_n(t)$ is a piecewise continuous function. Prove that $f(t)$ is of exponential order and that $\mathcal{L}\{f(t)\}(s) = \sum\limits_{n=1}^{\infty} F_n(s)$ for all $s > c$.*

10.3 Challenge Problems

Exercise 10.13. *Prove Theorem 10.3.*

10.4 Summary

- $\mathcal{L}\{f\}(s) = \int\limits_0^{\infty} e^{-st}f(t)\ dt.$
- If f is of exponential order and is piecewise continuous, then its Laplace converges for all large values of s.
- $\mathcal{L}$ has an inverse.

- Make sure you get yourself familiar with the Table of Laplace. You should be able to use it in evaluating Laplace and inverse Laplace.
- To solve initial value problems using the method of Laplace Transform:
  - If the initial value is at a point other than zero, by a change of variables move the initial condition to zero.
  - Take Laplace of both sides.
  - Use $\mathcal{L}\{y^{(n)}\} = s^nY(s) - s^{n-1}y(0) - \cdots - sy^{(n-2)}(0) - y^{(n-1)}(0)$.
  - Solve for $Y(s)$, the Laplace of the solution.
  - Take inverse Laplace to find the solution $y(t)$.
- To find the Laplace of a piecewise defined function, we need to write it using Heaviside function as follows:
  - For each condition $a \leq t < b$ consider the function $H(t-a) - H(t-b)$. This function is 1 when $a \leq t < b$ and zero otherwise.
  - Multiply each $H(t-a) - H(t-b)$ by its corresponding function, and add up all of these.
  - Rearrange the terms so each term is of the form $H(t-c)$(some function).
  - The goal is to write this in the form $H(t-c)j(t-c)$.
  - Set "some function=$j(t-c)$", and find $j(t)$ by substituting $t+c$ for t.
  - Take the Laplace of $j(t)$, and use $\mathcal{L}\{H(t-c)j(t-c)\} = e^{-sc}J(s)$.
- $\mathcal{L}\{f \star g\}(s) = F(s)G(s)$.

Chapter 11

Systems of Differential Equations

A first-order system is a system of equations with unknown functions $x_1, \dots, x_n$ of the form:

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(t, x_1, \dots, x_n) \\ \vdots \\ \dfrac{dx_n}{dt} = f_n(t, x_1, \dots, x_n) \end{cases}$$

We typically write the above system in a more compact form: $\dfrac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$, where $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{f} = (f_1, \dots, f_n)$. The vector function $\mathbf{f}$ is called "forcing".

We can write down any differential equation or system as a first-order system.

Example 11.1. Convert the system into a first-order system:

$$\begin{cases} x'' = x^2 + x' + tx \\ y'' = y'y + yt^3 \end{cases}$$

11.1 First-Order Linear Systems

A first-order n-dimensional linear system is a system of the following form:

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}(t)x_1(t) + \cdots + a_{1n}(t)x_n(t) + f_1(t) \\ \\ \dfrac{dx_2}{dt} = a_{21}(t)x_1(t) + \cdots + a_{2n}(t)x_n(t) + f_2(t) \\ \vdots \\ \dfrac{dx_n}{dt} = a_{n1}(t)x_1(t) + \cdots + a_{nn}(t)x_n(t) + f_n(t) \end{cases}$$

This system can be written in matrix form as follows:

$$\underbrace{\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}'} = \underbrace{\begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}}_{\text{coefficient matrix } A(t)} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} + \underbrace{\begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}}_{\text{forcing } \mathbf{f}(t)}.$$

This is often written in a more compact form $\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{f}(t)$. The sqaure matrix $A(t)$ is called the **coefficient matrix** and $\mathbf{f}(t)$ is called the **forcing**.

Example 11.2. Find the forcing and coefficient matrix of the system

$$\begin{cases} x_1' = 2x_1 - tx_2 + \sin t \\ x_2' = t^2 x_1 + (\cos t)\ x_2 \end{cases}$$

We can write equations or systems of higher order as first-order systems.

Example 11.3. Convert the equation $y''' - y'' + ty' + (\tan t)y = e^{t^2}$ into a first-order linear system.

Example 11.4. Convert the linear equation below into a first-order system:

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = f(t).$$

Similar to what we have seen before, the general solution to $\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{f}(t)$ can be obtained by finding the general solution $\mathbf{x}_H(t)$ to the equation $\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x}$, and a particular solution $\mathbf{x}_P(t)$ to the nonhomogeneous equation $\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{f}(t)$ and then adding them up.

Theorem 11.1 (Existence and Uniqueness Theorem)**.** *Consider the first-order n-dimensional equation*

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{f}(t).$$

Suppose all entries of $A(t)$ and $\mathbf{f}(t)$ are continuous over an open interval (a, b). Let $t_0 \in (a, b)$, and $\mathbf{x}_0 \in \mathbb{R}^n$. Then, there is a unique solution defined over (a, b) to the following initial value problem:

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{f}(t),\ \mathbf{x}(t_0) = \mathbf{x}_0$$

Example 11.5. Find the largest interval for which a unique solution to the following initial value problem is guaranteed to exist:

$$\begin{cases} t^2 x' = 2x - (\cos t)y + \tan t \\ (\sin t)y' = tx + y + \cos t \\ x(1) = y(1) = 0 \end{cases}$$

11.2 Homogeneous Linear Systems

In this section we will consider first-order n-dimensional homogeneous systems of the form

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x},$$

where all entries of $A(t)$ are continuous over an open interval (a, b).

Theorem 11.2. *Consider a first-order n-dimensional homogeneous system*

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x}.$$

Assume all entries of $A(t)$ are continuous over an open interval (a, b). Then, the set of all solutions to this equation defined over (a, b) is an n-dimensional vector space.

Suppose $\mathbf{\Phi}_1(t), \ldots, \mathbf{\Phi}_n(t)$ are solutions to the homogeneous system given above. For these to form a basis for the solution set of $\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x}$ we need to make sure all solutions of this system are of the form $c_1\mathbf{\Phi}_1(t) + \cdots + c_n\mathbf{\Phi}_n(t)$. Since each solution $\mathbf{\Phi}(t)$ is uniquely determined by an initial value $\mathbf{\Phi}(t_0) = \mathbf{c}_0$, in order for $c_1\mathbf{\Phi}_1(t) + \cdots + c_n\mathbf{\Phi}_n(t)$ to produce all solutions we need $c_1\mathbf{\Phi}_1(t_0) + \cdots + c_n\mathbf{\Phi}_n(t_0) = \mathbf{c}$ to have a solution for $c_1, \ldots, c_n$ for every $\mathbf{c} \in \mathbb{R}^n$. This can be written as

$$(\mathbf{\Phi}_1(t_0) \cdots \mathbf{\Phi}_n(t_0)) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{c}.$$

For this equation to have a solution for every $\mathbf{c} \in \mathbb{R}^n$ we need to have

$$\det\left(\mathbf{\Phi}_1(t_0) \cdots \mathbf{\Phi}_n(t_0)\right) \neq 0.$$

This brings us to the definition of **Wronskian** of $\mathbf{\Phi}_1(t), \ldots, \mathbf{\Phi}_n(t)$ as follows:

$$W[\mathbf{\Phi}_1(t), \ldots, \mathbf{\Phi}_n(t)] = \det\left(\mathbf{\Phi}_1(t) \cdots \mathbf{\Phi}_n(t)\right).$$

Theorem 11.3. *Suppose all entries of the coefficient matrix of a first-order n-dimensional homogeneous linear system are continuous over (a, b). Suppose the Wronskian of n solutions to this system is zero at one point $t_0 \in (a, b)$. Then the Wronskian must be zero everywhere on (a, b).*

Definition 11.1. Suppose all entries of the coefficient matrix of a first-order n-dimensional homogeneous linear system are continuous over (a, b). Solutions $\mathbf{\Phi}_1(t), \ldots, \mathbf{\Phi}_n(t)$ to a homogeneous first-order n-dimensional system are said to form a **Fundamental Set of Solutions** (FSoS) if they are a basis for this solution set. The matrix $(\mathbf{\Phi}_1(t) \cdots \mathbf{\Phi}_n(t))$ is called a **Fundamental Matrix** for this equation.

Example 11.6. Suppose $\mathbf{x}_1(t) = \begin{pmatrix} 1+t^2 \\ t \end{pmatrix}, \mathbf{x}_2(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$ are two solutions to a first-order 2-dimensional linear equation

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x}.$$

(a) Find the general solution of this system.

(b) Find a fundamental matrix for this system.

(c) Find the coefficient matrix $A(t)$.

11.3 More Examples

Example 11.7. Suppose $\mathbf{x}_1(t) = \begin{pmatrix} t+1 \\ t \end{pmatrix}, \mathbf{x}_2(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$ are two solutions to a first-order 2-dimensional linear system

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x},\ t \in (a,b).$$

Assume all entries of $A(t)$ are continuous over (a,b).

(a) Prove that $0 \notin (a,b)$.

(b) Find the general solution of this system.

(c) Find a fundamental matrix for this system.

(d) Find the coefficient matrix $A(t)$.

Solution. (a) Note that $W[\mathbf{x}_1, \mathbf{x}_2] = (t+1)t - t = t^2$. This is equal to 0 at $t = 0$. Therefore, if 0 were in (a,b), by Theorem 11.3 the Wronskian must be zero everywhere, which is a contradiction. Therefore, $0 \notin (a,b)$.

(b) Since $W[\mathbf{x}_1, \mathbf{x}_2] \neq 0$, the genereal solution is $c_1 \begin{pmatrix} t+1 \\ t \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ t \end{pmatrix}$.

(c) A fundamental matrix is $\begin{pmatrix} t+1 & 1 \\ t & t \end{pmatrix}$.

(d) Since $\mathbf{x}_1, \mathbf{x}_2$ are solutions to this system, we have

$$(\mathbf{x}_1' \mathbf{x}_2') = A(t)\,(\mathbf{x}_1 \mathbf{x}_2) \Rightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = A(t) \begin{pmatrix} t+1 & 1 \\ t & t \end{pmatrix} \Rightarrow A(t) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} t+1 & 1 \\ t & t \end{pmatrix}^{-1}$$

Therefore, $A(t) = \dfrac{1}{t^2} \begin{pmatrix} t & -1 \\ 0 & t \end{pmatrix}$ □

Example 11.8. Prove that $\mathbf{x}(t) = \begin{pmatrix} t^2+t \\ \sin t \end{pmatrix}$ cannot be a solution to a homogeneous, first-order, 2-dimensional linear system, all of whose coefficient matrix entries are continuous over $(-1,1)$.

Solution. Suppose on the contrary $\mathbf{x}$ is a solution to $\mathbf{x}' = A(t)\mathbf{x}$, where all entries of $A(t)$ are continuous over $(-1,1)$. Note that $\mathbf{x}(0) = \mathbf{0}$. Thus, $\begin{pmatrix} t^2+t \\ \sin t \end{pmatrix}$ is a solution to the IVP

$$\mathbf{x}' = A(t)\mathbf{x}, \mathbf{x}(0) = \mathbf{0}.$$

On the other hand $\mathbf{0}$ is a different solution to the same IVP. This is a contradiction! □

Example 11.9. Consider a linear system $\mathbf{x}' = A(t)\mathbf{x}$, where all entries of $A(t)$ are continuous over $\mathbb{R}$. Prove that if $\mathbf{x}$ is an odd solution to this system, then $\mathbf{x}$ is the trivial solution $\mathbf{0}$.

Solution. Note that since $\mathbf{x}$ is odd, we have $\mathbf{x}(-0) = -\mathbf{x}(0)$, and thus $\mathbf{x}(0) = \mathbf{0}$. Therefore, $\mathbf{x}$ satisfies the IVP $\mathbf{x}' = A(t)\mathbf{x}, \mathbf{x}(0) = \mathbf{0}$. Since $\mathbf{0}$ also satisfies the same IVP, by uniqueness we must have $\mathbf{x} = \mathbf{0}$, as desired. $\square$

Example 11.10. Consider the IVP

$$\begin{cases} x' = (\sin t)x + \sin t \\ y' = t^3 y + \sin t \\ x(0) = 0, y(0) = 1. \end{cases}$$

(a) Prove that there is a unique solution to this IVP defined over $\mathbb{R}$.

(b) Prove that this solution is even.

Solution. (a) The entries of the coefficient matrix and forcing are all continuous over $\mathbb{R}$. Therefore, by the Existence and Uniqueness Theorem, a unique solution exists.

(b) Let $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. We need to show $\mathbf{x}(-t) = \mathbf{x}(t)$. Let $\mathbf{z}(t) = \mathbf{x}(-t)$. Note that

$$\mathbf{z}'(t) = -\mathbf{x}'(-t) = -\begin{pmatrix} \sin(-t) & 0 \\ 0 & (-t)^3 \end{pmatrix} \mathbf{x}(-t) - \begin{pmatrix} \sin(-t) \\ \sin(-t) \end{pmatrix} = \begin{pmatrix} \sin t & 0 \\ 0 & t^3 \end{pmatrix} \mathbf{z}(t) + \begin{pmatrix} \sin t \\ \sin t \end{pmatrix}.$$

We also have $\mathbf{z}(0) = \mathbf{x}(-0) = \mathbf{x}(0)$. Therefore, by the uniqueness of the solution, we have $\mathbf{z}(t) = \mathbf{x}(t)$. Thus, $\mathbf{x}(-t) = \mathbf{x}(t)$, which means $\mathbf{x}$ is even. $\square$

Example 11.11. Suppose $\mathbf{x}_1(t) = \begin{pmatrix} t+e^t \\ 1 \end{pmatrix}, \mathbf{x}_2(t) = \begin{pmatrix} 1+e^t \\ 1 \end{pmatrix}$, and $\mathbf{x}_3(t) = \begin{pmatrix} e^t \\ t \end{pmatrix}$ are solutions to a non-homogeneous system with continuous coefficient matrix and forcing over an open interval (a, b) given by

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t).$$

Solve the initial value problem

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t), \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Solution. By linearity, we note that $\mathbf{x}_1(t) - \mathbf{x}_2(t) = \begin{pmatrix} t-1 \\ 0 \end{pmatrix}$, and $\mathbf{x}_2(t) - \mathbf{x}_3(t) = \begin{pmatrix} 1 \\ 1-t \end{pmatrix}$ are solutions of the corresponding homogeneous system $\mathbf{x}' = A(t)\mathbf{x}$. We also note that

$$W[\mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_2 - \mathbf{x}_3] = \det \begin{pmatrix} t-1 & 1 \\ 0 & 1-t \end{pmatrix} = -(1-t)^2 \neq 0.$$

Therefore, $\mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_2 - \mathbf{x}_3$ form a FSoS for the homogeneous system. Therefore, the general solution to the non-homogeneous system is $\mathbf{x} = c_1 \begin{pmatrix} t-1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1-t \end{pmatrix} + \begin{pmatrix} e^t \\ t \end{pmatrix}$. To solve the initial value problem we need to solve

$$\mathbf{x} = c_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} -c_1 + c_2 + 1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow c_1 = c_2 = 2.$$

The solution to the given IVP is thus, $\begin{pmatrix} 2t+e^t \\ 2-t \end{pmatrix}$. □

Example 11.12. Prove the Existence and Uniqueness Theorem for Linear Differential Equations follows from the Existence and Uniqueness Theorem for First-Order Systems.

Solution. Given a linear differential equation, we can turn it into a system, and then apply the Existence and Uniqueness Theorem for systems. □

11.4 Exercises

Exercise 11.1. *Find the largest interval for which the IVP is guaranteed to have a solution.*

(a)

$$\begin{cases} tx' = x + y + \tan t \\ (\sqrt{t^2-3})y' = (\cos t)x + |t|y + \sin t \\ x(2) = y(2) = 3. \end{cases}$$

(b)

$$\begin{cases} x' = x + y + \sqrt[3]{t^2-1} \\ |t^2-3|y' = (t^2+t)x + y + \csc t \\ x(-1) = y(-1) = 2. \end{cases}$$

Exercise 11.2. *Convert each equation into a first-order system. Find its coefficient matrix and forcing.*

(a) $y'' + ty' - 7y = \sin t.$

(b) $t^2y'' + (\sin t)y' + \cos t = 0.$

Exercise 11.3. *Consider the system of differential equations:*

$$\begin{cases} y'' + ty' - t^3y = 1 \\ z'' - z = \sin t \end{cases}$$

(a) Convert the system above into a first order linear system.

(b) What are coefficient matrix and forcing of this system?

Exercise 11.4. *Suppose* $\begin{pmatrix} e^t \\ e^{2t} \end{pmatrix}, \begin{pmatrix} 1 \\ t \end{pmatrix}$ *are solutions to a first-order 2-dimensional homogeneous linear system.*

(a) Find the coefficient matrix of this system.

(b) Find the general solution of this system.

Exercise 11.5. *Suppose* $\begin{pmatrix} 1+e^t \\ e^t \end{pmatrix}, \begin{pmatrix} 1+e^t \\ t+e^t \end{pmatrix}, \begin{pmatrix} e^t \\ e^t \end{pmatrix}$ *are solutions to a 2-dimensional linear system*

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{f}(t).$$

(a) Find $A(t)$ *and* $\mathbf{f}(t)$.

(b) Find the largest open interval containing $t = 1$ *for which all entries of the coefficient matrix and the forcing are continuous.*

(c) Find the general solution of this system.

(d) Solve the IVP: $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t), \mathbf{x}(1) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Exercise 11.6. *Suppose the* n*-dimensional linear system* $\mathbf{x}' = A(t)\mathbf{x}$ *has* n *linearly independent constant solutions. Prove that* $A(t) = 0$.

Exercise 11.7. *Suppose* $X(t)$ *is a fundamental matrix for the system* $\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x}$. *Show that:*

(a) If $Y(t)$ *is another fundamental matrix for the same system, then* $Y(t) = X(t)C$ *for some invertible constant matrix* C.

(b) $X(t)[X(t_0)]^{-1}\mathbf{x}_0$ *is the solution to the initial value problem:* $\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x},\ \mathbf{x}(t_0) = \mathbf{x}_0$.

Exercise 11.8. *Consider the initial value problem*

$$\begin{cases} x' = tx + t^3 y - \sin t \\ y' = tx + (t^3 + t)y + \tan^{-1} t \\ x(0) = 1, y(0) = 2 \end{cases}$$

(a) Show that this IVP has a unique solution defined over $\mathbb{R}$.

(b) Show this solution is even.

Exercise 11.9. *Prove that there is no first-order 2-dimensional homogeneous linear system with coefficients that are continuous over* $(-1, 1)$*, where one of its solutions is* $\begin{pmatrix} t + \sin t \\ t^3 - t\cos t \end{pmatrix}$.

Exercise 11.10. *Prove that there is no first-order 2-dimensional homogeneous linear system with coefficients that are continuous over* $\mathbb{R}$*, where one of its solutions is* $\begin{pmatrix} t^3 - 1 \\ \sin(t\pi) \end{pmatrix}$.

Exercise 11.11. *Consider a first-order* n*-dimensional homogeneous linear system. Assume* W *is the Wronskian of some FSoS. Prove that for every nonzero* $c \in \mathbb{R}$ *there is a FSoS for this system whose Wronskian is* cW.

The goal of the following exercise is to prove the Abel's Theorem for systems.

Exercise 11.12 (Abel's Theorem for Systems)**.** *Consider the first-order n-dimensional homogeneous linear system*

$$\mathbf{x}' = A(t)\mathbf{x}.$$

Suppose all entries of $A(t)$ are continuous over an open interval (a, b), and let $\mathbf{\Phi}_1, \dots, \mathbf{\Phi}_n$ be n solutions of this system. Let $W(t) = W[\mathbf{\Phi}_1(t), \dots, \mathbf{\Phi}_n(t)]$. Prove $W'(t) = (\text{tr } A(t))W(t)$.

Hint: Use Exercise 2.6.

Exercise 11.13. *Suppose $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$ are solutions to an n-dimensional homogeneous linear system $\mathbf{x}' = A(t)\mathbf{x}$. Assume (a, b) is an open interval for which $W[\mathbf{x}_1, \dots, \mathbf{x}_n](t) \neq 0$ for all $t \in (a, b)$. Prove that all entries of $A(t)$ are continuous over (a, b).*

11.5 Summary

- To convert a system into a first-order system, for every instance of $y^{(n)}$ introduce new variables: $x_1 = y, x_2 = y', \dots, x_n = y^{(n-1)}$.
- If all coefficients and forcing of a first-order linear system are continuous, then the system has a unique solution for each initial value.
- $W[\mathbf{\Phi}_1(t), \dots, \mathbf{\Phi}_n(t)] = \det(\mathbf{\Phi}_1(t) \dots \mathbf{\Phi}_n(t))$.
- When the Wronskian is nonzero we have a FSoS, and all solutions are given by $(\mathbf{\Phi}_1(t) \ \cdots \ \mathbf{\Phi}_n(t))\, \mathbf{c}$.
- The fundamental matrix is the matrix $(\mathbf{\Phi}_1(t) \ \cdots \ \mathbf{\Phi}_n(t))$, where the columns form a FSoS.
- The coefficient matrix $A(t)$ of a homogeneous system can be found by solving $\Phi'(t) = A(t)\Phi(t)$, i.e. $A(t) = \Phi'(t)[\Phi(t)]^{-1}$.

Chapter 12

Linear Systems with Constant Coefficients

12.1 Homogeneous Linear Systems with Constant Coefficients

In this section we will focus on systems of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where A is a constant square matrix.

Recall that for any scalar a, one solution to the differential equation $y' = ay$ is $y = e^{at}$. Recall also that

$$e^{tA} = 1 + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \cdots + \frac{t^nA^n}{n!} + \cdots.$$

Differentiating we obtain:

$$\begin{aligned}
\frac{d}{dt}(e^{tA}) &= \frac{A}{1!} + \frac{2tA^2}{2!} + \cdots + \frac{nt^{n-1}A^n}{n!} + \cdots \\
&= \frac{A}{0!} + \frac{tA^2}{1!} + \cdots + \frac{t^{n-1}A^n}{(n-1)!} + \cdots \\
&= A\left(I + \frac{tA}{1!} + \cdots + \frac{t^{n-1}A^{n-1}}{(n-1)!} + \cdots\right) \\
&= Ae^{tA}.
\end{aligned}$$

To summarize, we showed $\dfrac{d}{dt}(e^{tA}) = Ae^{tA}$, which means each column of e^{tA} is a solution to $\mathbf{x}' = A\mathbf{x}$. Since $e^{0A} = I$ is invertible, the matrix e^{tA} is a fundamental matrix.

We have already seen one way of finding e^{tA} by looking at the Jordan form of the matrix tA. This method is often too computational. We will find another methods of finding the matrix exponential e^{tA}.

As usual, we will use $D = \dfrac{d}{dt}$. We note that

$$D[e^{tA}] = Ae^{tA}, D^2[e^{tA}] = A^2e^{tA}, \ldots, D^n[e^{tA}] = A^n e^{tA}, \ldots.$$

Therefore, if $p(z)$ is a polynomial, then

$$p(D)[e^{tA}] = p(A)e^{tA}.$$

Assume $p(A) = 0$, then $p(D)[e^{tA}] = 0$. Thus, all entries of e^{tA} satisfy the differential equation $p(D)[y] = 0$. We will now find these entries by finding initial conditions for the matrix e^{tA}. We will use the following:

$$e^{tA}|_{t=0} = I,\ \frac{d}{dt}(e^{tA})|_{t=0} = A,\ \frac{d^2}{dt^2}(e^{tA})|_{t=0} = A^2, \ldots.$$

Thus, in order to find e^{tA} we need to do the following:

- Find a polynomial $p(z)$ for which $p(A) = 0$.
- Find a NFSoS $N_0(t), N_1(t), \ldots, N_m(t)$ for $p(D)[y] = 0$. (Here, the order of $p(D)[y] = 0$ is $m + 1$.)
- $e^{tA} = N_0(t)I + N_1(t)A + \cdots + N_m(t)A^m$.

One such polynomial $p(z)$ can be found using the Cayley-Hamilton Theorem that we saw in an earlier chapter, but any such nonzero polynomial $p(z)$ can be used.

Reminder: **Cayley-Hamilton Theorem:** If $p(z) = \det(A - zI)$, then $p(A) = 0$.

Example 12.1. Compute e^{tA}, where $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. Use that to solve the initial value problem

$$\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We know e^{tA} is a fundamental matrix. Suppose $\Phi(t)$ is another fundamental matrix of $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$. Then, since all columns of e^{tA} are solutions to this system, we must have $e^{tA} = \Phi(t)B$ for a constant square matrix B. Substituting $t = 0$ we obtain $I = \Phi(0)B$ which gives us $B = [\Phi(0)]^{-1}$ and hence

$$e^{tA} = \Phi(t)[\Phi(0)]^{-1}.$$

Example 12.2. Solve the initial value problem:

$$\begin{cases} x' = 2x + y \\ y' = x + 2y \end{cases}, x(0) = 1,\ y(0) = -1.$$

Example 12.3. Evaluate e^{tA}, where

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Example 12.4. Prove that the solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x}, \mathbf{x}(0) = \mathbf{x}_0$$

is $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$.

Example 12.5. Prove that if A and B are two square matrices of the same size and $AB = BA$, then $e^{A+B} = e^A e^B$.

12.1.1 Eigenpair Method

Looking back at differential eqautions of the form $y' = ay$, we know a solution is of the form e^{at}. We guess that a solution to $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ might be of the form $\mathbf{x} = e^{\lambda t}\mathbf{v}$. Substituting this into the system we obtain

$$\lambda e^{\lambda t}\mathbf{v} = Ae^{\lambda t}\mathbf{v} \Rightarrow e^{\lambda t}\lambda\mathbf{v} = e^{\lambda t}A\mathbf{v} \Rightarrow \lambda\mathbf{v} = A\mathbf{v}.$$

This means that $(\lambda, \mathbf{v})$ is an eigenpair for A. So, in order to find solutions for the system we will need to find eigenpairs.

Theorem 12.1. *Let $A \in M_n(\mathbb{R})$, $\lambda \in \mathbb{C}$, and $\mathbf{v} \in \mathbb{C}^n$. Then, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}$ if and only if $(\lambda, \mathbf{v})$ is an eigenpair for A.*

Example 12.6. Solve by eigenpair method:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\mathbf{x}.$$

Example 12.7. Solve by eigenpair method:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\mathbf{x}.$$

When A is diagonalizable, (e.g. when all eigenvalues are distinct) we can find a basis of $\mathbb{C}^n$ consisting of eigenvectors of A. Suppose the corresponding eigenpairs are $(\lambda_1, \mathbf{v}_1), \ldots, (\lambda_n, \mathbf{v}_n)$. For each real λ_j we will see that $e^{\lambda_j t}\mathbf{v}_j$ is a solution to $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$. For each nonreal pair of eigenpairs $(a \pm ib, \mathbf{v} \pm i\mathbf{w})$, where $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$, we will obtain two solutions by taking real part and imaginary part of $e^{(a+ib)t}(\mathbf{v}+i\mathbf{w})$.

When A is not diagonalizable, this method would only yield a partial Fundamental Set of Solutions.

Example 12.8. For every 2×2 constant matrix A find e^{tA}. Use that to solve all 2-dimensional homogeneous systems of the form $\mathbf{x}' = A\mathbf{x}$.

Example 12.9. Suppose $\begin{pmatrix} e^t \\ e^t \end{pmatrix}, \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$ form a FSoS for the equation $\mathbf{x}' = A\mathbf{x}$.

(a) Find A.

(b) Find e^{tA}.

12.2 Nonhomogeneous Systems with Constant Coefficients

12.2.1 Variation of Parameters

Suppose $\Phi(t)$ is a fundamental matrix for $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$. Thus, every solution can be obtained from $\mathbf{x}(t) = \Phi(t)\mathbf{c}$. Similar to what we did before, we assume a particular solution to the nonhomogeneous system $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}+\mathbf{f}(\mathbf{x})$ is of the form $\mathbf{x}_P(t) = \Phi(t)\mathbf{u}(t)$. Substituting into the nonhomogeneous system and using the fact that $\Phi' = A\Phi$ we obtain:

$$\Phi'\mathbf{u} + \Phi\mathbf{u}' = A\Phi\mathbf{u} + \mathbf{f} \Rightarrow \Phi\mathbf{u}' = \mathbf{f} \Rightarrow \mathbf{u}' = [\Phi]^{-1}\mathbf{f} \Rightarrow \mathbf{u}(t) = \mathbf{u}(t_0) + \int_{t_0}^{t} [\Phi(s)]^{-1}\mathbf{f}(s)\ ds.$$

Suppose we want to solve the initial value problem $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{f}(t),\ \mathbf{x}(t_0) = \mathbf{x}_0$. Substituting $\mathbf{x} = \Phi\mathbf{u}$ into the initial value we obtain $\Phi(t_0)\mathbf{u}(t_0) = \mathbf{x}_0$. Therefore, the solution to this initial value problem is

$$\mathbf{x}(t) = \Phi(t)\mathbf{u}(t) = \Phi(t)\mathbf{u}(t_0) + \Phi(t)\int_{t_0}^{t} [\Phi(s)]^{-1}\mathbf{f}(s)\ ds = \Phi(t)[\Phi(t_0)]^{-1}\mathbf{x}_0 + \Phi(t)\int_{t_0}^{t} [\Phi(s)]^{-1}\mathbf{f}(s)\ ds$$

Substituting $\Phi(t)$ by e^{tA} and using the fact that $e^{tA}(e^{sA})^{-1} = e^{(t-s)A}$ we obtain the following formula:

$$\boxed{\mathbf{x}(t) = e^{(t-t_0)A}\mathbf{x}_0 + \int_{t_0}^{t} e^{(t-s)A}\mathbf{f}(s)\ ds.}$$

Example 12.10. Solve the initial value problem $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 4 & 5 \\ -2 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 4e^t \cos t \\ 0 \end{pmatrix},\ \mathbf{x}(0) = \mathbf{0}$.

12.2.2 Laplace Transforms

For a vector valued function $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ let its Laplace be defined as

$$\mathcal{L}\{\mathbf{x}\} = \begin{pmatrix} \mathcal{L}\{x_1\} \\ \mathcal{L}\{x_2\} \\ \vdots \\ \mathcal{L}\{x_n\} \end{pmatrix}.$$

Suppose $\mathbf{x}$ is a solution to $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{f}(t)$. Using properties of Laplace transforms we obtain $\mathcal{L}\{\dfrac{d\mathbf{x}}{dt}\} = A\mathcal{L}\{\mathbf{x}\}+\mathcal{L}\{\mathbf{f}\}$. Similar to before we obtain $sX(s)-\mathbf{x}(0) = AX(s)+F(s)$, where $X(s), F(s)$ are the Laplace of

$\mathbf{x}(t)$ and $\mathbf{f}(t)$, respectively. This yields $(sI - A)X(s) = F(s) + \mathbf{x}(0)$. Multiplying by the inverse of $sI - A$ and then taking the Laplace inverse we can find the solution to the initial value problem $\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{f}, \mathbf{x}(0) = \mathbf{x}_0$, and thus a particular solution.

Example 12.11. Solve the initial value problem using Laplace method:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ 0 \end{pmatrix}, \ \mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

12.3 More Examples

Example 12.12. Evaluate e^{tA} for each matrix once using the NFS method and once by the eigenpair method.

(a) $A = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{pmatrix}$

Solution. (a) The characteristic polynomial is $p(z) = (4-z)(1-z) + 2 = z^2 - 5z + 6 = (z-2)(z-3)$. The general solution to $p(D)[y] = 0$ is $y = c_1 e^{2t} + c_2 e^{3t}$. We now solve the IVP $p(D)[y] = 0, y(0) = y_0, y'(0) = y_1$ to find a NFSoS at zero. This yields the following system:

$$\begin{cases} c_1 + c_2 = y_0 \\ 2c_1 + 3c_2 = y_1 \end{cases}$$

Subtracting twice the first equation from the second we obtain $c_2 = y_1 - 2y_0$. Substituting into the first equation we obtain $c_1 = 3y_0 - y_1$. Therefore,

$$y = (3y_0 - y_1)e^{2t} + (y_1 - 2y_0)e^{3t} \Rightarrow y = y_0(3e^{2t} - 2e^{3t}) + y_1(e^{3t} - e^{2t}) \Rightarrow N_0 = 3e^{2t} - 2e^{3t}, N_1 = e^{3t} - e^{2t}.$$

Therefore, $e^{tA} = N_0 I + N_1 A = \begin{pmatrix} -e^{2t} + 2e^{3t} & 2e^{3t} - 2e^{2t} \\ -e^{3t} + e^{2t} & 2e^{2t} - e^{3t} \end{pmatrix}$.

Now, we will use the eigenpair method. The eigenpairs of A are $(2, \binom{-1}{1})$ and $(3, \binom{-2}{1})$. Therefore, $e^{2t}\binom{-1}{1})$ and $e^{3t}\binom{-2}{1})$ for a FSoS. Thus, a fundamental matrix is given by

$$\begin{pmatrix} -e^{2t} & -2e^{3t} \\ e^{2t} & e^{3t} \end{pmatrix}.$$

Therefore,

$$e^{tA} = \begin{pmatrix} -e^{2t} & -2e^{3t} \\ e^{2t} & e^{3t} \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -e^{2t} + 2e^{3t} & 2e^{3t} - 2e^{2t} \\ -e^{3t} + e^{2t} & 2e^{2t} - e^{3t} \end{pmatrix}.$$

(b) The characteristic polynomial is $p(z) = (1-z)(2-z)(-1-z)$. The general solution to $p(D)[y] = 0$ is $y = c_1e^t + c_2e^{2t} + c_3e^{-t}$. We will now find a NFSoS for $p(D)[y] = 0$ at $t_0 = 0$. The equalities $y(0) = y_0, y'(0) = y_1, y''(0) = y_2$ turn into the following system which we solve for c_1, c_2, c_3.

$$\begin{cases} c_1 + c_2 + c_3 = y_0 \\ c_1 + 2c_2 - c_3 = y_1 \\ c_1 + 4c_2 + c_3 = y_2. \end{cases}$$

Solving this system we obtain $c_1 = y_0 + \dfrac{y_1 - y_2}{2}, c_2 = \dfrac{y_2 - y_0}{3}, c_3 = \dfrac{2y_0 - 3y_1 + y_2}{6}$. Therefore,

$$y = \frac{e^{-t} + 3e^t - e^{2t}}{3} y_0 + \frac{e^t - e^{-t}}{2} y_1 + \frac{e^{-t} - 3e^t + 2e^{2t}}{6} y_2$$

This implies, the NFSoS at $t_0 = 0$ is

$$N_0(t) = \frac{e^{-t} + 3e^t - e^{2t}}{3}, N_1(t) = \frac{e^t - e^{-t}}{2}, N_2(t) = \frac{e^{-t} - 3e^t + 2e^{2t}}{6}.$$

Thus,

$$e^{tA} = N_0(t)I + N_1(t)A + N_2(t)A^2.$$

This can be evaluated!

Three eigenpairs of A are $(1, (1, 0, 0)), (2, (2, 1, 0))$, and $(-1, (-1, 1, 3))$. Thus, a fundamental matrix is

$$\begin{pmatrix} e^t & 2e^{2t} & -e^{-t} \\ 0 & e^{2t} & e^{-t} \\ 0 & 0 & 3e^{-t} \end{pmatrix}.$$

Therefore,

$$e^{tA} = \begin{pmatrix} e^t & 2e^{2t} & -e^{-t} \\ 0 & e^{2t} & e^{-t} \\ 0 & 0 & 3e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}^{-1}.$$

□

Example 12.13. Solve the IVP using the Laplace transform method.

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}, \mathbf{x}(0) = \binom{1}{2}.$$

Solution. Let $X(s) = \mathcal{L}\{\mathbf{x}(t)\}(s)$. Taking the Laplace of both sides we obtain

$$sX(s) - \mathbf{x}(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X(s) \Rightarrow \begin{pmatrix} s & -1 \\ -1 & s \end{pmatrix} X(s) = \binom{1}{2} \Rightarrow X(s) = \begin{pmatrix} s & -1 \\ -1 & s \end{pmatrix}^{-1} \binom{1}{2}$$

Therefore,

$$X(s) = \begin{pmatrix} \dfrac{s+2}{s^2-1} \\ \\ \dfrac{2s+1}{s^2-1} \end{pmatrix}.$$

Using the method of partial fractions we obtain the following:

$$\frac{s+2}{s^2-1} = \frac{-1/2}{s+1} + \frac{3/2}{s-1} \Rightarrow \mathcal{L}^{-1}\{\frac{s+2}{s^2-1}\} = \frac{3e^t - e^{-t}}{2}.$$

Similarly

$$\frac{2s+1}{s^2-1} = \frac{1/2}{s+1} + \frac{3/2}{s-1} \Rightarrow \mathcal{L}^{-1}\{\frac{2s+1}{s^2-1}\} = \frac{e^{-t} + 3e^t}{2}.$$

Therefore,

$$\mathbf{x}(t) = \begin{pmatrix} \dfrac{3e^t - e^{-t}}{2} \\ \\ \dfrac{e^{-t} + 3e^t}{2} \end{pmatrix}$$

□

Example 12.14. Prove that if A is a constant square matrix for which $e^{tA} = I + tA$, then $A^2 = 0$.

Solution. Taking the derivative of both sides we obtain $Ae^{tA} = A$. Taking the derivative again we obtain $A^2 e^{tA} = 0$. Substituting $t = 0$ yields $A^2 = 0$. □

Example 12.15. Suppose $A(t)$ is a square matrix all of whose entries are functions of t that are differentiable over $\mathbb{R}$. Assume $A(s+t) = A(s)A(t)$ for all $s, t \in \mathbb{R}$, and assume $A(0) = I$. Prove that $A(t) = e^{tB}$ for some constant matrix B.

Scratch. We can determine B by differentiating both sides and thus obtain $A'(t) = Be^{tB}$, which means $B = A'(0)$. We will show $A(t)$ and $e^{tA'(0)}$ both satisfy the same IVP. Then we will apply the uniqueness.

Solution. Taking the derivative of both sides with respect to s we obtain $A'(s+t) = A'(s)A(t)$. Substituting $s = 0$ we obtain $A'(t) = A'(0)A(t)$. Therefore, columns of $A(t)$ satisfy the equation $\mathbf{x}' = A'(0)\mathbf{x}$. On the other hand all columns of $e^{tA'(0)}$ also satisfy the system $\mathbf{x}' = A'(0)\mathbf{x}$. Since $e^{0A'(0)} = I = A(0)$, by uniqueness we must have $A(t) = e^{tA'(0)}$, as desired. □

Example 12.16. Is there a nonzero vector $\mathbf{v} \in \mathbb{R}^n$ for which $(\cos t)\mathbf{v}$ is a solution to an n-dimensional linear homogeneous system with constant coefficients? How about $e^{t^2}\mathbf{v}$? How about $e^{5t}\mathbf{v}$?

Solution. We will substitute $\mathbf{x} = (\cos t)\mathbf{v}$ into $\mathbf{x}' = A\mathbf{x}$. This yields $-(\sin t)\mathbf{v} = A(\cos t)\mathbf{v}$, which can be written as $A\mathbf{v} = -(\tan t)\mathbf{v}$, if $\cos t \neq 0$. Since $\mathbf{v}$ is a nonzero vector, the quantity $-\tan t$ is an eigenvalue of A. However, A has at most n distinct eigenvalues. This is a contradiction. Similarly for e^{t^2} there is no such vector $\mathbf{v}$.

Applying a similar strategy to $e^{5t}\mathbf{v}$, we obtain

$$5e^{5t}\mathbf{v} = Ae^{5t}\mathbf{v} \Rightarrow A\mathbf{v} = 5\mathbf{v}.$$

We can set $A = 5I$, and $\mathbf{v} = \mathbf{e}_1$. So, there is a such a system for e^{5t}. $\square$

12.4 Exercises

Exercise 12.1. *Find e^{tA} for each matrix.*

(a) $A = \begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$

(c) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Exercise 12.2. *Let n be a positive integer and let A be the $n \times n$ matrix whose entries are all 1.*

(a) Find a polynomial $p(z)$ for which $p(A) = 0$. Hint: Evaluate A^2.

(b) Using the method of natural fundamental set of solutions find e^{tA}.

Exercise 12.3. *Suppose $(\lambda, \mathbf{v})$ is an eigenpair of a square real matrix A, where $\lambda \in \mathbb{C}$ is not real.*

(a) Prove that $(\overline{\lambda}, \overline{\mathbf{v}})$ is also an eigenpair of A. Hint: See Exercise 1.23.

(b) Using part (a), prove that $\mathrm{Re}(e^{\lambda t}\mathbf{v})$ and $\mathrm{Im}(e^{\lambda t}\mathbf{v})$ are linearly independent solutions of $\mathbf{x}' = A\mathbf{x}$.

Exercise 12.4. *Solve the initial value problem, once by the method of variation of parameters, and once using Laplace transform.*

$$\begin{cases} x' = x + 2y - 2 \\ y' = -x + 4y, \\ x(0) = 1, y(0) = 0 \end{cases}$$

Exercise 12.5. *Solve* $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$, *where* $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$.

Exercise 12.6. *Evaluate e^{tA}, where* $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$.

Exercise 12.7. *For a positive integer n, let A be the $(2n) \times (2n)$ matrix, shown below, all of whose entries are 1 except for the ones in positions (i, j), where $i > n$ and $j \leq n$. Using the NFS method evaluate e^{tA}. (Hint: Evaluate $A^2 - nA$.)*

$$A = \begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{pmatrix}_{(2n)\times(2n)}$$

Exercise 12.8. *Prove that* $\mathcal{L}\{\frac{d^n\mathbf{x}}{dt^n}\} = s^n X(s) - s^{n-1}\mathbf{x}(0) - s^{n-2}\mathbf{x}'(0) - \cdots - \mathbf{x}^{(n-1)}(0)$.

Exercise 12.9. *Suppose A, B are two square matrices of the same size. Assume $Ae^{tB} = e^{tB}A$ for all $t \in \mathbb{R}$. Prove that $AB = BA$.*

Exercise 12.10. *Suppose $A \in M_n(\mathbb{R})$ for which $e^{tA} = I + tA^2$ for all $t \in \mathbb{R}$. Prove that $A = 0$.*

Hint: See Example 12.14.

12.5 Challenge Problems

Exercise 12.11. *Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ for which there is a nonzero vector $\mathbf{v} \in \mathbb{R}^n$ for which $\mathbf{x}(t) = f(t)\mathbf{v}$ is a solution to some equation of the form $\mathbf{x}' = A\mathbf{x}$, where A is a constant $n \times n$ matrix.*

Exercise 12.12. *Suppose $A(t)$ is a square matrix for which $A(t)$ and $[A(t)]^{-1}$ are both infinitely many times differentiable for every $t \in \mathbb{R}$. Assume in addition that $A''(t) = A'(t)[A(t)]^{-1}A'(t)$ for all $t \in \mathbb{R}$. Prove that there are constant square matrices B and C for which $A(t) = e^{tB}C$.*

12.6 Summary

- To find e^{tA}:
  - Find a polynomial $p(z)$ for which $p(A) = 0$.
  - Find a NFSoS $N_0(t), N_1(t), \ldots, N_m(t)$ for $p(D)[y] = 0$. (Here, the order of $p(D)[y] = 0$ is $m + 1$.)
  - $e^{tA} = N_0(t)I + N_1(t)A + \cdots + N_m(t)A^m$.
- If $(\lambda, \mathbf{v})$ is an eigenpair for A, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}$.
- If λ is not real, then by taking the real and imaginary parts of $e^{\lambda t}\mathbf{v}$ we can find two linearly independent solutions.
- To find particular solution to $\mathbf{x}' = A\mathbf{x}+\mathbf{f}(\mathbf{x})$ we will find $\mathbf{u}$ from $\mathbf{u}' = [\Phi]^{-1}A$. This gives us a particular solution $\mathbf{x}_P(t) = \Phi(t)\mathbf{u}(t)$.

- The solution to the initial value problem $\mathbf{x}' = A\mathbf{x}, \mathbf{x}(t_0) = \mathbf{x}_0$ is given by

$$\mathbf{x}(t) = e^{(t-t_0)A}\mathbf{x}_0 + \int_{t_0}^{t} e^{(t-s)A}\mathbf{f}(s)\ ds.$$

- Solving initial value problems with the method of Laplace Transforms is done via solving the equation $sX(s) - \mathbf{x}_0 = AX(s) + F(s)$.

Chapter 13

Qualitative Theory of Differential Equations

13.1 Autonomous Systems

The main focus of this chapter is the study of solutions to systems of the form $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, called autonomous systems.

Definition 13.1. Any system of the form $\dfrac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$ is called **autonomous**.

Definition 13.2. A solution to a system $\dfrac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$ is called **stationary** or **equilibrium** (or a fixed point or a critical point), if it is a constant function.

Example 13.1. Find all stationary solutions of the system $x' = y^2 - 1,\ y' = xy^2 + x$.

Definition 13.3. A solution to a system $\dfrac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$ is said to be **semistationary** if all components of $\mathbf{x}(t)$, except for one, are constant.

Example 13.2. Find all stationary and semistationary solutions of $x' = x^2 - xy - x + y,\ y' = y(x^2 - 2x + 3)$.

We will now focus on first order 2-dimensional autonomous systems: $\dfrac{dx}{dt} = f(x, y), \dfrac{dy}{dt} = g(x, y)$.
There are several questions that we would like to answer for such systems.

- Are there stationary solutions?
- Are there semistationary solutions?
- What happens to a solution if the initial value is slightly modified?
- What happens to a solution over the long run, i.e. as t gets large?
- Are there periodic solutions?

13.2 Orbit Equation

Example 13.3. Solve the system $x' = 2y,\ y' = 2x - 4x^3$.

Definition 13.4. The **orbit equation** of the system $x' = f(x, y),\ y' = g(x, y)$ is the equation

$$g(x, y)\frac{dx}{dt} - f(x, y)\frac{dy}{dt} = 0.$$

Sometimes this equation is exact or can be solved using an integrating factor. Other times it is not possible to solve.

13.3 Stability of Solutions

Definition 13.5. A solution $\mathbf{\Phi}_0(t)$ to $\dfrac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$ is said to be **stable** if every solution $\mathbf{\Phi}(t)$ that starts sufficiently close to $\mathbf{\Phi}_0(t)$ remains close to $\mathbf{\Phi}_0(t)$ for all future values of t. In other words, we say $\mathbf{\Phi}_0(t)$ is stable if the following holds:

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ s.t. if } ||\mathbf{\Phi}(0) - \mathbf{\Phi}_0(0)|| < \delta, \text{ then } ||\mathbf{\Phi}(t) - \mathbf{\Phi}_0(t)|| < \epsilon \text{ for all solutions } \mathbf{\Phi}, \text{ and all future } t.$$

Otherwise, we say $\mathbf{\Phi}_0(t)$ is **unstable**.

Note that sometimes even though $\mathbf{f}(\mathbf{x})$ has derivatives of all orders, there are solutions that are not defined for all $t \geq 0$. This is why we say "for all future t" instead of saying for all $t \geq 0$. For example, the solution to $y' = y^2$ with initial condition $y(0) = 1$ is given by $y(t) = 1/(1-t)$, and thus we may only discuss the behavior of this function over the interval $(-\infty, 1)$. In this case "all future t" means $0 < t < 1$. This can be stated as follows by defining a "maximal" solution.

Definition 13.6. A solution $\mathbf{x}(t)$ of $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, defined over an open interval (a, b) is called **maximal**, if this solution cannot be extended to a solution over an open interval (α, β) properly containing (a, b).

In this chapter we assume all solutions are maximal.

Note also that $\mathbf{x} = \mathbf{0}$ is always a solution to any homogeneous equation $\mathbf{x}' = A\mathbf{x}$. We will start with some examples on stability of the solution $\mathbf{0}$.

Example 13.4. In each case check if $\mathbf{0}$ is a stable or unstable solution to the system $\mathbf{x}' = A\mathbf{x}$.

(a) $A = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix}$

(b) $A = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$

(d) $A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

Definition 13.7. A solution $\mathbf{\Phi}_0(t)$ is said to be **asymptotically stable** if it is stable and every solution that starts sufficiently close to $\mathbf{\Phi}_0(t)$ approaches $\mathbf{\Phi}_0(t)$ as t gets large. In other words, there exists $\delta > 0$ such that for every solution $\mathbf{\Phi}(t)$

$$\text{If } ||\mathbf{\Phi}(0) - \mathbf{\Phi}_0(0)|| < \delta, \text{ then } ||\mathbf{\Phi}_0(t) - \mathbf{\Phi}(t)|| \to 0 \text{ as } t \text{ gets as large as possible.}$$

Theorem 13.1. *Consider the system* $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$ $(*)$, *where A is a constant $n \times n$ matrix.*

- *If all eigenvalues of A have negative real parts, then all solutions of $(*)$ are asymptotically stable.*
- *If at least one eigenvalue of A has positive real part, then all solutions of $(*)$ are unstable.*
- *Suppose all eigenvalues of A have nonpositive real parts. Let $\lambda_1, \ldots, \lambda_k$ (with $k \geq 1$) be all distinct eigenvalues of A whose real parts are zero. Suppose the multiplicity of λ_j as a root of the characteristic polynomial of A is m_j.*
  - *If A has m_j linearly independent eigenvectors corresponding to λ_j for every j, then every solution to $(*)$ is stable, but not asymptotically stable.*
  - *Otherwise, every solution to $(*)$ is unstable.*

Example 13.5. Determine if $\mathbf{0}$ is a stable, unstable or asymptotically stable solution to the system $\mathbf{x}' = A\mathbf{x}$, where A is given below:

$$A = \begin{pmatrix} 0 & 4 & -1 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

13.4 Stability of Stationary Solutions to Nonlinear Systems

In order to understand the stability of a stationary solution to a system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, we will follow the steps below:

- Approximate the system by a linear system near the stationary solution. This can be done using the tangent plane approximation.
- Find out if zero is a stable solution to the linear system.
- Find out if the stability of zero for the approximated system and the stability of the stationary solution for the original system are equivalent.

Let us first consider the following system:

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

Tangent plane approximation for functions $f(x, y)$ and $g(x, y)$ are given below:

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$
$$g(x, y) \approx g(x_0, y_0) + g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0).$$

The system can now be approximated by the following system:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{pmatrix} + \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}.$$

Assuming (x_0, y_0) is a stationary point, we have $f(x_0, y_0) = g(x_0, y_0) = 0$, and thus we obtain the following:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}.$$

Setting $\tilde{x} = x - x_0, \tilde{y} = y - y_0$ we obtain the following linear system:

$$\frac{d}{dt}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}.$$

The above system is often called the **linearization** of the original system. This can be done for systems with any number of variables.

Example 13.6. Find and classify all stationary solutions to the following system:

$$\frac{dx}{dt} = 1 - xy, \ \frac{dy}{dt} = x - y^3.$$

Theorem 13.2. *Suppose* $\mathbf{x}_0$ *is a stationary solution to* $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ $(*)$. *Let* A *be the Jacobian matrix of* $\mathbf{f}(\mathbf{x})$ *at* $\mathbf{x}_0$.

- *If all eigenvalues of* A *have negative real parts, then* $\mathbf{x}_0$ *is an asymptotically stable solution to* $(*)$.
- *If at least one eigenvalue of* A *has positive real part, then* $\mathbf{x}_0$ *is an unstable solution to* $(*)$.
- *If none of the above happens, then* $\mathbf{x}_0$ *could be stable, unstable or asymptotically stable.*

Example 13.7. Characterize the stability of all stationary solutions to the following system

$$\frac{dx}{dt} = \sin(x + y), \ \frac{dy}{dt} = e^x - 1.$$

13.5 More Examples

Example 13.8. Determine if $\mathbf{0}$ is a stable, asymptotically stable, or unstable solution of the system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ in each of the following cases. Solve it once using the $\epsilon - \delta$ definition and once using an appropriate theorem.

(a) $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$

(c) $A = \begin{pmatrix} -4 & -2 \\ 1 & -1 \end{pmatrix}$

Solution. (a) This matrix only has one eigenvalue of 2 with multiplicity 2. Since 2 is positive, by Theorem 13.1 all solutions are unstable.

Now, we will prove $\mathbf{0}$ is unstable using the $\epsilon - \delta$ definition. Assume on the contrary $\mathbf{0}$ is stable. Set $\epsilon = 1$ in the definition of stability. Therefore, there is $\delta > 0$ for which

$$||\mathbf{\Phi}(0) - \mathbf{0}|| < \delta \text{ implies } ||\mathbf{\Phi}(t) - \mathbf{0}|| < 1 \text{ for all } t \geq 0, \text{ and all solutions } \Phi$$

We see that $(2, \binom{1}{1})$ is an eigenpair of A, and thus $\mathbf{\Phi}(t) = \frac{\delta}{2}e^{2t}\binom{1}{1}$ is a solution. Note that

$$||\mathbf{\Phi}(0) - \mathbf{0}|| = ||\frac{\delta}{2}\begin{pmatrix}1\\1\end{pmatrix}|| = \frac{\delta\sqrt{2}}{2} < \delta$$

By assumption we must have

$$||\frac{\delta}{2}e^{2t}\begin{pmatrix}1\\1\end{pmatrix}|| < 1, \text{ for all } t \geq 0 \Rightarrow \delta\sqrt{2}e^{2t} < 2.$$

Letting $t \to \infty$ we obtain a contradiction. Therefore, $\mathbf{0}$ is unstable.

(b) This matrix has precisely one eigenvalue of 0 with multiplicity 2. Its eigenspace is 1-dimensional. Thus, by Theorem 13.1 all solutions to this system are unstable.

Now, we will prove $\mathbf{0}$ is unstable using the $\epsilon - \delta$ definition. Note that $A^2 = 0$ since the characteristic polynomial of A is z^2. We will find a NFSoS at initial time $t_0 = 0$ for $y'' = 0$. The general solution to this equation is $Y_H = c_1 + c_2 t$. The initial values $y(0) = y_0, y'(0) = y_1$ yield $c_1 = y_0, c_2 = y_1$ and thus the general solution is $Y_H = y_0 + y_1 t$, which implies $\{1, t\}$ is a NFSoS at initial time $t_0 = 0$. Therefore,

$$e^{tA} = I + tA = \begin{pmatrix} 1+t & -t \\ t & 1-t \end{pmatrix}.$$

This means the general solution is

$$y = c_1\begin{pmatrix}1+t\\t\end{pmatrix} + c_2\begin{pmatrix}-t\\1-t\end{pmatrix}.$$

On the contrary assume $\mathbf{0}$ is stable. Let $\epsilon = 1$ in the definition of stability. Thus, there exists $\delta > 0$ for which the following holds:

$$\forall c_1, c_2 \in \mathbb{R}, ||c_1\begin{pmatrix}1+0\\0\end{pmatrix} + c_2\begin{pmatrix}-0\\1-0\end{pmatrix} - \mathbf{0}|| < \delta \text{ implies } ||c_1\begin{pmatrix}1+t\\t\end{pmatrix} + c_2\begin{pmatrix}-t\\1-t\end{pmatrix} - \mathbf{0}|| < 1 \text{ for all } t \geq 0.$$

Setting $c_1 = 0$ and $c_2 = \delta/2$ we see that

$$||0\begin{pmatrix}1\\0\end{pmatrix} + \frac{\delta}{2}\begin{pmatrix}0\\1\end{pmatrix} - \mathbf{0}|| = \frac{\delta}{2} < \delta$$

Therefore, we must have

$$||0\begin{pmatrix}1+t\\t\end{pmatrix}+\frac{\delta}{2}\begin{pmatrix}-t\\1-t\end{pmatrix}-\mathbf{0}||<1 \text{ for all } t\geq 0 \Rightarrow \frac{\delta}{2}\sqrt{t^2+(1-t)^2}<1 \text{ for all } t\geq 0.$$

Letting $t\to\infty$ we obtain a contradiction.

(c) The eigenvalues of A are -2 and -3. Both have negative real parts, and thus all solutions are asymptotically stable.

Now, we will prove $\mathbf{0}$ is stable using the $\epsilon-\delta$ definition. Two eigenpairs are $(-2,\binom{-1}{1})$ and $(-3,\binom{-2}{1})$. Thus, the general solution is $\mathbf{\Phi}(t)=c_1e^{-2t}\binom{-1}{1}+c_2e^{-3t}\binom{-2}{1}$. Let $\epsilon>0$ and set $\delta=\epsilon/100$. Suppose $||\mathbf{\Phi}(0)||<\delta$. We have:

$$||c_1\begin{pmatrix}-1\\1\end{pmatrix}+c_2\begin{pmatrix}-2\\1\end{pmatrix}||<\delta\Rightarrow\sqrt{(c_1+2c_2)^2+(c_1+c_2)^2}<\delta\Rightarrow|c_1+2c_2|<\delta, \text{ and } |c_1+c_2|<\delta.$$

Applying the Triangle Inequality we obtain:

$$|c_1|\leq|-2(c_1+c_2)|+|c_1+2c_2|<2\delta+\delta=3\delta$$
$$|c_2|\leq|c_1+c_2|+|-c_1|<\delta+3\delta=4\delta$$

Combining these and the Triangle Inequality we obtain, for every $t\geq 0$:

$$||c_1e^{-2t}\begin{pmatrix}-1\\1\end{pmatrix}+c_2e^{-3t}\begin{pmatrix}-2\\1\end{pmatrix}||\leq|c_1|e^{-2t}||\begin{pmatrix}-1\\1\end{pmatrix}||+|c_2|e^{-3t}||\begin{pmatrix}-2\\1\end{pmatrix}||<3\delta\sqrt{2}+4\delta\sqrt{5}=(3\sqrt{2}+4\sqrt{5})\delta.$$

This is less than ϵ, as $3\sqrt{2}+4\sqrt{5}<100$. Therefore, $\mathbf{0}$ is a stable solution to the given linear system. Also, similar to above

$$||c_1e^{-2t}\begin{pmatrix}-1\\1\end{pmatrix}+c_2e^{-3t}\begin{pmatrix}-2\\1\end{pmatrix}||\leq|c_1|e^{-2t}\sqrt{2}+|c_2|e^{-3t}\sqrt{5}\to 0, \text{ as } t\to\infty.$$

Therefore, by the Squeeze Theorem $||c_1e^{-2t}\binom{-1}{1}+c_2e^{-3t}\binom{-2}{1}||\to 0$, and thus $\mathbf{0}$ is asymptotically stable. □

Example 13.9. Determine if solutions to the system are stable, unstable or asymptotically stable.

$$\frac{d\mathbf{x}}{dt}=\begin{pmatrix}-1&1\\0&-2\end{pmatrix}\mathbf{x}+\begin{pmatrix}0\\3\end{pmatrix}$$

Solution. By Exercise 13.4, stability of any solution to the above system is equivalent to the stability of $\mathbf{0}$ as a solution to the homogeneous system

$$\frac{d\mathbf{x}}{dt}=\begin{pmatrix}-1&1\\0&-2\end{pmatrix}\mathbf{x}.$$

Since the coefficient matrix is upper triangular, its eigenvalues are its diagonal entries, -1 and -2. By Theorem 13.1 every solution to this system is stable. Thus, every solution to the nonhomogeneous system is also stable. □

Example 13.10. Suppose A and B are two matrices in $M_2(\mathbb{R})$ for which $\mathbf{0}$ is a stable solution of both systems $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, and $\frac{d\mathbf{x}}{dt} = B\mathbf{x}$. Is it true that $\mathbf{0}$ must be a stable solution of the system $\frac{d\mathbf{x}}{dt} = (A+B)\mathbf{x}$? Answer the same question if the word "stable" is replaced by "unstable" or "asymptotically stable". Do the same problem when we add the additional condition that $AB = BA$.

Solution. Let $A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, and $B = \begin{pmatrix} -1 & 0 \\ 5 & -1 \end{pmatrix}$. We have $A + B = \begin{pmatrix} -2 & 1 \\ 5 & -2 \end{pmatrix}$. Since all eigenvalues of A and B are negative, $\mathbf{0}$ is an asymptotically stable solution to both systems $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x}' = B\mathbf{x}$. On the other hand, characteristic polynomial of $A+B$ is $(z+2)^2 - 5$ which has a positive root and thus $\mathbf{0}$ is an unstable solution to $\mathbf{x}' = (A+B)\mathbf{x}$. Thus, the answer is no, for "stable" and "asymptotically stable" cases.

Similarly if we set $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then both A and B have an eigenvalue of 1 which is positive and thus $\mathbf{0}$ is an unstable solution to both systems $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x}' = B\mathbf{x}$. On the other hand $A + B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ has an eigenvalue of 0 whose eigenspace is 2-dimensional, and thus the system $\mathbf{x}' = (A+B)\mathbf{x}$ is stable. Note that in this case $AB = BA$ and thus the answer for "unstable" is negative even if $AB = BA$.

Suppose $AB = BA$ and $\mathbf{0}$ is a stable solution of both $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x}' = B\mathbf{x}$. If $(\lambda, \mathbf{v})$ is an eigenpair of A, then $A\mathbf{v} = \lambda\mathbf{v}$. We have

$$AB\mathbf{v} = BA\mathbf{v} = B\lambda\mathbf{v} = \lambda B\mathbf{v}.$$

Thus, $B\mathbf{v}$ lies in the eigenspace of A corresponding to λ. We will take two cases:

Case I: The dimension of the eigenspace of A corresponding to λ is 1. In this case, $B\mathbf{v}$ is a multiple of $\mathbf{v}$. If the eigenvalues of A are distinct, then we can repeat this argument and get a basis $\mathbf{v}, \mathbf{w}$ for $\mathbb{R}^2$ consisting of eigenvectors of A. Writing A and B in this basis we obtain:

$$A = \begin{pmatrix} \mathbf{v} & \mathbf{w} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \mathbf{v} & \mathbf{w} \end{pmatrix}^{-1}, \text{ and } B = \begin{pmatrix} \mathbf{v} & \mathbf{w} \end{pmatrix} \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{pmatrix} \begin{pmatrix} \mathbf{v} & \mathbf{w} \end{pmatrix}^{-1}.$$

Adding these up we obtain

$$A + B = \begin{pmatrix} \mathbf{v} & \mathbf{w} \end{pmatrix} \begin{pmatrix} \lambda_1 + \lambda_3 & 0 \\ 0 & \lambda_2 + \lambda_4 \end{pmatrix} \begin{pmatrix} \mathbf{v} & \mathbf{w} \end{pmatrix}^{-1}.$$

By assumption, all λ_j's have nonpositive real parts and thus real parts of $\lambda_1 + \lambda_3$ and $\lambda_2 + \lambda_4$ are both nonpositive. Since $A + B$ is diagonalizable, the multiplicity of each eigenvalue is equal to the dimension of the corresponding eigenspace. Thus, by Theorem 13.1, the solution $\mathbf{0}$ is stable.

If A has a unique eigenvalue λ, then by Theorem 13.1, since $\mathbf{0}$ is a stable solution for $\mathbf{x}' = A\mathbf{x}$, the real part of eigenvalue λ is negative. Let $\mathbf{v}, \mathbf{w}$ be a basis for $\mathbb{R}^2$ and write A and B in this basis. We obtain

$$A = \begin{pmatrix} \mathbf{v} & \mathbf{w} \end{pmatrix} \begin{pmatrix} \lambda & * \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \mathbf{v} & \mathbf{w} \end{pmatrix}^{-1}, \text{ and } B = \begin{pmatrix} \mathbf{v} & \mathbf{w} \end{pmatrix} \begin{pmatrix} \lambda_3 & * \\ 0 & \lambda_4 \end{pmatrix} \begin{pmatrix} \mathbf{v} & \mathbf{w} \end{pmatrix}^{-1}.$$

Adding these up we obtain

$$A + B = \begin{pmatrix} \mathbf{v} & \mathbf{w} \end{pmatrix} \begin{pmatrix} \lambda + \lambda_3 & * \\ 0 & \lambda_2 + \lambda_4 \end{pmatrix} \begin{pmatrix} \mathbf{v} & \mathbf{w} \end{pmatrix}^{-1}.$$

Since the real parts of λ_3, λ_4 are nonpositive and the real part of λ is negative, the real parts of $\lambda + \lambda_3$ and $\lambda + \lambda_4$ are both negative. Therefore, the real parts of eigenvalues of $A + B$ are negative. Thus $\mathbf{0}$ is a stable solution for $\mathbf{x}' = (A + B)\mathbf{x}$.

Case II: The dimension of the eigenspace of A corresponding to λ is 2. Therefore, writing A in a basis consisting of two eigenvectors we obtain $A = P(\lambda I)P^{-1} = \lambda I$.

Similar to Case I, if B has an eigenvalue whose corresponding eigenspace is 1-dimensional, then we are done. So, we may assume B also has a unique eigenvalue γ whose corresponding eigenspace is 2-dimensional, and thus $B = \gamma I$. By Theorem 13.1, the real parts of both λ and of γ are nonpositive. We see $A + B = (\lambda + \gamma)I$, and the real part of $\lambda + \gamma$ is nonpositive. Thus, by Theorem 13.1, the solution $\mathbf{0}$ is stable for $\mathbf{x}' = (A + B)\mathbf{x}$.

The argument for the asymptotically stable case is similar.

We summarize the results in the following table:

$\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x}' = B\mathbf{x}$	Require $AB = BA$?	$\mathbf{x}' = (A + B)\mathbf{x}$
Unstable	No	No Conclusion
Stable	No	No Conclusion
Asymptotically Stable	No	No Conclusion
Unstable	Yes	No Conclusion
Stable	Yes	Stable
Asymptotically Stable	Yes	Asymptotically Stable

□

Example 13.11. Find all stationary solutions of each system.

(a) $x' = x^3 - y, y' = \sin(xy)$.

(b) $x' = x - y, y' = x^2 - y^2$.

Solution. (a) All stationary solutions satisfy $y = x^3$ and $xy = \pi k$ for some $k \in \mathbb{Z}$. Thus, $x^4 = \pi k$, i.e. $x = \pm\sqrt[4]{\pi k}, y = \pm\sqrt[4]{\pi^3 k^3}$. Therefore, all stationary solutions are of the form

$$(\sqrt[4]{\pi k}, \sqrt[4]{\pi^3 k^3}), \text{ and } (-\sqrt[4]{\pi k}, -\sqrt[4]{\pi^3 k^3}), k \geq 0 \text{ is an integer}$$

(b) We need $y = x$ and $x^2 = y^2$. Thus, all points of the form (c, c), where $c \in \mathbb{R}$ are stationary solutions. $\square$

Example 13.12. Find all real numbers c for which $\mathbf{0}$ is a stable solution to the following system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} c+1 & 2 \\ 5 & -1 \end{pmatrix}.$$

Solve the problem when "stable" is replaced by "asymptotically stable".

Solution 1. The characteristic equation is $p(z) = z^2 - cz - c - 11$. The eigenvalues are $\dfrac{c \pm \sqrt{c^2 + 4c + 44}}{2}$. The discriminant is equal to $(c+2)^2 + 40$ and thus it is positive. Therefore, the eigenvalues are distinct and real. For $\mathbf{0}$ to be stable we need both roots to be nonpositive. The larger one is $\dfrac{c + \sqrt{c^2 + 4c + 44}}{2}$, which means we need to have

$$c + \sqrt{c^2 + 4c + 44} \leq 0 \Rightarrow \sqrt{c^2 + 4c + 44} \leq -c$$

For this inequality to hold we need $-c > 0$ and

$$c^2 + 4c + 44 \leq c^2 \Rightarrow 4c + 44 \leq 0 \Rightarrow c \leq -11.$$

Thus, $\mathbf{0}$ is stable if and only if $c \leq -11$, and it is asymptotically stable if and only if $c < -11$. $\square$

Solution 2. Similar to above, the two eigenvalues are real. For both to be nonnegative we need their sum to be nonnegative and their product to be nonpositive. Their sum is c and their product is $-c - 11$. Thus $\mathbf{0}$ is stable if and only if $-c - 11 \geq 0$ and $c \leq 0$. This yields $c \leq -11$. $\square$

13.6 Exercises

Exercise 13.1. *Find all stationary solutions of each system.*

(a) $x' = e^y - 1, y' = x^2 - x.$

(b) $x' = \sin(xy), y' = x^2 - y^2.$

(c) $x' = x^2 y - xy + x, y' = xy^2 - y.$

Exercise 13.2. *Find all semistationary solutions of each system.*

(a) $x' = 2x + yx, y' = \cos(y).$

(b) $x' = xy + x + y + 1, y' = x^2 y - y.$

Exercise 13.3. *Determine if* $\mathbf{0}$ *is a stable, asymptotically stable, or unstable solution of the system* $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$ *in each of the following cases. Solve it once using the* $\epsilon - \delta$ *definition and once using an appropriate theorem.*

(a) $A = \begin{pmatrix} 0 & 2 \\ -8 & 0 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & -1 & -2 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$

Exercise 13.4. *Suppose* $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$ *is continuous. Consider the linear nonhomogenuous system with constant coefficients given below:*

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t) \qquad (*)$$

(a) Using the $\epsilon - \delta$ *definition, prove that* $\mathbf{0}$ *is a stable solution to the homogeneous system* $\mathbf{x}' = A\mathbf{x}$ *if and only if every solution to* $(*)$ *is stable.*

(b) Using the $\epsilon - \delta$ *definition, prove that* $\mathbf{0}$ *is an asymptotically stable solution to the homogeneous system* $\mathbf{x}' = A\mathbf{x}$ *if and only if every solution to* $(*)$ *is asymptotically stable.*

Exercise 13.5. *Consider the system* $\dfrac{dx}{dt} = x - y$, $\dfrac{dy}{dt} = e^x + y$.

(a) Prove that this system has a unique equilibrium solution.

(b) Find the stability of this equilibrium solution.

Hint: You do not need to find the stationary solution.

Exercise 13.6. *Determine if* $\mathbf{0}$ *is a stable, asymptotically stable, or unstable solution of the system* $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$ *in each of the following cases. Solve it once using the* $\epsilon - \delta$ *definition and once using an appropriate theorem.*

(a) $A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$ *(Eigenpairs are* $(2, \binom{1}{1})$ *and* $(-1, \binom{2}{1})$*.)*

(b) $A = \begin{pmatrix} 0 & 2 \\ -1 & -3 \end{pmatrix}$ *(Eigenpairs are* $(-2, \binom{-1}{1})$ *and* $(-1, \binom{-2}{1})$*.)*

(c) $A = \begin{pmatrix} -3 & -1 \\ 2 & 0 \end{pmatrix}$ *(Eigenpairs are* $(-1, \binom{-1}{2})$ *and* $(-2, \binom{-1}{1})$*.)*

(d) $A = \begin{pmatrix} 4 & -1 & 5 \\ 4 & 0 & 4 \\ 0 & 0 & -1 \end{pmatrix}$

Exercise 13.7. *Consider the following system:*

$$x' = x + 2xy, y' = x + y - y^2.$$

(a) Find all stationary solutions of this system.

(b) Find the linearization of the system near each stationary solution.

(c) Determine if each stationary solution is stable, asymptotically stable, or unstable.

13.7 Summary

- A solution $\mathbf{\Phi}_0$ to a system is said to be stable if any solution that starts near $\mathbf{\Phi}_0$ stays near $\mathbf{\Phi}_0$.
- A solution $\mathbf{\Phi}_0$ is said to be asymptotically stable if it is stable and every solution that starts near $\mathbf{\Phi}_0$ approaches $\mathbf{\Phi}_0$.
- To prove $\mathbf{0}$ is a stable solution of $\mathbf{x}' = A\mathbf{x}$ using the $\epsilon - \delta$ definition:
  - Find the general solution $\mathbf{\Phi}(t)$.
  - Start with the inequality $||\mathbf{\Phi}(t) - \mathbf{\Phi}_0(t)|| < \epsilon$. Using the Triangle Inequality, eliminate t, and find out what upper bounds for $c_1, c_2, \ldots$ you need.
  - Use the inequality $||\mathbf{\Phi}(0)|| < \delta$ to get upper bounds for $c_1, c_2, \ldots$ in terms of δ.
  - Combine this information to find δ in terms of ϵ.
- To prove $\mathbf{0}$ is an unstable solution to $\mathbf{x}' = A\mathbf{x}$ using the $\epsilon - \delta$ definition:
  - On the contrary assume $\mathbf{0}$ is stable.
  - Write down the definition of stability with some small $\epsilon > 0$. (For linear systems any positive ϵ would work.)
  - Yield a contradiction as $t \to \infty$.
- Theorem 13.1 helps determine if solutions to a linear system are stable, unstable, or asymptotically stable.
- To find statinary solutions of a system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, we solve the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$.
- To find semi-stationary solutions of the system $x' = f(x, y), y' = g(x, y)$ we will find all values of x_0 for which $f(x_0, y) = 0$. Then we will solve the equation $y' = g(x_0, y)$. We will do the same for $y = y_0$.
- To check stability of stationary solutions to system $x' = f(x, y)$, $y' = g(x, y)$:
  - Write down the Jacobian matrix of f, g.
  - Evaluate the Jacobian matrix at the stationary solutions.
  - Find all eigenvalues of the Jacobian matrix.
  - If all eigenvalues have negative real parts, then the stationary solution is asymptotically stable.
  - If there is an eigenvalue with positive real part, then the stationary solution is unstable.
  - In all other cases, we cannot infer the stability of the stationary solution.

Chapter 14

Orbits and Phase Plane Portraits

14.1 Orbits and Their Properties

Definition 14.1. Any solution $(x(t), y(t))$ to a first order autonomous system

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y)$$

can be represented with a curve in the xy-plane. This curve is called an **orbit** of this system. When samples of these orbits are drawn in the xy-plane we obtain what we call a **phase plane portrait**. An orbit consisting of a single stationary point is called a **trivial orbit**. Any other orbit is called **nontrivial**.

Theorem 14.1 (Existence and Uniqueness Theorem)**.** *Let U be an open subset of $\mathbb{R}^n$. Suppose all components of the vector field*

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})), \ \textit{where } \mathbf{x} = (x_1, x_2, \ldots, x_n),$$

have continuous first partials with respect to $x_1, \ldots, x_n$ over U. Then, for every $t_0 \in \mathbb{R}$ and every $\mathbf{x}_0 \in U$, the initial value problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \ \mathbf{x}(t_0) = \mathbf{x}_0$$

has a unique solution defined over some open interval containing t_0, and the graph of this solution remains in U.

Theorem 14.2 (Properties of Orbits)**.** *Suppose all components of the vector field $\mathbf{f}(\mathbf{x})$ have continuous first partials. Then,*

(a) No two distinct orbits intersect.

(b) If $\mathbf{\Phi}(t)$ is a solution for which $\mathbf{\Phi}(t_0) = \mathbf{\Phi}(t_0 + T)$ for some $t_0, T \in \mathbb{R}$, with $T > 0$ then $\mathbf{\Phi}(t) = \mathbf{\Phi}(t + T)$ for all $t \geq t_0$. In other words, $\mathbf{\Phi}$ is periodic.

(c) If an orbit lies on a closed curve C, where C contains no stationary points, then the orbit must be periodic.

Example 14.1. Prove that every solution to the following system is periodic.

$$\frac{dx}{dt} = ye^{1+x^2+y^2}, \ \frac{dy}{dt} = -xe^{1+x^2+y^2}.$$

Example 14.2. Prove that every solution to the following system that starts in the right half plane $x > 0$ stays there.

$$\frac{dx}{dt} = xy, \ \frac{dy}{dt} = e^x + y^2.$$

Example 14.3. Show that every solution to the system is periodic: $\dfrac{dx}{dt} = y, \ \dfrac{dy}{dt} = -2x - 4x^3.$

14.2 Phase Plane Portrait for Linear Systems

Example 14.4. In each case draw a phase plane portrait for the system $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$.

(a) $A = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$ (Two eigenpairs are $(-3, \binom{-1}{1})$ and $(3, \binom{2}{1})$.)

(b) $A = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}$ (Two eigenpairs are $(1, \binom{-1}{1})$ and $(2, \binom{-2}{3})$.)

(c) $A = \begin{pmatrix} 1 & 6 \\ -1 & -4 \end{pmatrix}$ (Two eigenpairs are $(-1, \binom{-3}{1})$ and $(-2, \binom{-2}{1})$.)

Example 14.5. In each case draw the phase plane portrait of system $\mathbf{x}' = A\mathbf{x}$:

(a) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ (One eigenpair is $(1+i, \binom{-i}{1})$.)

(b) $\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$ (One eigenpair is $(2i, \binom{-i}{2})$).

(c) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

14.3 Phase Plane Portrait for Nonlinear Systems Near Stationary Solutions

Example 14.6. Consider the system

$$\frac{dx}{dt} = y, \ \frac{dy}{dt} = 4x - x^3.$$

(a) Find all stationary solutions of this system.

(b) Write down the linearization of this system near each stationary solution.

(c) Sketch the phase plane portrait of this system near each stationary solution.

(d) Classify each stationary solution as stable or unstable, and also as attracting or not attracting.

Theorem 14.3 (Poincare-Bendixon Theorem)**.** *Let R be a closed and bounded region of the xy-plane. Suppose $f(x, y)$ and $g(x, y)$ have continuous first partials over an open region containing R. Assume a solution $x(t), y(t)$ to a system of equations*

$$\frac{dx}{dt} = f(x, y),\ \frac{dy}{dt} = g(x, y)$$

remains in R for all future $t \geq 0$. Suppose further that R contains no stationary solutions. Then, either the orbit $(x(t), y(t))$ is itself a closed curve or it spirals into a simple close curve which itself is an orbit of a periodic solution. Therefore, any such system has a periodic solution.

Example 14.7. Prove that the equation $z'' + (z^2 + 2z'^2 - 1)z' + z = 0$ has a nontrivial periodic solution.

Example 14.8. Prove that the following system has a nontrivial periodic solution.

$$x' = x(1 - 4x^2 - y^2) - \frac{1}{2}y(1 + x),\ y' = y(1 - 4x^2 - y^2) + 2x(1 + x).$$

Example 14.9. Prove that the following system has a nontrivial periodic solution.

$$x' = -y + x(1 - x^2 - y^2),\ y' = x + y(1 - x^2 - y^2).$$

Can you find one such solution?

14.4 More Examples

Example 14.10. Find the orbits of each system:

(a) $x' = y(1 + x + y), y' = -x(1 + x + y)$.

(b) $x' = x^2 + \cos y, y' = -2xy$.

Sketch. We generally find the orbits by solving the orbit equation, however after that we need to see if each curve is a complete orbit or a union of orbits. For that we will find the stationary solutions. If there is no stationary solution on that curve, then the curve is a single orbit.

Solution.(a) First, we will find all stationary solutions. $y(1 + x + y) = -x(1 + x + y) = 0$ yields, either $1 + x + y = 0$ or $x = y = 0$. So, each point of the form $(a, -1 - a)$ or the origin $(0, 0)$ is an orbit.

The orbit equation is $y(1 + x + y)dy + x(1 + x + y)dx = 0$. If $1 + x + y \neq 0$, we obtain $ydy + xdx = 0$. This exact equation yields the circle $x^2 + y^2 = C$, for some nonnegative constant C.

We find find all C's for which a stationary solution is on this circle. The stationary solution $(0,0)$ is not on this circle if $C > 0$. If $(a, -1-a)$ is on $x^2+y^2 = C$ we obtain $a^2+1+2a+a^2 = C$. This yields $2a^2+2a+1-C = 0$, which yields $a = \dfrac{-1 \pm \sqrt{2C-1}}{2}$. Thus, all orbits can be listed as:

- The origin.
- All points of the form $(a, -1-a)$.
- All circles given by $x^2+y^2 = C$, where $C \in (0, 1/2)$.
- The circle $x^2+y^2 = 1/2$, excluding the point $(-1/2, -1/2)$.
- Each of the two arcs of the circle $x^2+y^2 = C$, bwteeen the points
$$\left(\frac{-1+\sqrt{2C-1}}{2}, \frac{-1-\sqrt{2C-1}}{2}\right), \text{ and } \left(\frac{-1-\sqrt{2C-1}}{2}, \frac{-1+\sqrt{2C-1}}{2}\right),$$
where $C \in (1/2, \infty)$.

(b) First, we will find stationary solutions:

$$\begin{cases} x^2+\cos y = 0 \\ -2xy = 0 \Rightarrow x = 0, \text{ or } y = 0 \end{cases}$$

If $x = 0$, then the first equation yields $\cos y = 0$, i.e. $y = \pi k + \dfrac{\pi}{2}$ for some integer k. If $y = 0$, the first equation yields $x^2+1 = 0$, which has no solutions. Therefore, the stationary solutions are of the form $(0, k\pi+\pi/2)$.

The orbit equation is $(x^2+\cos y)dy + 2xydx = 0$. Solving we obtain $x^2y+\sin y = C$. Substituting by the stationary solutions we obtain $\sin y = C$, which yields $C = \pm 1$ depending on if k is even or odd. So, the orbits are of the form

- All points of the form $(0, k\pi+\pi/2)$, where k is an integer.
- Parts of the curves $x^2y+\sin y = \pm 1$ that lie between the stationary solutions above.
- Curves of the form $x^2y+\sin y = C$, where $C \neq \pm 1$ is a real number.

$\square$

Example 14.11. Consider the systems

$$(1) \begin{cases} x' = y + x^3 \\ y' = -x + y^3 \end{cases} \qquad (2) \begin{cases} x' = y - x^3 \\ y' = -x - y^3 \end{cases}$$

(a) Prove each system has precisely one stationary solution: the origin.

(b) Prove that both systems have the same linearization near the origin.

(c) Show that the phase plane for this linearization is center.

(d) Show $\mathbf{0}$ is an unstable solution of (1), but it is an asymptotically stable solution for (2).

Solution. (a) The stationary solutions for each system must satisfy

$$(1)\ \begin{cases} y + x^3 = 0 \Rightarrow y = -x^3 \\ -x + y^3 = 0 \Rightarrow x = y^3 = -x^9 \Rightarrow x(1 + x^8) = 0 \Rightarrow x = 0 \Rightarrow y = 0 \end{cases}$$

$$(2)\ \begin{cases} y - x^3 = 0 \Rightarrow y = x^3 \\ -x - y^3 = 0 \Rightarrow -x - x^9 = 0 \Rightarrow -x(1 + x^8) = 0 \Rightarrow x = 0 \Rightarrow y = 0 \end{cases}$$

(b) The Jacobian matrices for the systems are

$$(1)\begin{pmatrix} 3x^2 & 1 \\ -1 & 3y^2 \end{pmatrix}, \text{ and } (2)\begin{pmatrix} -3x^2 & 1 \\ -1 & -3y^2 \end{pmatrix}$$

The linearization near $(0, 0)$ is given by

$$\frac{d}{dt}\binom{x}{y} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\binom{x}{y}$$

(c) The eigenvalues of the Jacobian matrix are $\pm i$, and thus the phase plane portrait is center.

(d) For (1) we have $\frac{d}{dt}(x^2 + y^2) = 2xx' + 2yy' = 2(x^4 + y^4) \geq 0$. Thus, $x^2 + y^2$ is increasing. Assume $\mathbf{x}$ is a nonstationary solution. Hence, $\mathbf{x}(t) \neq \mathbf{0}$ for all t. Thus, $\frac{d}{dt}(x^2 + y^2) = 2(x^4 + y^4)$ is positive and hence, $||\mathbf{x}(t)||$ is strictly increasing. We will show $||\mathbf{x}(t)||$ approaches infinity as t gets large, and thus $\mathbf{x}(t)$ is unstable. Assume $||\mathbf{x}(t)||$ does not approach infinity. Since it is strictly increasing, it must be bounded by a real number M. Therefore,

$$\mathbf{x}(0) \leq ||\mathbf{x}(t)|| \leq R, \text{ for all future } t \geq 0.$$

Therefore, by the Poincare-Bendixon Theorem, $\mathbf{x}$ must approach a periodic solution inside the annulus $||\mathbf{x}(0)||^2 \leq x^2 + y^2 \leq R^2$. However, since $x^2 + y^2$ is strictly increasing, there cannot be any periodic solutions. This contradiction shows $\mathbf{0}$ is unstable.

Similarly, for (2) we have $\frac{d}{dt}(x^2 + y^2) = 2xx' + 2yy' = -2(x^4 + y^4) \leq 0$. By an argument similar to above $x^2 + y^2$ must be strictly decreasing and thus it must approach a periodic solution. On the other hand since $x^2 + y^2$ is strictly decreasing no solution is periodic, except for the stationary solution $\mathbf{0}$. Thus $x^2 + y^2 \to 0$ as t gets large.

Let ϵ be a positive real number. If $||\mathbf{x}(0)|| < \epsilon$, then since $||\mathbf{x}(t)||$ is decreasing, for all future $t \geq 0$ we have $||\mathbf{x}(t)|| \leq ||\mathbf{x}(0)|| < \epsilon$. Thus, $\mathbf{0}$ is stable. Combining this with the fact that $\mathbf{x}(t)$ approaches $\mathbf{0}$ we conclude that $\mathbf{0}$ is asymptotically stable. □

Example 14.12. Show that each solution to the following system that starts in the interior of the unit circle remains there.

$$\begin{cases} x' = y + x(1 - x^2 - y^2) \\ y' = -x - y(1 - x^2 - y^2) \end{cases}$$

Sketch. We need to show the unit circle is a union of orbits, and then use the fact that orbits do not intersect. The unit circle can be parametrized by $x = \cos t$, $y = \sin t$. However this does not yield a solution to the system. So, let's try something more generic. Let $x = \cos(\theta(t)), y = \sin(\theta(t))$. Since $x^2 + y^2 = \cos^2(\theta(t)) + \sin^2(\theta(t)) = 1$ we obtain the following:

$$-\sin(\theta(t))\theta'(t) = \sin(\theta(t)), \text{ and } \cos(\theta(t))\theta'(t) = -\cos(\theta(t)).$$

This means we need $\theta'(t) = -1$. We can use $\theta(t) = -t$.

Solution. First, note that the forcing is C^1 everywhere. In order to show the claim, we will show the unit circle is itself an orbit, and hence since orbits do not intersect, every solution that starts in the interior of the unit circle will stay there. $x = \cos t, y = -\sin t$ satisfies the system, since

$$y + x(1 - x^2 - y^2) = -\sin t = x', \text{ and } -x - y(1 - x^2 - y^2) = -\cos t = y'.$$

Therefore, $(\cos t, -\sin t)$ is a solution. However, we know $(\cos t, -\sin t)$ parametrizes the unit circle. Therefore, the unit circle is an orbit, which completes the proof. $\square$

14.5 Exercises

Exercise 14.1. *In each case draw a phase-plane portrait for* $\mathbf{x}' = A\mathbf{x}$.

(a) $A = \begin{pmatrix} 2 & 7 \\ 2 & -3 \end{pmatrix}$ *(Eigenpairs are* $(-5, \binom{-1}{1})$ *and* $(4, \binom{7}{2})$).

(b) $A = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$ *(Eigenpairs are* $(1, \binom{1}{0})$ *and* $(2, \binom{-2}{1})$.*)*

(c) $A = \begin{pmatrix} -13 & 4 \\ -30 & 9 \end{pmatrix}$ *(Eigenpairs are* $(-3, \binom{2}{5})$ *and* $(-1, \binom{1}{3})$.*)*

(d) $A = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$ *(Eigenpairs are* $(1, \binom{2}{1})$ *and* $(0, \binom{1}{1})$.*)*

(e) $A = \begin{pmatrix} -4 & 2 \\ -4 & 2 \end{pmatrix}$ *(Eigenpairs are* $(-2, \binom{1}{1})$ *and* $(0, \binom{1}{2})$.*)*

(f) $A = \begin{pmatrix} -1 & -1 \\ 5 & 1 \end{pmatrix}$ *(One eigenpair is* $(2i, \binom{-1+2i}{5})$*.)*

(g) $A = \begin{pmatrix} -1 & 1 \\ 3 & 1 \end{pmatrix}$ *(Eigenpairs are* $(-2, \binom{-1}{1})$ *and* $(2, \binom{1}{3})$*.)*

(h) $A = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}$ *(One eigenpair is* $(i, \binom{1-i}{1})$*.)*

(i) $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$

(j) $A = 2I$

(k) $A = -I$

(l) $A = 0$

Exercise 14.2. *Show that each solution to the system*

$$x' = x^2 + y\sin x, y' = -1 + xy + \cos y$$

that starts in the first quadrant stays there.

Exercise 14.3. *Consider the 2-dimensional system*

$$x' = f(x,y),\ y' = g(x,y),$$

where f, g *have continuous first partials on a open region* R*.*

(a) *Suppose the trajectory of an orbit on the* xy*-plane is a close simple curve* C *that lies in* R*. Let* D *be the region enclosed by* C*. Using the Green's Theorem prove that* $\iint\limits_D f_x + g_y \ \mathrm{d}A = 0$*.*

(b) *Show that if* $f_x + g_y$ *is always positive on* R*, or always negative on* R*, except possibly at some isolated points, then there are no nontrivial periodic solutions to this system on* R*.*

Exercise 14.4. *Consider the systems*

$$(1)\ \begin{cases} x' = -y - x^2 \\ y' = x \end{cases} \qquad\qquad (2)\ \begin{cases} x' = -y + x^3 \\ y' = x \end{cases}$$

(a) *Prove each system has precisely one stationary solution: the origin.*

(b) *Prove that both systems have the same linearization near the origin.*

(c) *Show that the phase plane for this linearization is center.*

(d) *Find all orbits of* (1) *and show these orbits are all periodic near the origin, but none are periodic when you get farther away from the origin.*

(e) Using Green's Theorem show (2) *does not have any nontrivial periodic solutions.*

(f) Prove that $\mathbf{0}$ *is a stable solution for* (1).

(g) Prove that $\mathbf{0}$ *is an unstable solution to* (2) *by showing* $x^2(t)+y^2(t) \to \infty$ *as* $t \to \infty$, *for any nonstationary solution* (x, y).

Hint: For the last part, prove $x^2 + y^2$ is increasing. Then use Poincare-Bendixon Theorem.

Exercise 14.5. *Show that the following equation has a nontrivial periodic solution:*

$$z'' + [\ln(z^2 + 4z'^2)]z' + z = 0$$

Solution.

□

Exercise 14.6. *Consider the following system:*

$$x' = y, y' = -\frac{1}{2}[x^2 + (x^4 + 4y^2)^{1/2}]x$$

Suppose every solution is uniquely determined once $x(0), y(0)$ *are given.*

(a) Prove that $x(t) = c\sin(ct+d), y(t) = c^2\cos(ct+d)$ *with* $c, d \in \mathbb{R}$ *yields the general solution to this system.*

(b) Show that $\mathbf{0}$ *is a stable solution to this system, but it is not asymptotically stable.*

(c) Show that every nonzero solution to this system is unstable.

Exercise 14.7. *Prove that all solutions of each equation is periodic:*

(a) $z'' + e^z = 1$

(b) $z'' + z^3/(1 + z^4) = 0$

(c) $z'' + z + z^7 = 0$

(d) $z'' + az + bz^3 = 0$, *where* a, b *are positive real numbers.*

Exercise 14.8. *Show that each solution to the following system that starts in the interior of the unit circle remains there.*

$$\begin{cases} x' = 2y - 3x(1 - x^2 - y^2) \\ y' = -2x - y(1 - x^2 - y^2) \end{cases}$$

Exercise 14.9. *Show that each solution to the following system that starts in the interior of the ellipse* $x^2 + 4y^2 = 4$ *remains there.*

$$\begin{cases} x' = -2y + 3x(4 - x^2 - 4y^2) \\ y' = \dfrac{x}{2} + 2y(4 - x^2 - 4y^2) \end{cases}$$

Exercise 14.10. *Given a nonautonomous 2-dimensional system* $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ *a solution* $\mathbf{x}(t)$ *describes a curve in the xy-plane, where* $\mathbf{x}(t) = (x(t), y(t))$*. By an example show that two such curves may intersect. Also, show that it is possible that for a solution* $\mathbf{x}$ *to satisfy* $\mathbf{x}(t_0 + T) = \mathbf{x}(t_0)$ *for some* $t_0, T > 0$ *without being periodic.*

Exercise 14.11. *Prove that all nontrivial orbits of the following system are ellipses.*

$$x' = 2y, y' = -\frac{x}{2}.$$

Exercise 14.12. *Suppose* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ *are nonzero vectors, and* $0 \le \lambda_1 < \lambda_2$ *are two real numbers, and* c_1, c_2 *are two nonzero real numbers. Consider the curve* C *given by* $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v} + c_1 e^{\lambda_2 t}\mathbf{w}$ *in the xy-plane. Let* ℓ_t *be the line tangent to* C *at* $\mathbf{x}(t)$*. Prove that as* $t \to \infty$*, the slope of* ℓ_t *approaches the slope of lines parallel to* $\mathbf{w}$*.*

Exercise 14.13. *Prove that every nontrivial orbit of the following system is either a ray or a half parabola.*

$$x' = x, y' = -2x + 2y.$$

Exercise 14.14. *Suppose* $f(x, y)$ *and* $g(x, y)$ *have continuous first partials over* $\mathbb{R}^2$*. Assume* $\mathbf{x}(t) = (x(t), y(t))$ *is a solution to the system*

$$x' = f(x, y), y' = g(x, y)$$

for which $\mathbf{x}(t) \to (x_0, y_0)$ *as* t *gets large. Prove that* (x_0, y_0) *is a stationary solution.*

Hint: Use proof by contradiction. Apply the Poincare-Bendixon Theorem.

Exercise 14.15. *Consider the systems*

$$(1) \begin{cases} x' = y - x(x^2 + y^2) \\ y' = -x - y(x^2 + y^2) \end{cases} \qquad (2) \begin{cases} x' = y + x(x^2 + y^2) \\ y' = -x - y(x^2 + y^2) \end{cases}$$

(a) Find all stationary solutions of both systems.

(b) Prove that both systems have the same linearization near the origin. (Note that the origin is a stationary solution for both systems.)

(c) Show that the phase plane for this linearization is center, and thus $\mathbf{0}$ *is stable, but not asymptotically stable solution for the linearization.*

(d) Write down the orbit equation of (1) *and solve. Use that to prove* $\mathbf{0}$ *is an asymptotically stable solution for* (1)*. Do the same for* (2) *and show* $\mathbf{0}$ *is an unstable solution for* (2)*.*

Exercise 14.16. *Consider the system*

$$x' = x^2, y' = 2xy$$

Prove that $\mathbf{0}$ *is a stable solution for the linearization of this system, but it is an unstable solution for this system.*

Exercise 14.17. *Consider the systems*

$$(1)\ x' = x^3, y' = y^3, \text{ and } (2)\ x' = -x^3, y' = -y^3$$

(a) Prove that $\mathbf{0}$ *is a stable, but not asymptotically stable solution for the linearization of both systems near the origin.*

(b) Prove that $\mathbf{0}$ *is an unstable solution for* (1)*, while it is an asymptotically stable solution for* (2)*.*

Exercise 14.18. *Prove that* $x = \sin(t^2), y = \sin(t^3), t \in \mathbb{R}$ *is not a solution to any autonomous system of the form*

$$x' = f(x, y), y' = g(x, y),$$

where f, g *are* C^1 *functions over* $\mathbb{R}^2$*.*

Exercise 14.19. *Prove that* $x = \cos(t^2), y = \sin(t^2), t \in \mathbb{R}$ *is not a solution to any autonomous system of the form*

$$x' = f(x, y), y' = g(x, y),$$

where f, g *are* C^1 *functions over* $\mathbb{R}^2$*.*

Exercise 14.20. *Find one nontrivial periodic solution to the system in Example 14.8.*

Exercise 14.21. *Suppose* $f(x, y), g(x, y)$ *are* C^1*. Prove that no nontrivial periodic solution to the system*

$$x' = f(x, y), y' = g(x, y),$$

is asymptotically stable.

14.6 Challenge Problems

Exercise 14.22. *Suppose* $A(t) = (a_{ij}(t))_{n\times n}$ *is a matrix whose entries* $a_{ij}(t)$ *are all continuous over* $\mathbb{R}$*. Suppose* $\int_0^\infty |a_{ij}(t)|\, dt$ *converges for all* i, j *and let* $B(t) = \int_0^t A(s)\,\mathrm{d}s$*. Suppose* $B(t)$ *and* $A(t)$ *commute. Prove that* $\mathbf{0}$ *is a stable solution of* $\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x}$*.*

14.7 Summary

- To prove all solutions of a 2-dimensional system are periodic:
  - Find all stationary solutions of the system.
  - Write and solve the orbit equation.
  - Prove that the orbit equation gives a closed curve.
  - Show stationary solutions only lie on their own closed curves obtained above.

- To draw the phase plane portrait of a 2-dimensional linear system $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$:
  - Find eigenpairs of A.
  - For real eigenpairs draw the eigensolution orbits $ce^{\lambda t}\mathbf{v}$.
  - For each eigensolution orbit indicate with arrows the behavior of the solution as t increases.
  - Sketch sample orbits and indicate with arrows the behavior of the solution as t increases.
  - When we have nonreal eigenvalues $a \pm bi$, orbits will be spirals (when $a \neq 0$) or ellipses (when $a = 0$).
  - Indicate the direction of spirals and their orientations. If $a < 0$ then the arrows should be inwards, towards the origin, and if $a > 0$ the arrows should point outwards. To determine the orientation of spiral pick a sample point (say $(1, 0)$ or $(0, 1)$) and see if x or y increases or decreases. Put these two pieces of information together to determine if the spiral is clockwise or counter-clockwise, and then specify how the solution behaves with arrows.
- Every phase plane portrait is one of the following:
  - Distinct real e-values:
    * Both positive: Nodal source
    * Both negative: Nodal sink
    * One positive, one negative: Saddle
    * One zero, one positive: Linear source
    * One zero, one negative: Linear sink
  - Nonreal e-values $a \pm ib$
    * $a \neq 0$: Spiral
    * $a = 0$: Center
    * $a < 0$: Sink
    * $a > 0$: Source
    * $A_{21} < 0$: Clockwise.
  - One real e-value and $A = \lambda I$:
    * $\lambda = 0$: Zero.
    * $\lambda > 0$: Radial source.
    * $\lambda < 0$: Radial sink.
  - One real e-value and $A \neq \lambda I$:
    * $\lambda \neq 0$: Twist.
    * $\lambda = 0$: Parallel Shear.

- To draw phase plane portraits for nonlinear systems:
  - Find all stationary solutions of the system. You could also find semistationary solutions for a more accurate diagram, but that is not absolutely necessary.
  - Find the Jacobian matrix.
  - Evaluate the Jacobian matrix at every stationary solution.
  - Draw the phase plane portrait near each stationary solution.
  - This gives an approximation for the phase plane port near the stationary solution, if either at least one eigenvalue is positive or all real parts of eigenvalues are negative then this is a good approximation for the phase plane portrait.
- To prove a system or an equation has a periodic solution use the Poincare-Bendixon Theorem:
  - Write down the equation as a first order system if it is not already in that form.
  - Find all stationary solutions of the system. (You could skip this step if you can show later that stationary solutions are not inside the region R.)
  - Find a closed and bounded region R for which solutions that start in that region stay in that region.
  - Most of the times the region R is a region between two curves C_1 and C_2.
  - Make sure $f(x, y)$ and $g(x, y)$ have continuous partial derivatives on an open set containing R.