Math 340 - Final Exam

December 22, 2020

Time: 150 minutes

- Please one problem per page.
- Justify everything.
- No calculators!

1. (10 pts) Consider the transformation $T : \mathbb{P}_n \to \mathbb{R}^3$ given by

$$T(p(t)) = (p(0), p'(1), p''(2)).$$

Prove or disprove that this transformation is linear.

2. (10 pts) Let

$$f(x,y) = \begin{cases} \dfrac{|y|}{x^2} e^{-|y|/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Using the definition of limit show $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

3. (10 pts) Suppose C is a subset of $\mathbb{R}^n$ that is not compact. Prove that there is a continuous function $f : C \to \mathbb{R}$ that is unbounded. (i.e. the image of f is unbounded in $\mathbb{R}$.)

Hint: (a) What do we know about sets that are not compact? (b) Consider using a function of the form $\dfrac{1}{|\mathbf{x} - \mathbf{a}|}$.

4. Classify $\mathbf{0}$ as a local minimum, local maximum or a saddle point of the following quadratic form, in two ways:

$$f(x,y,z) = x^2 - y^2 - z^2 + 4xy + 6xz$$

(a) (7 pts) Using an appropriate Theorem.

(b) (8 pts) By evaluating e-values.

5. Use the Stokes' Theorem to find the work done by the following vector field on a particle that travels on the line segments from $(0,0,1)$ to $(0,2,1)$ to $(2,2,1)$ to $(2,0,1)$ and back to $(0,0,1)$.

$$\mathbf{F} = (xyz^2 - e^x)\mathbf{i} - (xy + \sin(y^2))\mathbf{j} + e^{z^2}\mathbf{k}$$

6. Let Σ be the sphere of radius 2 centered at $(0,0,1)$ oriented inward. Use Gauss' Theorem to evaluate

$$\iint_{\Sigma} ((x^3 - y^4)\mathbf{i} - (3x^2y - e^z)\mathbf{j} + (z - 7x^6)\mathbf{k}) \cdot d\mathbf{S}.$$

7. Let R be the triangle in the xy-plane whose vertices are $(0,0), (1,0)$, and $(0,1)$. Using an appropriate change of variables evaluate $\iint_R e^{(x+y)^2} \mathrm{d}A$.

Hint: $u = x + y$ might help.

8. Consider $\mathbf{X} : [0,1] \times [1,2] \times [-1,1] \to \mathbb{R}^5$ given by $\mathbf{X}(u_1, u_2, u_3) = (u_1u_3, u_2, u_3, u_3^2, u_1u_2 + u_3)$.

(a) (7 pts) Prove that $\mathbf{X}$ is a manifold.

(b) (7 pts) Prove that this manifold is smooth.

(c) (6 pts) Evaluate $\int_{\mathbf{X}} (x_1 + x_2 - x_4) dx_1 \wedge dx_3 \wedge dx_5$. Write your final answer as an iterated triple integral. You do not need to evaluate the integral.

9. (10 bonus pts) Let $\mathcal{P}$ be a plane in $\mathbb{R}^3$ that is not parallel to the yz-plane. Let R be a bounded region in $\mathcal{P}$. Suppose D is the projection of R onto the xy-plane, and D is an elementary region. Let $\mathbf{n}$ be an upward normal vector to the plane $\mathcal{P}$, and θ be the angle between $\mathbf{n}$ and the vector $\mathbf{k}$. Prove that the area of R is equal to the area of D divided by $\cos\theta$.

(For this bonus problem, partial credit will only be awarded for significant progress towards a solution.)