Math 341 - May 14, 2022 - Dr. Ebrahimian

Final Exam - Time: 120 minutes

Calculators and notes are not allowed. **Completely and clearly justify your answers.** Solutions to different problems should go on separate pages and in order. Different parts of a problem may not be related. If something is unclear feel free to ask. Show work completely and clearly!

1. (10 pts) Suppose $A \in M_n(\mathbb{C})$ is a square matrix for which $A^2 = A$. Prove that A is diagonalizable.

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2. Solve each equation. Your solution may be implicit.

2a. (5 pts) $\dfrac{dy}{dt} = \dfrac{y+t}{t-y}, t > 0.$

2b. (5 pts) $y^{(4)} - y = 0.$

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3. (10 pts) Prove that the following initial value problem has a unique solution defined over $[0, 10]$.

$$y' = \frac{\sin(1 + t^2 y)}{1 + y^2}, y(0) = 0.$$

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4. (10 pts) Given that $y = e^t$ is a solution to $y'' + (e^t - 3)y' + (2 - e^t)y = 0$, find the general solution to this equation. The final answer should not contain integrals.

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5. (10 pts) For the following equation, find the largest interval for which all solutions are guaranteed to equal their Taylor series centered at $t_0 = 0$.

$$(1 + 4t^2)y'' + y' + ty = 0.$$

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6. (10 pts) Evaluate the Laplace of $\dfrac{\sin t}{t}$. You may assume $\dfrac{\sin t}{t}$ is piecewise continuous and of exponential order.

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7. Suppose $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}, \mathbf{x}_2(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$ are two solutions to a first-order 2-dimensional linear system

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x},\ t \in (a, b).$$

Assume all entries of $A(t)$ are continuous over an interval (a, b).

(a) (5 pts) Prove that neither -1 nor 1 is in (a, b).

(b) (5 pts) Find the coefficient matrix $A(t)$. Simplify your answer.

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8. (10 pts) Find e^{tA} using the method of Natural Fundamental Set, where

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}.$$

You do not need to simplify your answer.

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9. (10 pts) Prove that the origin is the only stationary solution of the following system, and determine its stability, (i.e. if **0** is stable, unstable, or asymptotically stable.)

$$x' = e^x - x - 1, y' = xy + y$$

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10. (10 pts) Prove that each nontrivial orbit of the following system is an ellipse:

$$x' = 3y, y' = -4x$$

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11. (10 bonus pts) Are there functions $f(x, y), g(x, y)$ with continuous first partials on the entire xy-plane for which $x(t) = e^t - t$, $y(t) = \cos t, t \in \mathbb{R}$ is a solution to the system $x' = f(x, y), y' = g(x, y)$?