Math 341 - May 18, 2019 - Dr. Ebrahimian

Final Exam - Time: 120 minutes

Calculators and notes are not allowed. **Completely and clearly justify your answers.** Solutions to different problems should go on separate pages and in order. Different parts of a problem may not be related. If something is unclear feel free to ask.

1. (10 pts) Let V be a vector space over $\mathbb{C}$ and let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. For every two vectors $u = \sum_{j=1}^{n} c_j v_j \in V$ and $w = \sum_{j=1}^{n} d_j v_j \in V$, define $\langle u, w \rangle = \sum_{j=1}^{n} c_j \overline{d}_j$. Prove $\langle u, w \rangle$ is an inner product for V.

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2. Solve each IVP. Your answers may be implicit.

2a. (5 pts) $\dfrac{dy}{dt} = \dfrac{y}{y+t}$, $y(1) = 1$.

2b. (5 pts) $\dfrac{dy}{dt} + y = e^t$, $y(0) = 0$.

3a. (5 pts) Suppose $p(t)$ and $q(t)$ are functions that are continuous over $\mathbb{R}$. Assume y_1 and y_2 are nonzero solutions to the equation $y'' + p(t)y' + q(t)y = 0$, for which $y_1(0) = y_2(0) = 0$. Prove that there is a constant c for which $y_1(t) = cy_2(t)$ for all $t \in \mathbb{R}$.

3b. (5 pts) Suppose $\binom{t}{t+1}$ and $\binom{t-1}{t}$ are solutions to the 2-dimensional linear system $\dfrac{d\mathbf{x}}{dt} = A(t)\mathbf{x}$. Find $A(t)$. Write your final answer as one matrix and simplify your final answer.

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4. (10 pts) Suppose $\phi_1(t) = e^t + e^{t^2}, \phi_2(t) = 2e^t + e^{t^2}$, and $\phi_3(t) = e^{-t} + e^{t^2}$ are solutions to the differential equation $y'' + p(t)y' + q(t)y = f(t)$. Solve the initial value problem:

$$y'' + p(t)y' + q(t)y = f(t), y(0) = 0, y'(0) = 2.$$

5a. (5 pts) Find the Laplace $\mathbf{X}(s)$ of the solution to the IVP: $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x} + \binom{t}{\sin t}$, $\mathbf{x}(0) = \binom{0}{1}$, where

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}.$$

You do not need to find $\mathbf{x}(t)$. You do <u>not</u> need to perform any of the matrix operations or simplify your answer.

5b. (5 pts) Find the inverse Laplace of $\dfrac{2}{s^2 - 1}$.

More problems on the back!

6. (10 pts) Consider the equation $(t-1)y'' - ty' + y = 0$. Given that one solution to this equation is $y = e^t$, find a general solution to this equation. (Guessing is not acceptable.)

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7. Consider the 2-dimensional system $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$, where

$$A = \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}.$$

7a. (5 pts) Determine if $\mathbf{0}$ is a stable or unstable solution using a theorem.

7b. (5 pts) Prove your answer in part (a) using the definition of stability.

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8. (10 pts) Using the Poincaré-Bendixson Theorem prove that the equation $\ddot{z} + [\ln(z^2 + 2\dot{z}^2)]\dot{z} + z = 0$ has a nontrivial periodic solution.

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9. (10 pts) Prove that if a solution $(x(t), y(t))$ to the system $\dfrac{dx}{dt} = 1 + x^3$, $\dfrac{dy}{dt} = 1 + xy^2$ satisfies $x(0) \neq y(0)$, then for all t in the domain of this solution we have $x(t) \neq y(t)$.

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10. Consider the system $\dfrac{dx}{dt} = e^y - 1$, $\dfrac{dy}{dt} = \sin(x+y)$.

10a. (2 pts) Find all stationary solutions of this system.

10b. (4 pts) Determine the stability of each stationary solution.

10c. (4 pts) Draw a phase plane portrait near each stationary solution.

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11. (10 bonus points) Suppose $A(t)$ is a square matrix for which $A(t)$ and $[A(t)]^{-1}$ are both infinitely many times differentiable for every $t \in \mathbb{R}$. Assume in addition that $A''(t) = A'(t)[A(t)]^{-1}A'(t)$ for all $t \in \mathbb{R}$. Prove that there are constant square matrices B and C for which $A(t) = e^{tB}C$.
