Math 340 - December 21, 2021 - Dr. Ebrahimian

Final Exam - Time: 120 minutes

Calculators and notes are not allowed. **Completely and clearly justify your answers.** Solutions to different problems should go on separate pages and in order. Different parts of a problem may not be related. If something is unclear feel free to ask.

1. (10 pts) Determine if each of the following is a linear mapping. If it is, find its kernel.

(a) $T : \mathbb{P}_3 \to \mathbb{R}$ given by $T(f)(t) = f(0)$.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(x, y) = (x, 2y, x + y)$.

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2. (10 pts) Using the Lagrange Multipliers Theorem, find the shortest distance from $(3, 4)$ to the points on the circle $x^2 + y^2 = 100$. (As usual, make sure you justify such a minimum exists.)

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3. (10 pts) Prove that for every positive integer n, the set of all points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfying $\cos x_1 + \cos x_2 + \cdots + \cos x_n = 0$ is closed, but it is not compact.

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4. (10 pts) Using the $\epsilon - \delta$ definition of limits prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2} = 0$.

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5. (10 pts) Consider the function $f(x, y, z) = x + 2y + xyz$. Find the directional derivative of f at the origin in the direction of the vector $(1, -2, 2)$. (Note that a direction is a unit vector.) What is the maximum directional derivative of f at the origin?

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6. (10 pts) Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are such that each pair of them are linearly independent. Is it true that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ must be linearly independent?

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7. (10 pts) Using an appropriate change of variables evaluate $\iint_D x^2 + y^2 \, dA$, where D is the region enclosed by the ellipse $4x^2 + 9y^2 = 36$. Stop when you get an iterated double integral with constant limits of integration.

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8. (10 pts) Let E be the region in $\mathbb{R}^3$ that lies inside the sphere $x^2 + y^2 + z^2 + 2z = 0$. Evaluate $\iiint_E \sqrt{x^2 + y^2 + z^2} \, dV$. Evaluate and simplify your final answer.

More Problems on the Back!

9. (10 pts) Evaluate $\int_C \frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy$, where C is the circle given by $x^2+y^2=9$ in the xy-plane, oriented clockwise. Evaluate and simplify.

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10. (10 pts) Use Gauss' Theorem to evaluate $\iint\limits_{\Sigma}((7xy^6+e^z)\mathbf{i} - y^7\mathbf{j} + z\mathbf{k}) \cdot d\mathbf{S}$, where Σ is the union of the five upper faces of the unit cube $[0,1]\times[0,1]\times[0,1]$ oriented outward from the cube. Evaluate and simplify.

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11. (10 bonus pts) Let n be a positive integer and let $M_n(\mathbb{R})$ be the vector space of all $n \times n$ matrices with real entries. Define a function $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by $f(A) = A^2$. Find the differential of this function f at a "point" $A \in M_n(\mathbb{R})$.

Here we treat matrices of $M_n(\mathbb{R})$ as vectors of $\mathbb{R}^{n^2}$ by placing the entries of each matrix into components of vectors starting from the upper left corner and moving to the right and down. For example, when $n = 2$, the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is identified with the vector $(a, b, c, d) \in \mathbb{R}^4$.

(For this problem partial credit will only be given for significant progress towards a solution.)