Math 341 - May 17, 2018 - Dr. Ebrahimian

Final Exam - Time: 120 minutes

Calculators and notes are not allowed. **Completely and clearly justify your answers.** Solutions to different problems should go on separate pages and in order. Different parts of a problem may not be related. If something is unclear, feel free to ask.

1. (10 pts) Find the Laplace Transform (i.e. $Y(s)$) of the solution to $y'' - 2y' + y = f(t), y(0) = 0, y'(0) = 1$, where

$$f(t) = \begin{cases} 1 + 2t & \text{if } 0 \leq t < 1 \\ t - 1 & \text{if } t \geq 1 \end{cases}$$

(You do not need to evaluate $y(t)$.)

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2. (10 pts) Suppose $\begin{pmatrix} t+1 \\ 1-t \end{pmatrix}, \begin{pmatrix} 1+t^2 \\ t \end{pmatrix}$ and $\begin{pmatrix} t \\ t \end{pmatrix}$ are solutions to a nonhomogeneous 2-dimensional system $\dfrac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{f}(t)$. Find the solution of the IVP: $\dfrac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{f}(t), \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

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3. (10 pts) Consider the system $\dfrac{dx}{dt} = 1 + x^2, \dfrac{dy}{dt} = xy + 1$. Show that if for a solution $(x(t), y(t))$ of this system we have $x(0) \neq y(0)$, then $x(t) \neq y(t)$ for all t in the domain of the solution.

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4. Suppose y_1, y_2 form a fundamental set of solutions for a second order linear differential equation.

4a. (5 pts) Prove that y_1 and y_2 cannot vanish at the same point.

4b. (5 pts) Prove that between every two consecutive zeros of y_1, there must be at least one zero of y_2. (Hint: Consider y_1/y_2.)

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5. (10 pts) Find a general solution for $t^2y'' - (t^2+2t)y' + (t+2)y = 0$ with $t > 0$. (Hint: $y(t) = t$ is a solution.)

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6. Consider the system $\dfrac{dx}{dt} = x^2 + xy - 3x,\ \dfrac{dy}{dt} = 2x - y$.

6a. (2 pts) Find all stationary solutions of this system.

6b. (4 pts) Find the linearization of the system near each stationary solution.

6c. (4 pts) Determine the stability of each stationary solution.

More problems on the back!

7. (10 pts) Show all solutions of $\frac{d^2z}{dt^2} + z + 2z^3 = 0$ are periodic.

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8. (10 pts) For every two vectors $z_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, z_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{C}^2$, define $\langle z_1, z_2 \rangle = x_1 x_2 + y_1 y_2$. Does this define an inner product on $\mathbb{C}^2$? If yes, prove it. If no, clearly state *all* condition(s) of the inner product that this satisfies and *all* of those that it does not satisfy. Make sure you prove the ones that it satisfies and provide counterexamples for the ones that it does not satisfy.

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9. (10 pts) Solve the IVP: $\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \begin{pmatrix} e^{-t} \\ e^t \end{pmatrix}, \mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

10a. (5 pts) Find a general (real-valued) solution for $y'' + 4y = 0$.

10b. (5 pts) Let α and β be given constants. Show that if $y'' + 4y = 0, y(\alpha) = y_0, y'(\beta) = y_1$ has more than one solution for some y_0, y_1, then $4\alpha - 4\beta = (2k+1)\pi$ for some integer k.

(Hint: $\cos(u - v) = \cos u \cos v + \sin u \sin v$.)

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11. (10 bonus points) Let n be a positive integer, evaluate e^A, where A is the $n \times n$ matrix with t in all entries. Your final answer must be a single matrix and it must be simplified.
