Math 341 - May 17, 2020 - Dr. Ebrahimian

Please read the instructions carefully before getting started.

- All final answers must be simplified, unless specified otherwise.
- You can use your notes, the textbook or any online resource that you would like, however you are not allowed to get help from anybody. You must work on the problems alone
- Only use what has been discussed in this class. Anything else must be proved.
- You may use any theorem or homework problem but please cite them.
- Make sure your solutions are legible and vertical.
- Show all of your work. Keep your work clean and organized. Take your time and submit solutions that are well thought out.
- Your work will be graded based on its accuracy, its completeness and how thorough it is.

1. (10 pts) Solve the equation $\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{ty + y + y^2}{t + 2y}$. Your solution may be implicit.

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2. (10 pts) Suppose $A, B \in M_n(\mathbb{C})$, where n is a positive integer, such that $AB = BA$. Suppose A has n distinct eigenvalues. Prove that there is an invertible matrix P for which $A = PD_1P^{-1}$ and $B = PD_2P^{-1}$, where $D_1, D_2 \in M_n(\mathbb{C})$ are both diagonal.

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3. (10 pts) Find a general solution for $\frac{dy}{dt} = \frac{y}{t + y}$, given $t > 0$ and $y > 0$.

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4. (10 pts) Solve the initial value problem: $\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t \\ 0 \end{pmatrix}$, and $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

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5. Let y_1 and y_2 form a fundamental set of solutions for a standard homogeneous linear equation whose coefficients are continuous over $\mathbb{R}$. Prove or disprove each of the following statements:

5a. (5 pts) There cannot be a value $t \in \mathbb{R}$ for which $y_1''(t) = y_2''(t) = 0$.

5b. (5 pts) If $W[y_1, y_2]$ is constant, then the Wronskian of each two solutions of the same equation is also constant.

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6. (10 pts) Find the inverse Laplace of $\frac{3}{s^2 + 3s}$. You may only use the table of Laplace transform posted on Elms.

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7. (10 pts) Determine if $\{\sin t, \, \cos t, \sin(2t)\}$ is a linearly independent set of elements of $C(\mathbb{R})$, the vector

space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

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8. (10 pts) For all constants a solve the equation $t^2y'' + ty' + ay = 0$, with $t > 0$.

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9. (10 pts) Suppose $\begin{pmatrix} t+e^t \\ t \end{pmatrix}$, $\begin{pmatrix} t \\ t \end{pmatrix}$, and $\begin{pmatrix} t \\ t+e^{-t} \end{pmatrix}$ are solutions of $\dfrac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{f}(t)$. Find $A(t)$ and $\mathbf{f}(t)$.

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10. Consider the system $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -7 & 3 \\ -18 & 8 \end{pmatrix} \mathbf{x}$.

10a. (3 pts) Using a theorem determine if $\mathbf{0}$ is a stable, unstable or asymptotically stable solution.

10b. (7 pts) Using the $\epsilon - \delta$ definition prove the result that you found in part (a). Please only include the proof and not the scratch work.

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11. (10 bonus points) Suppose $A(t) = (a_{ij}(t))_{n\times n}$ is a matrix whose entries $a_{ij}(t)$ are all continuous over $\mathbb{R}$. Suppose $\displaystyle\int_0^\infty |a_{ij}(t)|\, dt$ converges for all i, j and let $B(t) = \displaystyle\int_0^t A(s)\, ds$. Suppose $B(t)$ and $A(t)$ commute. Prove that $\mathbf{0}$ is a stable solution of $\dfrac{d\mathbf{x}}{dt} = A(t)\mathbf{x}$.

For this bonus problem only meaningful progress towards a solution will get credit.

(As far as I can tell this is a difficult problem, but I would be happy to be proven wrong by one or several of you!)
