Math 341 - Final Exam - May 15, 2021

Dr. Ebrahimian

- Show all work completely. Everything must be fully justified.
- For each problem you must upload a different file containing ONLY the solution to that problem. Do NOT upload the same gigantic file for all the problems! Make sure your submission is legible and vertical.
- Notes are allowed. You should work on the problems alone.
- You have **120 minutes** to work on the problems. You must start uploading your solutions at the 135 minute mark, at the latest.

1. (10 pts) Suppose a square matrix A satisfies $A^3 = 4A$. Prove that A is diagonalizable.
Hint: You may want to use the Jordan form.

2. Solve each of the following equations. Your answers may be implicit:
(a) (10 pts) $\dfrac{dy}{dt} = (y-2t)^2 + 1$.
(b) (10 pts) $(xe^{xy}+1)\dfrac{dy}{dx} + (ye^{xy}+2x) = 0$.

3. Consider the initial value problem

$$\frac{dy}{dt} = 1 - \cos y, y(0) = \pi.$$

(a) (10 pts) Find the first three Picard iterates (i.e. y_0, y_1, and y_2).
(b) (10 pts) Prove there is a unique solution to this initial value problem defined for all $t \geq 0$.

4. (15 pts) Using the method of Laplace transforms find the solution to the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

5. Consider the system

$$\frac{dx}{dt} = x^2 y - x, \ \frac{dy}{dt} = x - y$$

(a) (10 pts) Find all stationary solutions of this system.
(b) (10 pts) Determine if each stationary solution is stable, asymptotically stable or unstable.

6. (15 pts) Using Poincare-Bendixon Theorem prove that the equation

$$6z'' + z'(z^2 + 3z'^2 - 1) + 2z = 0$$

has a non-trivial (i.e. non-stationary) periodic solution.

7. (10 bonus points) Prove that if A, B are square matrices of the same size for which $A + B = AB$, then $AB = BA$.

(Partial credit will only be given for significant progress towards a solution.)