

Multivariable Calculus

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**These notes may contain occasional typos or errors. Feel free to bring them to my attention
by sending me an email at ebrahimi@umd.edu**

1 Lecture 1

Recall: In 2-D or \mathbb{R}^2 : To each point, we assign a pair of real numbers (x, y) . This allows us to represent lines, circles, parabolas and other curves with equations such as $y = 2x - 1$, $x^2 + y^2 = 1$, etc. We would like to do something similar in 3-D space.

In 3-D or \mathbb{R}^3 : We need the origin O , and three axes, x , y , and z axes. Note that x , y , and z axes must be pairwise perpendicular (aka. orthogonal) and must satisfy the right hand rule. To each point in 3-D we assign a triple (x, y, z) of real numbers. This triple is found by dropping perpendiculars from the point to the three axes.

The three axes create three planes called the **coordinate planes**. These planes divide the space into eight **octants**. The **first octant** is the one that consists of all points with $x, y, z > 0$. Other octants do not have names.

The distance between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$|PQ| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

The equation of the sphere of radius r centered at (a, b, c) is given by $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$.

Example 1.1. Show that $x^2 + y^2 + z^2 + 2x - y + 1 = 0$ represents a sphere. Find its center and its radius.

A **closed ball** is all points inside or on a sphere. A closed ball of radius r centered at (a, b, c) is given by the inequality $(x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2$. An **open ball** is all points inside a sphere. Thus, it is given by $(x - a)^2 + (y - b)^2 + (z - c)^2 < r^2$.

1.1 Summary

- x , y , and z axes must satisfy the right hand rule.
- Make sure you know how to determine coordinates of a point.
- The distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.
- The equation of a sphere is given by $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$, where r is the radius and (a, b, c) is the center of the sphere. To find the center and radius of a sphere, you would need to complete the squares.

2 Lecture 2

A **vector** in 3-D is an ordered triple $\mathbf{v} = (a, b, c)$ of real numbers. a, b, c are called the **components** of \mathbf{v} . Every vector has a **direction** and a magnitude. Vectors in 2-D are defined similarly.

Notation: Points are generally denoted by capital letters: P, Q , etc., whereas vectors are either denoted by lower case letters with an arrow on top: \vec{u}, \vec{v} , etc., or by lower case letters in bold: \mathbf{u}, \mathbf{v} , etc.

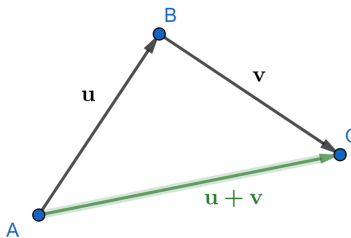
The **magnitude** or **length** of the vector $\mathbf{v} = (a, b, c)$ is given by $\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}$.

Given points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ the vector whose initial point is P and whose terminal point is Q is denoted by \vec{PQ} and is evaluated by $\vec{PQ} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

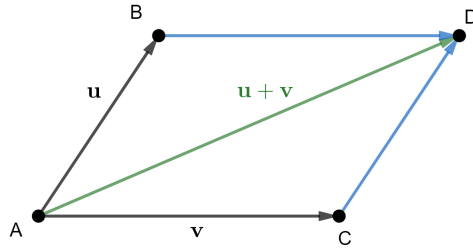
Sum, and difference of vectors are done componentwise:

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

The **Triangle Law** states, the sum of vectors \vec{AB} and \vec{BC} is the vector \vec{AC} .



The **Parallelogram Law** states, the sum of vectors \vec{AB} and \vec{AC} is the diagonal \vec{AD} in the parallelogram $ABDC$.



If r is a real number, then the **scalar multiple** $r\mathbf{v}$ is given by (ra, rb, rc) .

A vector is called a **unit vector** if its length is one.

The unit vectors in the directions of positive x , y , and z axes are $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$.

Every vector $\mathbf{v} = (a, b, c)$ can be written as $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Since the notations for vectors and points are the same, we prefer to use $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ for vectors, but be aware that (a, b, c) could both refer to a point and a vector.

To find the unit vector in the direction of vector \mathbf{v} , we divide \mathbf{v} by its magnitude $\|\mathbf{v}\|$.

Example 2.1. Consider the points $P = (1, 0, \sqrt{3})$, and $Q = (3, 3, 0)$.

- (a) Find the unit vector in the direction of \vec{PQ} .
- (b) Find the vector in the opposite direction of \vec{PQ} whose length is 7.

Two vectors are **parallel** if they are scalar multiples of each other.

Example 2.2. Find x and y for which the vectors $(3, 6, 2)$ and $(x, 2y, x + 9)$ are parallel.

The Dot Product

The **dot product** of two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is given by $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$.

Main properties of the dot product:

1. The dot product of two vectors is a scalar, i.e. a real number.
2. Commutativity, and distributive law both hold for dot products: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$, and $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.
3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$, where c is a scalar.
4. $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$.

5. $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$, where θ is the angle between \mathbf{u} and \mathbf{v} .

Example 2.3. Find the angle between the vector $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and the negative x -axis.

In general to find the angle between two vectors we use $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$.

2.1 Summary

- To find a vector \overrightarrow{PQ} we should subtract the coordinates of the initial point P from the coordinates of the terminal point Q .
- The dot product is given by $(u_1, u_2, u_3) \cdot (v_1, v_2, v_3) = u_1v_1 + u_2v_2 + u_3v_3$.
- The formula $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$ can be used to find the angle θ between two vectors \mathbf{u} and \mathbf{v} .

3 Lecture 3

Recall: The dot product of two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ can be found in two different ways:

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3, \quad \text{and} \quad \mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta,$$

where θ is the angle between the two vectors \mathbf{u} and \mathbf{v} . This can be used to find θ .

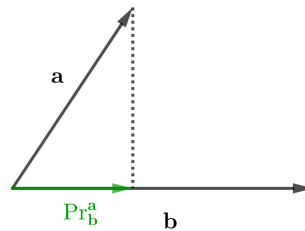
Note that $\cos \theta = 0$ whenever $\theta = \pi/2$ or when the vectors are perpendicular. Therefore,

To check if vectors \mathbf{u} and \mathbf{v} are perpendicular we need to see if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example 3.1. Find all constants c for which the vectors $(2, c, -3)$ and $(1, c, 2)$ are perpendicular.

The **projection** of a vector \mathbf{a} onto a vector \mathbf{b} is given by

$$\text{Pr}_{\mathbf{b}}^{\mathbf{a}} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \mathbf{b}$$



Example 3.2. Find the projection of $\mathbf{i} - 2\mathbf{j}$ onto $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

The work done by a force \mathbf{F} with displacement \mathbf{D} is $\mathbf{F} \cdot \mathbf{D}$.

The Cross Product

Now, we will discuss a different type of product of vectors. The outcome of this product is a vector instead of a scalar. For that we need the notion of determinant.

A 2×2 **determinant** is defined as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

To find a 3×3 determinant we start with the first row and multiply each entry by a 2×2 determinant obtained by removing the row and column of that entry. We then alternate the sign. This is shown in the following example.

Example 3.3. Evaluate $\begin{vmatrix} -1 & 1 & 2 \\ 0 & -2 & 3 \\ -1 & -3 & 2 \end{vmatrix}$.

Solution. The first entry of the first row is -1. Removing the first row and column we obtain the 2×2 determinant $\begin{vmatrix} -2 & 3 \\ -3 & 2 \end{vmatrix}$.

For the second entry of the first row, we need to remove the first row and the second column to obtain the 2×2 determinant: $\begin{vmatrix} 0 & 3 \\ -1 & 2 \end{vmatrix}$.

For the third entry of the first row, we need to remove the first row and the third column to obtain: $\begin{vmatrix} 0 & -2 \\ -1 & -3 \end{vmatrix}$.

Thus the given 3×3 determinant is

$$\begin{vmatrix} -1 & 1 & 2 \\ 0 & -2 & 3 \\ -1 & -3 & 2 \end{vmatrix} = -1 \begin{vmatrix} -2 & 3 \\ -3 & 2 \end{vmatrix} - 1 \begin{vmatrix} 0 & 3 \\ -1 & 2 \end{vmatrix} + 2 \begin{vmatrix} 0 & -2 \\ -1 & -3 \end{vmatrix} = -(-4 + 9) - (0 + 3) + 2(0 - 2) = -12.$$

□

The **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector given by the 3×3 determinant obtained by placing \mathbf{i}, \mathbf{j} , and \mathbf{k} in the first row, the components of \mathbf{u} in the second row and the components of \mathbf{v} in the third row.

Make sure you do not swap the rows of this determinant or you could get a wrong answer!

Example 3.4. Evaluate $(\mathbf{i} + 2\mathbf{j}) \times (\mathbf{j} - \mathbf{k})$.

Properties of the Cross Product:

1. $\mathbf{u} \times \mathbf{v}$ is a vector (NOT a scalar.)
2. $\mathbf{u} \times \mathbf{v}$ is perpendicular to both \mathbf{u} and \mathbf{v} .
3. The vectors \mathbf{u}, \mathbf{v} and $\mathbf{u} \times \mathbf{v}$ satisfy the right-hand rule.
4. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$.

5. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.
6. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.
7. $(c \mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v})$.
8. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.

The area of the parallelogram formed by the vectors \mathbf{u} and \mathbf{v} is equal to $\|\mathbf{u} \times \mathbf{v}\|$.

Example 3.5. Find the area of the triangle whose vertices are $(2, 3, -1), (1, 2, -1), (0, 0, 1)$.

The volume of the parallelepiped based on vectors \mathbf{u}, \mathbf{v} , and \mathbf{w} is the absolute value of $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. Note that you could change the order of the vectors and you would end up with the same answer. In other words, $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})| = |\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|$.

Example 3.6. Find the volume of the parallelepiped formed by the vectors $\mathbf{u} = (1, 2, 1), \mathbf{v} = (0, 1, 0)$, and $\mathbf{w} = (2, 1, 3)$.

Example 3.7. Show that the points $A = (1, 1, -2), B = (-1, 2, -1), C = (2, 1, -3)$, and $D = (0, 1, -1)$ lie on a plane.

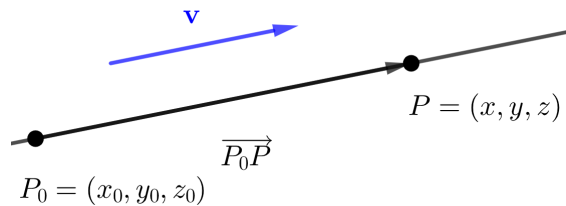
3.1 Summary

- To check if vectors \mathbf{u} and \mathbf{v} are perpendicular we need to see if $\mathbf{u} \cdot \mathbf{v} = 0$.
- $\text{Pr}_{\mathbf{b}}^{\mathbf{a}} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \mathbf{b}$.
- $\mathbf{u} \times \mathbf{v}$ is a vector obtained by evaluating a 3×3 determinant whose first row is $(\mathbf{i} \ \mathbf{j} \ \mathbf{k})$, and whose second and third rows are \mathbf{u} and \mathbf{v} .
- $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.
- $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$.
- $\mathbf{u} \times \mathbf{v}$ is perpendicular to both \mathbf{u} and \mathbf{v} and the three vectors \mathbf{u}, \mathbf{v} and $\mathbf{u} \times \mathbf{v}$ satisfy the right-hand rule.
- The area of the parallelogram formed by the vectors \mathbf{u} and \mathbf{v} is $\|\mathbf{u} \times \mathbf{v}\|$.
- The volume of the parallelepiped formed by vectors \mathbf{u}, \mathbf{v} , and \mathbf{w} is given by $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$.

4 Lecture 4

Lines in Space

A line ℓ can be determined by a point $P_0 = (x_0, y_0, z_0)$ on the line and a vector $\mathbf{v} = (a, b, c)$ parallel to the line. If $P = (x, y, z)$ is a point on ℓ , then the vectors $\overrightarrow{P_0P}$ and \mathbf{v} must be parallel. Thus, $\overrightarrow{P_0P} = t\mathbf{v}$ for some real number t .



Thus, we can write $(x - x_0, y - y_0, z - z_0) = (ta, tb, tc)$, which gives us a **parametric equation of ℓ**

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

Solving for t , we obtain $t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$. A **symmetric equation of ℓ** is thus as follows:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

If O is the origin then, the equation $\overrightarrow{P_0P} = t\mathbf{v}$ can be written as $\overrightarrow{OP} - \overrightarrow{OP_0} = t\mathbf{v}$. Letting $\overrightarrow{OP} = \mathbf{r}(t)$ and $\overrightarrow{OP_0} = \mathbf{r}_0$, we obtain a **vector equation of ℓ**

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

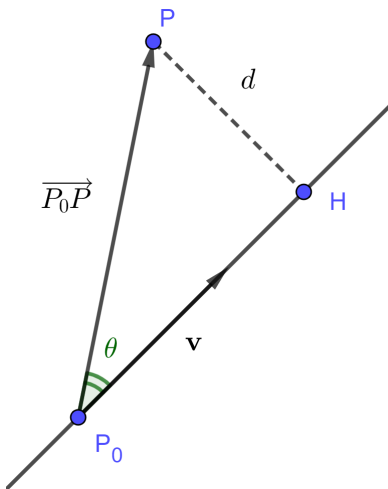
Example 4.1. Find a parametric equation of the line passing through the points $A = (1, 1, 1)$ and $B = (-1, 2, 3)$.

Example 4.2. Find a symmetric equations of the line parallel to the line given by $\frac{x - 1}{2} = \frac{y - 1}{3} = z$ that passes through $(2, -1, 1)$.

The distance from a point P to the line ℓ is given by

$$\frac{\|\mathbf{v} \times \overrightarrow{P_0P}\|}{\|\mathbf{v}\|},$$

where P_0 is a point on the line ℓ , and \mathbf{v} is the direction vector of ℓ .



Example 4.3. Find the distance from point $(1, 2, -1)$ to the line given by $\frac{x-1}{3} = \frac{y+1}{2} = -z$.

Example 4.4. Show that the lines given by

$$\ell : \frac{x-1}{2} = \frac{y}{3} = -z, \text{ and}$$
$$k : \begin{cases} x = 1 - 2t \\ y = 2 - 3t \\ z = t \end{cases}$$

are parallel and find the distance between them.

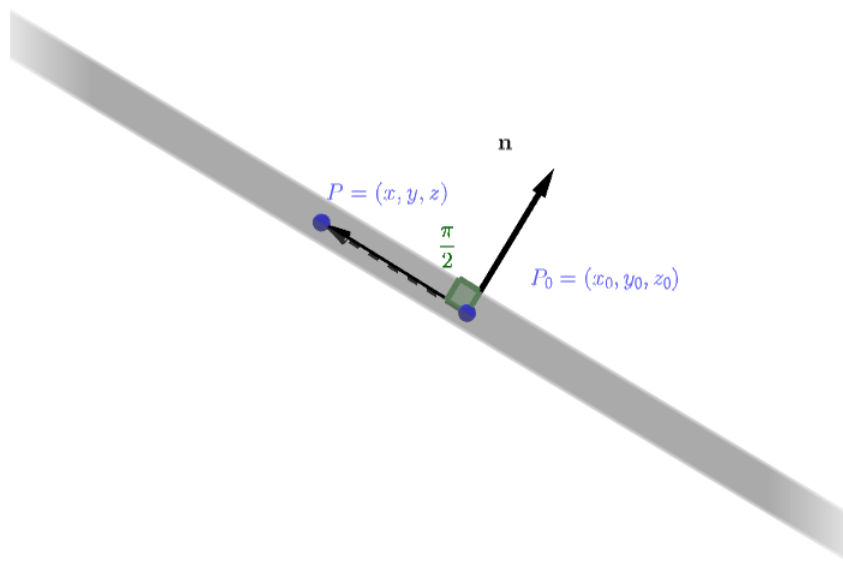
4.1 Summary

- To find an equation of a line we need a point $P_0 = (x_0, y_0, z_0)$ on the line and a vector $\mathbf{v} = (a, b, c)$, called its direction vector, parallel to the line.
- Symmetric equation of the line is given by $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$.
- Parametric equation of the line is given by $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$.
- Vector equation of the line is given by $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{r}_0 = (x_0, y_0, z_0)$.
- To find the distance from point P to line ℓ
 - Find a point P_0 on ℓ . (Any point would work!)
 - Find a direction vector \mathbf{v} of ℓ .
 - The distance is given by $D = \frac{\|\mathbf{v} \times \overrightarrow{P_0P}\|}{\|\mathbf{v}\|}$.

5 Lecture 5

Planes in Space

A plane \mathcal{P} can be identified by a point $P_0 = (x_0, y_0, z_0)$ on \mathcal{P} and a vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ perpendicular to \mathcal{P} , called a **normal vector** of the plane.



If a point $P = (x, y, z)$ is on the plane, then the vectors $\overrightarrow{P_0P}$ and \mathbf{n} must be perpendicular. This implies $\overrightarrow{P_0P} \cdot \mathbf{n} = 0$, which gives us

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Example 5.1. Find an equation of the plane passing through the point $(-1, 3, -2)$ and perpendicular to the vector $2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

Example 5.2. Find an equation of the plane perpendicular to the line $\frac{x+1}{2} = \frac{y-1}{3} = z$ that passes through the point $(1, -1, 2)$.

Example 5.3. Find an equation of the plane that passes through points $(0, 0, 2)$, $(1, 2, 1)$, and $(1, 0, -1)$.

Example 5.4. Sketch the planes $x = 2$, $x + y = 1$, and $x + y + z = 1$.

The distance from a point P to a plane containing point P_0 with \mathbf{n} as its normal vector is

$$\frac{|\mathbf{n} \cdot \overrightarrow{P_0P}|}{\|\mathbf{n}\|}$$

Example 5.5. Find the distance from the origin to the plane $2x + 3y + \sqrt{3}z = 5$.

5.1 Summary

- To find an equation of a plane we need a point (x_0, y_0, z_0) on the plane and a normal vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.
- The equation of the plane is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.
- A normal vector of a plane can be found by looking at the coefficients of x, y , and z .
- To find the distance from a point P to a line ℓ , we need to
 - Find a point P_0 on ℓ .

- Find a normal vector \mathbf{n} for the plane. The coefficients of x, y, z give us a normal vector.
- Use the formula $\frac{|\mathbf{n} \cdot \overrightarrow{P_0P}|}{\|\mathbf{n}\|}$ to find the distance.

6 Lecture 6

A **vector-valued function** (VVF for short) is a function that assigns vectors to numbers. The trace of each VVF will be a curve in 3D space. This trace is called a **space curve**.

Example 6.1. Identify the trace of each of the vector-valued functions below:

(a) $\mathbf{F}(t) = (1 - 2t, t, 2 - 3t)$.

(b) $\mathbf{G}(\theta) = (\cos \theta, \sin \theta, 2)$.

Operations between vector-valued functions can be performed similar to vectors. Same properties still hold.

Example 6.2. Consider the two vector-valued functions $\mathbf{F}(t) = (t, 2 \sin t, 1)$, and $\mathbf{G}(t) = (t - 1, \cos t, 0)$. Evaluate $(\mathbf{F} \cdot \mathbf{G})(t)$ and $(\mathbf{F} \times \mathbf{G})(t)$.

Example 6.3. Find all points of intersection of the cylinder $x^2 + y^2 = 1$ and the curve traced out by $\mathbf{F}(t) = (t, 2t - 1, 1)$.

To take the limit of a vector-valued function we apply the limit to each component.

Example 6.4. Let $\mathbf{F}(t) = \frac{\sin(t-2)}{t^2-4} \mathbf{i} + \frac{t^2+2t-8}{t-2} \mathbf{j} + t^t \mathbf{k}$. Evaluate $\lim_{t \rightarrow 2} \mathbf{F}(t)$.

The derivative and integral of a vector-valued function is evaluated by taking the derivative or integral of each component. Derivative of VVF's has similar properties to that of real-valued functions. For example a similar product rule holds:

$$(\mathbf{F} \cdot \mathbf{G})'(t) = \mathbf{F}'(t) \cdot \mathbf{G}(t) + \mathbf{F}(t) \cdot \mathbf{G}'(t) \text{ and } (\mathbf{F} \times \mathbf{G})'(t) = \mathbf{F}'(t) \times \mathbf{G}(t) + \mathbf{F}(t) \times \mathbf{G}'(t)$$

Let $\mathbf{r}(t)$ be the position of a particle at time t . The displacement of this particle from time t_0 to time t is $\mathbf{r}(t) - \mathbf{r}(t_0)$. Thus, the average velocity of this particle in this time interval is given by $\frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{t - t_0}$. The instantaneous velocity at time t_0 , thus, is given by $\mathbf{r}'(t_0) = \lim_{t \rightarrow t_0} \frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{t - t_0}$.

The (instantaneous) **velocity** of a particle whose position vector is $\mathbf{r}(t)$ is given by $\mathbf{v}(t) = \mathbf{r}'(t)$. Its **speed** is the magnitude of its velocity $\|\mathbf{r}'(t)\|$, and its **acceleration** is the derivative of its velocity $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Example 6.5. The position of a particle in space is given by $\mathbf{r}(t) = (t + 1)\mathbf{i} + \cos t \mathbf{j} + \mathbf{k}$. Find its velocity, acceleration and speed when the particle is located at $(1, 1, 1)$.

Example 6.6. Suppose the velocity of a particle in space is given by $\mathbf{v}(t) = (t, t^2, t - 1)$. Assuming the particle is located at $(1, -1, 2)$ when $t = 0$, find the position of this particle when $t = 1$.

6.1 Summary

- A VVF is a function that allocates vectors to real numbers. The trace of a VVF determines a curve in space.
- If $\mathbf{r}(t)$ is the position vector, then the velocity, speed, and acceleration are given by $\mathbf{r}'(t)$, $\|\mathbf{r}'(t)\|$, $\mathbf{r}''(t)$.
- To find the position we integrate velocity. To find the velocity, we integrate acceleration.

7 Lecture 7

Recall: A space curve is the trace of a vector-valued function $\mathbf{r}(t)$.

A space curve given by $\mathbf{r}(t)$ with $a \leq t \leq b$ is called **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$.

Example 7.1. The curve given by $\mathbf{r}(t) = (\cos t, \sin t, 0)$ with $0 \leq t \leq 2\pi$ is closed, but the curve $\mathbf{r}(t) = (\cos t, \sin t, t)$ is not closed.

A curve given by $\mathbf{r}(t)$ is called **smooth** if $\mathbf{r}'(t) \neq \mathbf{0}$, except possibly at the endpoints. A curve given by $\mathbf{r}(t)$ with t in $[a, b]$ is called **piecewise smooth** if the interval $[a, b]$ can be divided into a finite number of sub-intervals in which $\mathbf{r}(t)$ is smooth over all of these sub-intervals.

Example 7.2. Show that $\mathbf{r}(t) = (\cos t, \sin t, 1)$ is a smooth vector-valued function.

Example 7.3. Show the curve given by $\mathbf{r}(t) = (t - \sin t) \mathbf{i} + (1 - \cos t) \mathbf{j} + \mathbf{k}$ with $0 \leq t \leq 4\pi$ is piecewise smooth.

Example 7.4. Find a smooth parametrization for the segment joining $(1, -1, 0)$ and $(1, 2, 1)$.

Example 7.5. Find a smooth parametrization for the circle in the plane $x = 2$ with center $(2, 1, -1)$ and radius 4.

Example 7.6. Find a smooth parametrization for the ellipse $x^2 + 4y^2 = 9$ in the xy -plane.

Example 7.7. Let $f(x)$ be a differentiable function over \mathbb{R} . Find a smooth parametrization for the graph of $f(x)$ in the xy -plane.

Example 7.8. Sketch the curve given by $\mathbf{r}(t) = (\cos t, \sin t, t)$ where $0 \leq t \leq 2\pi$. With an arrow indicate the direction in which the curve is traversed as t increases.

7.1 Summary

- $\mathbf{r}(t)$ is smooth if $\mathbf{r}'(t) \neq \mathbf{0}$, except possibly at the endpoints.
- $\mathbf{r}(t)$ is piecewise smooth if it can be broken into a finite number of smooth curves.
- To parametrize a circle or an ellipse we should use polar coordinates.

- The graph of a function $f(x)$ in the xy -plane can be parametrized by $(x, f(x))$.
- Segments in the space can be parametrized using the parametric equation of a line.
- To sketch a space curve find relations between x, y, z .

8 Lecture 8

Arc Length

Recall: In single variable calculus we saw that if a function f is differentiable, then for every $a < b$, there is c between a and b for which $f'(c) = \frac{f(b) - f(a)}{b - a}$. This is called the Mean-Value Theorem.

We also saw that to estimate the integral $\int_a^b f(x) dx$ we must divide $[a, b]$ into n subintervals of equal width Δx , pick a real number x_k in the k -th subinterval and then evaluate the sum $\sum_{k=1}^n f(x_k)\Delta x$. As we let $n \rightarrow \infty$ we approach the value of the definite integral $\int_a^b f(x) dx$.

Suppose \mathcal{C} is a curve given by $\mathbf{r}(t)$ where $a \leq t \leq b$. To estimate the length of this curve we divide $[a, b]$ into n subintervals as follows:

$$a = t_0 < t_1 < \dots < t_n = b$$

The length of the i -th piece of this curve is approximately $\|\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})\|$. By the Mean-Value Theorem, this can be estimated by $\|\mathbf{r}'(c_i)\Delta t\|$. Summing these up and letting n to go to infinity we get the following formula for the arc length.

$$L = \int_a^b \|\mathbf{r}'(t)\| dt$$

Example 8.1. Find the length of the curve given by $\mathbf{r}(t) = (\cos t, \sin t, t)$ between the points $(1, 0, 0)$ and $(1, 0, 2\pi)$.

Measuring the arc length from $t_0 = a$ to t yield a function $s(t)$, called the **arc length function**. In other words $s(t) = \int_a^t \|\mathbf{r}'(u)\| du$. The fundamental theorem of calculus tells us that $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$.

Example 8.2. Find the arc length function for the curve given by $\mathbf{r}(t) = (e^t, e^{-t}, \sqrt{2}t)$ measured from $(1, 1, 0)$.

Tangent and Normal Vectors

For a curve \mathcal{C} given by a smooth parametrization $\mathbf{r}(t)$, we define the **(unit) tangent vector** to \mathcal{C} at $\mathbf{r}(t)$ by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}$$

Note that dividing $\mathbf{r}'(t)$ by its length creates a unit vector and thus eliminates the length of this vector. In other words we would like for the tangent vector to not depend on the parametrization and only depend on the curve itself. This means the only thing that changes the tangent vector is if we change the orientation of the parametrization.

Example 8.3. Find the tangent vector to the circle of intersection of the plane $z = 5$ and the cylinder $x^2 + y^2 = 4$.

We know $\|\mathbf{T}(t)\| = 1$, and thus $\mathbf{T}(t) \cdot \mathbf{T}(t) = 1$. Taking the derivative we obtain $\mathbf{T}'(t) \cdot \mathbf{T}(t) + \mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$, and thus $2\mathbf{T}'(t) \cdot \mathbf{T}(t) = 0$. This shows $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal. This motivates us to define the **(unit) normal vector** as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

Example 8.4. Find the tangent and normal vectors of the curve given by $\mathbf{r}(t) = (2 \cos t, 2 \sin t, t)$ at point $(2, 0, 0)$.

Next time we will learn how tangent and normal vectors are closely related to the acceleration vector.

8.1 Summary

- The length of the curve given by $\mathbf{r}(t)$ with $a \leq t \leq b$ is $L = \int_a^b \|\mathbf{r}'(t)\| dt$.
- The arc length function is given by $s(t) = \int_a^t \|\mathbf{r}'(u)\| du$. and it satisfies $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$.
- The tangent vector is given by $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.
- The normal vector is given by $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$.

9 Lecture 9

Last time we saw that $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}$, and $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$, which implies $\mathbf{v}(t) = \|\mathbf{v}(t)\|\mathbf{T}(t)$, and $\mathbf{T}'(t) = \|\mathbf{T}'(t)\|\mathbf{N}(t)$. Differentiating $\mathbf{v}(t)$ we get the acceleration as follows:

$$\mathbf{a}(t) = \frac{d}{dt}(\|\mathbf{v}(t)\|)\mathbf{T}(t) + \|\mathbf{v}(t)\|\mathbf{T}'(t) = \underbrace{\left(\frac{d}{dt}(\|\mathbf{v}(t)\|)\right)}_{a_T} \mathbf{T}(t) + \underbrace{\left(\|\mathbf{v}(t)\|\|\mathbf{T}'(t)\|\right)}_{a_N} \mathbf{N}(t) \quad (*)$$

Note that the quantities $\frac{d}{dt}(\|\mathbf{v}(t)\|)$ and $\|\mathbf{v}(t)\|\|\mathbf{T}'(t)\|$ are scalars while $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are vectors. In other words we can write $\mathbf{a}(t) = a_T\mathbf{T}(t) + a_N\mathbf{N}(t)$, where a_T and a_N are two scalars (and a_N is non-negative.)

This means the acceleration lies in the plane of tangent and normal vectors and its normal component is never negative. The scalars a_T and a_N are called **tangential and normal components of acceleration** $\mathbf{a}(t)$.

Using the Pythagorean Theorem we obtain $\|\mathbf{a}\|^2 = a_T^2 + a_N^2$, which helps us find $a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2}$. Recall that $a_N \geq 0$. However the formula for a_T is not simple. An easier method would be using the projection of \mathbf{a} onto \mathbf{v} to find a_T . If θ is the angle between \mathbf{a} and \mathbf{v} , then $a_T = \|\mathbf{a}\| \cos \theta$ and $a_N = \|\mathbf{a}\| \sin \theta$. Therefore,

$$\boxed{a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}, \text{ and } a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}} \quad (**)$$

Example 9.1. Find the tangential and normal components of acceleration for the curve given by $\mathbf{r}(t) = (t, 2t, t^2)$.

Curvature

We would like **curvature** to measure the amount by which a curve deviates from being a line. In other words we would like the curvature of a line to be zero and as a curve bends faster we would like it to have a larger curvature. For example we expect the curvature of a great circle of the earth to be near zero but the curvature of the great circle of a soccer ball must be much larger. The way we can measure this is by measuring how much the tangent vector changes relative to arc length. We denote the curvature by $\kappa(t)$ (pronounced kappa) and its definition is $\kappa(t) = \left\| \frac{d\mathbf{T}}{ds} \right\|$. Recall that $\frac{ds}{dt} = \|\mathbf{r}'(t)\| = \|\mathbf{v}(t)\|$. Using the chain rule we obtain $\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \frac{ds}{dt}$, which implies $\kappa = \frac{\|\mathbf{T}'\|}{\|\mathbf{v}\|}$. Note that as we saw in (*) and (**) $a_N = \|\mathbf{v}\| \|\mathbf{T}'\| = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}$, which implies $\|\mathbf{T}'\| = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^2}$. Therefore, we obtain our main formula for the curvature.

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$$

Example 9.2. Find the curvature of a circle of radius R .

Example 9.3. Find the curvature of the graph $y = x^2$ in the xy -plane at point $(1, 1)$.

9.1 Summary

- $\mathbf{a}(t) = a_T \mathbf{T}(t) + a_N \mathbf{N}(t)$, where a_T and a_N are scalars. a_T and a_N are called tangential and normal components of acceleration.
- $a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}$, and $a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2}$.
- $\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$.

10 Lecture 10

Functions of Several Variables

Functions of form $f(x, y)$ or $f(x, y, z)$ that assign real numbers to points in 2D or 3D are called **functions of several variables**, or multivariable functions. Domain of a function of several variable is the set of all acceptable inputs and its range is the set of all possible outputs.

Example 10.1. Find and sketch the domain and range of each function:

(a) $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$

(b) $g(x, y) = \frac{1}{\sqrt[4]{x^2 + y^2 - 4}}$

Example 10.2. Write the function $f(x, y) = \ln(x^2 + y^2)$ as a composition of two functions.

The **graph** of a function $f(x, y)$ of two variables is in \mathbb{R}^3 and is the set of all points (x, y, z) for which $z = f(x, y)$. The graph of a function $f(x, y, z)$ of three variables is in 4-dimensional space \mathbb{R}^4 .

Example 10.3. Sketch the graph of each function below:

(a) $f(x, y) = 3 - x - y$.

(b) $g(x, y) = \sqrt{4 - x^2 - y^2}$.

Example 10.4. Sketch the graph of each of the following functions:

(a) $f(x, y) = x^2 + y^2$

(b) $g(x, y) = \sqrt{x^2 + y^2}$

As seen above, to understand the graph of a function $f(x, y)$ we need to look at different curves that create this graph. For every constant c , the curve $f(x, y) = c$ is called a **level curve** of the function $f(x, y)$. For a function of three variables $f(x, y, z)$, its **level surfaces** are all surfaces given by $f(x, y, z) = c$, where c is a constant.

Example 10.5. Investigate all level curves or level surfaces of each function below for various constants c :

(a) $f(x, y) = x^2 - 2x + y^2$

(b) $g(x, y, z) = 1 - x^2 - y^2 - z^2$

10.1 Summary

- Level curves and level surfaces are obtained by setting the multivariable function equal to a constant c .
- To sketch a surface find its level surfaces by letting $z = c$, $y = c$ and $x = c$, where c is a constant. Putting these information together we would be able to get a better understanding of what the surface must look like.

11 Lecture 11

Quadratic Surfaces

Example 11.1. Sketch and name each of the following surfaces:

(a) $x^2 + \frac{y^2}{4} + 9z^2 = 1$

(b) $x^2 + 4y^2 = 1$

(c) $z + 9x^2 + y^2 = 1$

(d) $z = y^2$

Limits and Continuity

If the value of $f(x, y)$ approaches L when (x, y) approaches (a, b) we write $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$. Note that in that case the value of $f(x, y)$ must approach L if (x, y) approaches (a, b) in *any* direction. The function $f(x, y)$ is called **continuous at** (x_0, y_0) if $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0)$.

Example 11.2. Evaluate each of the limits below or show they do not exist.

(a) $\lim_{(x,y) \rightarrow (1,\pi)} \frac{\sin(x^2 + y)}{x^2 - y^2}$.

(b) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + y^2}$.

(c) $\lim_{(x,y) \rightarrow (0,0)} \frac{y^3}{x^2 + y^2}$.

Example 11.3. Evaluate the limit or show it does not exist.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^5}{x^4 + y^4}.$$

Example 11.4. Find the value of constant c for which the following function is continuous at $(0, 0)$.

$$f(x, y) = \begin{cases} \frac{x^2 + y^2 + 3xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ c & \text{if } (x, y) = (0, 0) \end{cases}$$

Solution. We use polar coordinates. $\frac{x^2 + y^2 + 3xy^2}{x^2 + y^2} = \frac{r^2 + 3r^3 \cos \theta \sin^2 \theta}{r^2}$ We know $-1 \leq \cos \theta \sin^2 \theta \leq 1$, which implies $\frac{r^2 - 3r^3}{r^2} \leq \frac{x^2 + y^2 + 3xy^2}{x^2 + y^2} \leq \frac{r^2 + 3r^3}{r^2}$. Note that $\frac{r^2 - 3r^3}{r^2} = 1 - 3r \rightarrow 1$ and $\frac{r^2 + 3r^3}{r^2} = 1 + 3r \rightarrow 1$, as $r \rightarrow 0$. Therefore, by the Squeeze Theorem, $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 1$. Thus, for $f(x, y)$ to be continuous at $(0, 0)$, we need $c = 1$. \square

Example 11.5. Evaluate $\lim_{(x,y) \rightarrow (0,0)} \frac{y^4}{x^2 + y^4}$ or show it does not exist.

11.1 Summary

- You should be comfortable with the usual quadratic surfaces: ellipsoids, elliptic cylinders, elliptic paraboloids, elliptic double cones, parabolic sheets, hyperbolic paraboloids, two hyperbolic sheets, hyperboloids of one sheet, and hyperboloids of two sheets. (See p. 821-823)
- To evaluate $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$:
 - Find the limit along the lines $y = mx$ and $x = 0$. If these limits depends on m , then the limit does not exist.
 - If the limits in the previous step are all the same, you must use squeeze theorem to find the limit. Using polar coordinates is often helpful.
 - Note that if all the limits along different lines are the same that does NOT imply the limit exists.

12 Lecture 12

Partial Derivatives

Imagine you are climbing a mountain. At any given point there are many directions that you can choose to climb. Each direction could have a different steepness. Thus, to find the steepness you should first choose your direction. Mathematically speaking, the steepness of surface $z = f(x, y)$ depends on which direction we are choosing. Two particular directions are perhaps the easiest ones: directions of x - and y -axes.

Given x_0 , the slope of the curve $z = f(x_0, y)$ in the plane $x = x_0$ is given by

$$f_y(x_0, y_0) = \lim_{y \rightarrow y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$$

This is also denoted by $\frac{\partial f}{\partial y}(x_0, y_0)$. Similarly we define

$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = \lim_{x \rightarrow x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

To find f_x , we differentiate the function with respect to x , treating y as a constant. Similar for f_y .

Example 12.1. Evaluate $f_x(1, -1)$, and $f_y(0, 2)$, where $f(x, y) = x^3y + 2x + y^2$.

Example 12.2. Let

$$f(x, y) = \begin{cases} \frac{x^4y + y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Evaluate $f_x(0, 0)$ and $f_y(0, 0)$.

By taking derivatives of the first partial derivatives we obtain the second partial derivatives. So, for a function $f(x, y)$ of two variables we have four second partial derivatives: f_{xx} , f_{xy} , f_{yx} , and f_{yy} . Note that $f_{xy} = (f_x)_y$.

In other words, to find f_{xy} we differentiate f with respect to x first, and with respect to y next. Second partials are often denoted by $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$, and $\frac{\partial^2 f}{\partial y^2}$. Note that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)$. In other words

$$\frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

Example 12.3. Find all second partial derivatives of $f(x, y) = e^x \cos(xy)$.

You see that in the above example $f_{xy} = f_{yx}$. This is in fact always the case.

Theorem 12.1 (Clairaut's Theorem). $f_{xy} = f_{yx}$ as long as both of these second derivatives are continuous.

Example 12.4. Let \mathcal{C} be the curve of intersection of the paraboloid $z = x^2 + y^2$ and the plane $x = 1$. Find the equation of the line tangent to \mathcal{C} at $(1, 1, 2)$.

Example 12.5. Evaluate $f_x(0, 0, 0)$, where

$$f(x, y, z) = \begin{cases} \frac{x^3 + y^3 + z^4}{x^2 + y^2 + z^2} & \text{if } (x, y, z) \neq (0, 0, 0) \\ 0 & \text{if } (x, y, z) = (0, 0, 0) \end{cases}$$

Solution. By definition we have

$$f_x(0, 0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0, 0) - f(0, 0, 0)}{x} = \lim_{x \rightarrow 0} \frac{x^3/x^2}{x} = 1.$$

□

12.1 Summary

- To evaluate f_x (or $\frac{\partial f}{\partial x}$), we treat all variables except x as constants and differentiate the function with respect to x . Similar for f_y and f_z .
- When the function is piecewise defined, to find the partial derivatives we must use the definition: $f_x(x_0, y_0, z_0) = \lim_{x \rightarrow x_0} \frac{f(x, y_0, z_0) - f(x_0, y_0, z_0)}{x - x_0}$. Similar for y , and z .
- For all “usual” functions that you see in this class the order of partial derivative does not matter. For example $f_{xyz} = f_{zyx} = f_{xzy}$.

13 Lecture 13

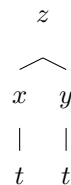
Chain Rule

Recall: Chain Rule for functions of single variable states that if y is a function of u and u is a function of x , then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

For functions of several variables we can apply a similar Chain Rule. Here are a few examples.

Example 13.1. Suppose $z = f(x, y)$, $x = g(t)$ and $y = h(t)$. Evaluate $\frac{dz}{dt}$.

Solution. z is in terms of two variables x and y , and each of these variables are in terms of t . We can illustrate that in the following tree:



To evaluate $\frac{dz}{dt}$, we move along all paths starting with z and ending at t . We multiply all the corresponding derivatives for each path and sum them up. This gives us the following:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

□

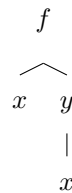
Example 13.2. Given $z = x^2 \ln(y)$, $x = t^3$, $y = t \sin(t)$, evaluate $\frac{dz}{dt}$ at $t = 1$.

Example 13.3. Given $z = x^2 + 2y^2$, $y = x + \ln x$, find $\frac{dz}{dx}$.

Example 13.4. Given $w = xy + z$, $x = u^2 + v^2$, $y = \ln(u + v)$, $z = 2uv$, find $\frac{\partial w}{\partial u}$, and $\frac{\partial^2 w}{\partial v \partial u}$ when $u = v = 1$.

Chain Rule can be used to find a formula for implicit differentiation as follows.

Suppose x and y are related by $f(x, y) = 0$. If y can be considered as a function of x , then we will have the following tree:



Using the Chain Rule we obtain $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$. This gives us $\boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$.

Note that when y is a considered a function of x , we have $f(x, y(x)) = 0$, and thus $\frac{df}{dx} = 0$. However $\frac{\partial f}{\partial x}$ is NOT zero, and must be evaluated when x and y are considered independent variables.

Example 13.5. Given the relation $x^3 + y^3 = 3xy$, find $\frac{dy}{dx}$.

13.1 Summary

- To evaluate the derivative $\frac{\partial z}{\partial t}$:
 - Draw a tree diagram with z as its top vertex (called the root).
 - Each variable y is connected to those variables in the next row that y depends on.
 - For each path starting with z and ending at t write a product of derivatives along that path.
 - Add up all the products formed in the previous step. That is equal to $\frac{\partial z}{\partial t}$.
- If $f(x, y) = 0$ and y is a function of x , then $\frac{dy}{dx} = -\frac{f_x}{f_y}$.

14 Lecture 14

Directional Derivatives

We learned about finding the derivatives of a function in the directions of coordinate axes. A similar process gives us the derivatives in other directions.

Let $\mathbf{u} = (a, b)$ be a unit vector and $f(x, y)$ be a function of two variables. Given a point (x_0, y_0) the directional derivative of $f(x, y)$ in the direction \mathbf{u} at (x_0, y_0) is given by

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{t \rightarrow 0} \frac{f(x_0 + ta, y_0 + tb) - f(x_0, y_0)}{t}$$

Note that if \mathbf{u} is not a unit vector, then we need to divide by its magnitude before using the above formula for the directional derivative. A similar formula can be used when f is a function of three variables.

The directional derivatives for $\mathbf{u} = \mathbf{i}, \mathbf{j}, \mathbf{k}$ are $f_x, f_y,$ and $f_z,$ respectively.

As long as $f_x,$ and f_y are continuous, $D_{\mathbf{u}}f(x_0, y_0) = af_x(x_0, y_0) + bf_y(x_0, y_0),$ where $\mathbf{u} = (a, b).$ Similarly, if $\mathbf{u} = (a, b, c)$ is a unit vector then, $D_{\mathbf{u}}f(x_0, y_0, z_0) = af_x(x_0, y_0, z_0) + bf_y(x_0, y_0, z_0) + cf_z(x_0, y_0, z_0).$

Example 14.1. Find the directional derivative of $f(x, y) = x \cos y$ at point $(1, \pi)$ in the direction $3\mathbf{i} + 4\mathbf{j}.$

Example 14.2. Evaluate the directional derivative of $f(x, y, z) = xy e^{xz}$ in the direction $\mathbf{i} + \mathbf{j} + \mathbf{k}$ at point $(1, -1, 0).$

As seen above $D_{\mathbf{u}}f = af_x + bf_y.$ This looks a lot like a dot product. In other words, $D_{\mathbf{u}}f = \mathbf{u} \cdot (f_x\mathbf{i} + f_y\mathbf{j}).$ Because of the importance of the vector $(f_x\mathbf{i} + f_y\mathbf{j})$ we have a name for that. The **gradient** of a function $f(x, y)$ is defined by $\nabla f = (f_x, f_y).$ If f is a function of three variables, we define the gradient as $\nabla f = (f_x, f_y, f_z).$

Note that

$$D_{\mathbf{u}}f = \mathbf{u} \cdot \nabla f = \|\mathbf{u}\| \|\nabla f\| \cos \theta = \|\nabla f\| \cos \theta,$$

where θ is the angle between \mathbf{u} and $\nabla f.$ This directional derivative is maximized when $\cos \theta = 1,$ i.e. $\theta = 0,$ and minimized when $\cos \theta = -1,$ i.e. $\theta = \pi.$ Therefore, the maximum directional derivative occurs when \mathbf{u} and ∇f are in the same direction, and the minimum directional derivative occurs when \mathbf{u} is in the opposite direction of $\nabla f.$ When $\theta = 0,$ the directional derivative is $\|\nabla f\|,$ and when $\theta = \pi,$ the directional derivative is $-\|\nabla f\|.$

Example 14.3. Given the function $f(x, y) = x^2 + 2xy - y^2,$ find the direction in which $f(x, y)$ increases most rapidly at point $(1, 2).$ Find this maximum directional derivative.

Normal lines and Tangent Planes

Assume the level surface $f(x, y) = k$ is given by $\mathbf{r}(t) = (x(t), y(t)),$ where k is a constant. Differentiating both sides and applying the Chain Rule we obtain $\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = 0.$ This can be written as

$$\underbrace{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)}_{\nabla f} \cdot \underbrace{\left(\frac{dx}{dt}, \frac{dy}{dt} \right)}_{\mathbf{r}'(t)} = 0.$$

Therefore, $\nabla f \cdot \mathbf{r}'(t) = 0.$ This means ∇f is perpendicular to the tangent vector $\mathbf{r}'(t),$ which means ∇f is a normal vector to the level curve $f(x, y) = k.$

Example 14.4. Suppose a curve on the xy -plane is given by $x^3 - 2xy + y^4 = 4$. Find a vector perpendicular to this curve at $(1, -1)$.

A similar argument can be made for level surfaces. Suppose a surface is given by $f(x, y, z) = k$, where k is a constant, and (x_0, y_0, z_0) is a point on this surface. Similar to above, let \mathcal{C} , given by $\mathbf{r}(t) = (x(t), y(t), z(t))$, be a curve on this surface through (x_0, y_0, z_0) . By the Chain Rule, we have

$$\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} = 0.$$

Thus, $\nabla f \cdot \mathbf{r}'(t) = 0$. Note that $\mathbf{r}'(t)$ is a vector tangent to \mathcal{C} . This means ∇f is perpendicular to all tangent vectors to curves that lie on the level surface through (x_0, y_0, z_0) . Thus ∇f is a normal vector of the tangent plane.

Example 14.5. Find the equation of the plane tangent to the ellipsoid $4x^2 + 9y^2 + z^2 = 14$ at point $(1, -1, 1)$.

Example 14.6. Find all points on the surface $x^3 + y^3 + z^3 + 3xyz = 6$ for which the tangent plane is parallel or identical to the plane $x + y + z = 1$.

Solution. The normal vector to this surface is $(3x^2 + 3yz, 3y^2 + 3xz, 3z^2 + 3xy)$. For the tangent plane to be parallel to the plane $x + y + z = 1$, we need this normal vector to be parallel to $(1, 1, 1)$. Thus, we need $3x^2 + 3yz = 3y^2 + 3xz = 3z^2 + 3xy$ in addition to the fact that $x^3 + y^3 + z^3 + 3xyz = 6$. We will now solve this system.

$$x^2 + yz = y^2 + xz \text{ implies } (x - y)(x + y) = z(x - y), \text{ which means either } x = y \text{ or } z = x + y.$$

Similarly $x^2 + yz = z^2 + xy$ implies either $x = z$ or $y = x + z$.

Case I. $x = y$ and $x = z$. Substituting into the equation of the surface we obtain $6x^3 = 6$, i.e. $x = 1$. This yields the point $(1, 1, 1)$.

Case II. $x = y$ and $y = x + z$. This implies $z = 0$. Substituting into the surface we obtain $2x^3 = 6$, i.e. $x = \sqrt[3]{3}$. This yields the point $(\sqrt[3]{3}, \sqrt[3]{3}, 0)$.

Case III. $z = x + y$ and $x = z$. This implies $y = 0$. Substituting into the surface we obtain $2x^3 = 6$, i.e. $x = \sqrt[3]{3}$. This yields the point $(\sqrt[3]{3}, 0, \sqrt[3]{3})$.

Case IV. $z = x + y$ and $y = x + z$. This implies $y = x + x + y$, i.e. $x = 0$ and $y = z$. Substituting into the surface we obtain $2y^3 = 6$, i.e. $y = \sqrt[3]{3}$. This yields the point $(0, \sqrt[3]{3}, \sqrt[3]{3})$.

Therefore, we obtain four points:

$$(1, 1, 1), (\sqrt[3]{3}, \sqrt[3]{3}, 0), (\sqrt[3]{3}, 0, \sqrt[3]{3}), (0, \sqrt[3]{3}, \sqrt[3]{3}).$$

□

14.1 Summary

- $D_{\mathbf{u}}f = \mathbf{u} \cdot \nabla f$.
- The maximum directional derivative occurs when $\mathbf{u} = \nabla f / \|\nabla f\|$ and this maximum directional derivative is equal to $\|\nabla f\|$.
- The minimum directional derivative occurs when $\mathbf{u} = -\nabla f / \|\nabla f\|$ and this minimum directional derivative is equal to $-\|\nabla f\|$.
- ∇f is the normal vector to level curves and level surfaces.

15 Lecture 15

Tangent Planes and Approximations

Last time we saw that ∇f is the normal vector to level curves and level surfaces. This normal vector can be used to find the equation of the line perpendicular to a level surface or level curve and also the equation of the tangent plane to a level surface.

Recall that for a function $f(x)$, the tangent line at $(x_0, f(x_0))$ is given by $y - f(x_0) = f'(x_0)(x - x_0)$, which is equivalent to $y = f(x_0) + f'(x_0)(x - x_0)$. Since the tangent line is close to the curve we have the tangent line approximation: $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$, which is valid when x is near x_0 . A similar approximation can be done for functions of two and three variables.

Example 15.1. Find the equation of the plane tangent to the surface $z = \sqrt[3]{x^3 + y}$ at $(0, 1, 1)$. Use that to approximate $\sqrt[3]{(0.01)^3 + 0.97}$.

In general, the surface given by $z = f(x, y)$ can be written as $f(x, y) - z = 0$. Thus, a normal vector to this surface is given by $(f_x, f_y, -1)$. This means the equation of the tangent plane at $(x_0, y_0, f(x_0, y_0))$ is given by $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0$. This can be written as $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. Thus, we get the tangent plane approximation

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

A similar formula is valid for functions of three variables.

$$f(x, y, z) \approx f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0)$$

Example 15.2. Approximate $\sqrt{(3.01)^2 + (0.01)^2 + (3.99)^2}$.

15.1 Summary

- ∇f is normal to the level curve $f(x, y) = k$ or the level surface $f(x, y, z) = k$.
- Tangent plane approximation: $f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. Similar for functions of three variables.

16 Lecture 16

Extreme Values

We say $f(x, y)$ has a **local (or relative) maximum** at a point (x_0, y_0) if there is a disk D centered at (x_0, y_0) for which $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in D . Similarly, $f(x, y)$ has a **local (or relative) minimum** at point (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in D . If f has either a local maximum or a local minimum at (x_0, y_0) we say f has an **extreme value** at (x_0, y_0) . We say $f(x, y)$ has an **absolute (or global) maximum** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in the domain of f . We say f has an **absolute (or global) minimum** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in the domain of f .

Example 16.1. $\cos(x^2 + y^2)$ has an absolute maximum at $(0, 0)$, because $\cos(x^2 + y^2) \leq 1$ and $1 = \cos(0^2 + 0^2)$, for all (x, y) .

In single-variable calculus we learned that if a function $f(x)$ has an extreme value at a point c , then $f'(c) = 0$ or $f'(c)$ does not exist. These points were called critical points. A similar approach could be used for multivariable functions. When a function $f(x, y)$ has an extreme value at a point (x_0, y_0) , then all directional derivatives must be zero.

A **critical point** for the function $f(x, y)$ is a point (x_0, y_0) for which both of the following hold:

- $f_x(x_0, y_0) = 0$ or it does not exist; and
- $f_y(x_0, y_0) = 0$ or it does not exist.

Example 16.2. Find all critical points of

(a) $f(x, y) = 1 - x^2 + 2x - y^2 + 3y$.

(b) $g(x, y) = |x| + y^2$.

Example 16.3. $f(x, y) = y^2 - x^2$ has a critical point at $(0, 0)$, but $f(x, y)$ does not have a relative extreme value at $(0, 0)$.

The second partials test tells us when a critical point is a relative minimum, relative maximum or a saddle point.

The Second Partial Test. Assume $f(x, y)$ has a critical point at (x_0, y_0) . To find out if (x_0, y_0) is a relative maximum, minimum or a saddle point, evaluate the 2×2 determinant $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ at (x_0, y_0) .

- (a) If $D > 0$ and $f_{xx} > 0$, then f has a relative minimum at (x_0, y_0) .
- (b) If $D > 0$ and $f_{xx} < 0$, then f has a relative maximum at (x_0, y_0) .
- (c) If $D < 0$, then f has a saddle point at (x_0, y_0) .
- (d) If $D = 0$, then the test is inconclusive.

Example 16.4. Let $f(x, y) = x^2 + y^3/3 + 2xy + 8x$. Find and categorize all critical points of f .

A subset D of the plane or the 3-dimensional space is called **bounded** if it can be fit into a circle of a sphere. It is called **closed** if it contains its boundary.

Extreme Value Theorem tells us that any continuous function $f : D \rightarrow \mathbb{R}$ whose domain D is closed and bounded has an absolute maximum and an absolute minimum. To find these values,

- Find all critical points of f in D .
- Find all extreme values of f on the boundary of D .
- The maximum of f is the largest value of f on points found in the previous parts, and the minimum is the smallest.

Example 16.5. Find the maximum and minimum values of $f(x, y) = x^2 + 2y^2 - 2xy$ on the set $R = \{(x, y) \mid 0 \leq x \leq 1, \text{ and } 0 \leq y \leq 1\}$.

16.1 Summary

- A local minimum for a function f is a point \mathbf{x}_0 for which $f(\mathbf{x}_0)$ is smaller than or equal to $f(\mathbf{x})$ for points \mathbf{x} close to \mathbf{x}_0 . Similar for local maximum.
- An absolute minimum for a function f is a point \mathbf{x}_0 for which $f(\mathbf{x}_0)$ is smaller than or equal to $f(\mathbf{x})$ for all points \mathbf{x} in the domain of f . Similar for absolute maximum.
- We say f has an extreme value at a point if it has either a local minimum or a local maximum.
- Critical points are all points for which $\nabla f = \mathbf{0}$ or ∇f does not exist.
- To classify the critical points we use the Second Partial Test.
- To find the absolute maximum and minimum values of f over a closed and bounded set we:
 - find all critical points in the given region.
 - find the extreme values of f on the boundary.
 - compare the value of f at the points found above.

17 Lecture 17

Lagrange Multipliers

Suppose we want to find the maximum of $f(x, y, z)$ under the constraint given by $g(x, y, z) = k$, where k is a constant. The Lagrange Multipliers Method tells us that an extreme value must satisfy $\nabla f = \lambda \nabla g$. In other words, it yields the following system:

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = k \end{cases}$$

Similar when we want to find the extreme value of $f(x, y)$ subject to $g(x, y) = k$.

Example 17.1. Find maximum and minimum values of $f(x, y) = x^3 + 2y^2$ given $x^2 + 3y^2 = 1$.

Example 17.2. Find the minimum of $x^2 + 2y^2 + z^2$ given $3x + 2y + z = 6$. Assume this minimum exists.

Example 17.3. Find the minimum distance from a point on the surface $3z^2 + x^2 + 2x + y^2 = 1$ to the origin. Assume this minimum distance exists.

Solution. We will use the Lagrange Multipliers method.

The distance from a point (x, y, z) to the origin is given by $\sqrt{x^2 + y^2 + z^2}$. Since the square root function is increasing, minimizing this square root is the same as minimizing $f(x, y, z) = x^2 + y^2 + z^2$. The constraint is given by $g(x, y, z) = 3z^2 + x^2 + 2x + y^2 = 1$. We will then set up the system for $\nabla f = \lambda \nabla g$.

$$\begin{cases} 2x = \lambda(2x + 2) \\ 2y = \lambda 2y \Rightarrow y = 0 \text{ or } \lambda = 1. \\ 2z = \lambda 6z \Rightarrow z = 0, \text{ or } \lambda = 1/3. \\ 3z^2 + x^2 + 2x + y^2 = 1 \end{cases}$$

We started from the second and third equations, since those are the simplest ones.

There are three possible cases:

Case I. $y = z = 0$. Substituting into the constraint we obtain $x^2 + 2x = 1$, which implies $x^2 + 2x - 1 = 0$. The quadratic formula yields $x = -1 \pm \sqrt{2}$. We see $f(-1 \pm \sqrt{2}, 0, 0) = (-1 \pm \sqrt{2})^2$. The smaller value is $(-1 + \sqrt{2})^2$.

Case II. $y = 0$, and $\lambda = 1/3$. Substituting into the first equation we get $2x = (2x + 2)/3$. Multiplying by 3 yields $4x = 2$, i.e. $x = 1/2$. The constraint gives us $3z^2 + 1/4 + 1 + 0 = 1$, which has no solutions.

Case III. $\lambda = 1$, and $z = 0$. Substituting $\lambda = 1$ into the first equation we get $2x = 2x + 2$, which is impossible.

Therefore, the smallest value of f subject to $g = 1$ is $(-1 + \sqrt{2})^2$. The answer to the problem is thus, $\sqrt{(-1 + \sqrt{2})^2} = \boxed{\sqrt{2} - 1}$. \square

17.1 Summary

- To show extreme values of a function f given a constraint $g = k$ exist we use the Extreme Value Theorem. We will have to show f is continuous and the region given by $g = k$ is closed and bounded.
- To find these extreme values of a function f under the constraint $g = k$, we solve the system $\nabla f = \lambda \nabla g$, $g = k$. Then find the values of f at the solutions of this system. The smallest value is the minimum and the largest is the maximum.

18 Lecture 18

Double Integrals

Recall that given a positive function $f(x)$, the definite integral $\int_a^b f(x) dx$ evaluates the area underneath the curve $y = f(x)$ between $x = a$ and $x = b$. Similarly given a positive function $f(x, y)$, the volume of the solid below the graph $z = f(x, y)$ and above the region R in the xy -plane is given by $\iint_R f(x, y) dA$.

Example 18.1. Let R be the triangle whose vertices are $(0, 0, 0)$, $(0, 1, 0)$, and $(1, 0, 0)$. Evaluate $\iint_R 1 dA$.

To evaluate double integrals we need a notion called **iterated integrals**, which are basically two integrals one nested inside the other.

Example 18.2. Evaluate the iterated double integral:

(a) $\int_0^1 \int_1^2 x + 2y dx dy$.

(b) $\int_{-1}^1 \int_0^x 3x^2 + 2y dy dx$.

Now, we are going to discuss the relation between double integrals and iterated integrals.

R is called **simple** if R is a rectangle given by $a \leq x \leq b$ and $c \leq y \leq d$, in which case

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy.$$

R is called **vertically simple** if R is given by $a \leq x \leq b$, $g(x) \leq y \leq h(x)$, in which case

$$\iint_R f(x, y) dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y) dy dx.$$

R is called **horizontally simple** if R is given by $c \leq y \leq d$, and $g(y) \leq x \leq h(y)$, in which case

$$\iint_R f(x, y) \, dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y) \, dx \, dy.$$

Note that in all cases the final answer must be a constant.

Example 18.3. Let R be the region in the xy -plane bounded by the x -axis, the line $x = 1$, and the curve $y = x^2$. Find the volume of the solid that lies above R and below the surface $z = x^2 + y^2$.

Example 18.4. Evaluate $\iint_R x \, dA$, where R is the region in the xy -plane bounded by the line $y = x + 2$ and the parabola $y = x^2$.

Solution. The intersection points of these two graphs satisfy $x^2 = x + 2$. Solving this we obtain $x = -1, 2$. Thus, the region is bounded by $x = -1$ and $x = 2$ on the sides. The region is vertically simple. We have

$$\begin{aligned} \iint_R x \, dA &= \int_{-1}^2 \int_{x^2}^{x+2} x \, dy \, dx \\ &= \int_{-1}^2 xy \Big|_{y=x^2}^{y=x+2} \, dx \\ &= \int_{-1}^2 x^2 + 2x - x^3 \, dx \\ &= \frac{x^3}{3} + x^2 - \frac{x^4}{4} \Big|_{x=-1}^{x=2} \\ &= \frac{8}{3} + 4 - \frac{16}{4} - \left(\frac{-1}{3} + 1 - \frac{1}{4} \right) = \frac{9}{4}. \end{aligned}$$

□

Example 18.5. Evaluate $\int_0^{\pi/2} \int_y^{\pi/2} \sin(x^2) \, dx \, dy$.

Note. Area of a region R is evaluated by $\iint_R 1 \, dA$.

18.1 Summary

- $\iint_R f(x, y) \, dA$ evaluates the volume of the solid that lies above the region R in the xy -plane and below the surface $z = f(x, y)$.
- Let R be a vertically simple region, i.e. a region that is bounded by two vertical lines $x = a$ and $x = b$ on the sides and two curves $y = g(x)$ and $y = h(x)$ at the bottom and on top. Then $\iint_R f(x, y) \, dA$ can be evaluated using the Fubini's Theorem: $\iint_R f(x, y) \, dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y) \, dy \, dx$.

- Some iterated double integrals can be evaluated by swapping the order of integration. To do that we turn the iterated double integral into a double integral by drawing its corresponding region, and then switch it back into an iterated integral by swapping dx and dy .

19 Lecture 19

Double Integrals in Polar Coordinates.

If R is a polar region, e.g. a circle, a washer, or a sector, we can evaluate the double integral over R using polar coordinates. If R is given by $a \leq \theta \leq b$ and $g(\theta) \leq r \leq h(\theta)$, with a, b , constants, then

$$\iint_R f(x, y) \, dA = \int_a^b \int_{g(\theta)}^{h(\theta)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.$$

The factor r appears because we switch from $dx \, dy$ to $dr \, d\theta$.

Example 19.1. Let R be the region between circles of radius 1 and 2 centered at the origin. Evaluate $\iint_R x^2 \, dA$.

Example 19.2. Find the volume of the solid that lies above the xy -plane and below the surface given by $z = 4 - x^2 - y^2$.

Example 19.3. Find the volume of the solid that lies inside the cylinder $(x - 1)^2 + y^2 = 1$, above the xy -plane and below the cone $z = \sqrt{x^2 + y^2}$.

Example 19.4. Evaluate $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{x^2+y^2} \, dy \, dx$.

Example 19.5. Find the area of the region enclosed by the polar curve $r = \cos(2\theta)$.

Solution. Let R be this region. The area of a region is $\iint_R 1 \, dA$. (When the height of a solid is 1, the volume of the solid is the same as the area of its base.) This region consists of four rose pedals. Thus we have

$$\iint_R 1 \, dA = 4 \int_{-\pi/4}^{\pi/4} \int_0^{\cos(2\theta)} r \, dr \, d\theta = 4 \int_{-\pi/4}^{\pi/4} \left. \frac{r^2}{2} \right|_{r=0}^{r=\cos(2\theta)} \, d\theta = 4 \int_{-\pi/4}^{\pi/4} \frac{\cos^2(2\theta)}{2} \, d\theta = \int_{-\pi/4}^{\pi/4} 1 + \cos(4\theta) \, d\theta.$$

Here, we used the half angle formula. Integrating we obtain $\frac{\pi}{2}$. □

19.1 Summary

- To evaluate $\iint_R f(x, y) \, dA$ where R is a polar region given by $a \leq r \leq b$, and $g(r) \leq \theta \leq h(r)$, we use the iterated double integral $\int_a^b \int_{g(r)}^{h(r)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta$.
- Some iterated double integrals can be evaluated by writing the iterated integral as a double integral over a region and switching to polar coordinates.

20 Lecture 20

Triple Integrals

Suppose the mass density of the solid E at point (x, y, z) is $f(x, y, z)$. The triple integral $\iiint_E f(x, y, z) \, dV$ evaluates the total mass of this solid. Let R be the projection of E onto the xy -plane, and suppose E lies between the surfaces $z = g(x, y)$ and $z = h(x, y)$. Then,

$$\iiint_E f(x, y, z) \, dV = \iint_R \left(\int_{g(x,y)}^{h(x,y)} f(x, y, z) \, dz \right) \, dA$$

This turns the problem into evaluating a single integral followed by a double integral. The double integral can be turned into an iterated double integral if R is a horizontally simple region, a vertically simple region, or a polar region.

Remark. When the density is 1, the mass and volume are the same. Therefore,

$$\text{Volume of } E = \iiint_E 1 \, dV$$

Example 20.1. Let E be the solid bounded by the planes $x = 2, x = 5, y = 0, y = 3, z = 0$, and $z = 1$. Evaluate $\iiint_E xy \, dV$.

Example 20.2. Evaluate the volume of the solid that lies above the plane $z = -2$, below the surface $z = 1 - x^2$, in front of the plane $x = 0$, to the right of the plane $y = 0$, and behind the plane $x + y = 1$.

Example 20.3. Consider the solid E below the paraboloid $z = 1 - x^2 - y^2$ and above the plane $z = -3$. Evaluate $\iiint_E x + 1 \, dV$.

20.1 Summary

- $\iiint_E f(x, y, z) \, dV$ evaluates the mass of an object at E whose mass density is $f(x, y, z)$.
- $\iiint_E 1 \, dV$ evaluates the volume of E .
- To evaluate $\iiint_E f(x, y, z) \, dV$ using Cartesian Coordinates:
 - Draw the solid E and its projection R onto the xy -plane.
 - Turn dV into $dz \, dx \, dy$ or $dz \, dy \, dx$ depending on whether R is horizontally simple or vertically simple.
 - Limits of z are the bottom and top functions that bound E . These limits may depend on x and y .
 - Limits of x (or y) are the left and right functions of R (or top and bottom of R). This may depend on y (or x).

- Limits of the last variable are two constants that are the smallest and largest possible values of that variable.
- To summarize we get

$$\iiint_E f(x, y, z) \, dV = \int_a^b \int_{\text{bottom of } R}^{\text{top of } R} \int_{\text{bottom of } E}^{\text{top of } E} f(x, y, z) \, dz \, dy \, dx$$

or

$$\iiint_E f(x, y, z) \, dV = \int_c^d \int_{\text{left of } R}^{\text{right of } R} \int_{\text{bottom of } E}^{\text{top of } E} f(x, y, z) \, dz \, dx \, dy$$

21 Lecture 21

Triple Integrals in Cylindrical Coordinates

As seen in the last example from the previous lecture, when the projection of the solid E onto the xy -plane is a polar region, using dz followed by $r \, dr \, d\theta$ simplifies our computation. This, in effect is using a new coordinate system called **cylindrical coordinates**.

Every point in the 3D space can be represented using (r, θ, z) . The variables r and θ represent polar coordinates of the projection of each point on the xy -plane.

Example 21.1. Find the volume of the solid in the first octant inside both cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

Example 21.2. Let E be the solid under the cone $z = \sqrt{x^2 + y^2}$, inside the surface given by $x^2 + y^2 = 1$, and above the xy -plane. An object whose mass density at every point is the distance to the xy -plane is placed at E . Evaluate the mass of this object.

Example 21.3. Find the volume of the solid that lies above the surface given by $z = r$, and inside the surface given by $x^2 + y^2 + z^2 = 2$.

21.1 Summary

- To evaluate $\iiint_E f(x, y, z) \, dV$ using cylindrical coordinates:
 - Draw the solid E and its projection R onto the xy -plane.
 - Write down the integrand in terms of r, θ, z .
 - Turn dV into $r \, dz \, dr \, d\theta$.
 - Limits of z are the bottom and top functions that bound E . These limits may depend on r and θ .

- Limits of r are the closest and farthest points to the origin on region R . These may depend on θ .
- Limits of θ are two constants that are the smallest and largest possible values θ .
- To summarize we get

$$\iiint_E f(x, y, z) \, dV = \int_{\alpha}^{\beta} \int_{\text{closest to the origin in } R}^{\text{farthest from the origin in } R} \int_{\text{bottom of } E}^{\text{top of } E} f(r \cos \theta, r \sin \theta, z) r \, dz \, dr \, d\theta$$

22 Lecture 22

Triple Integrals in Spherical Coordinates

So far we have learned two different representations of a point in space: Cartesian coordinates (x, y, z) , and cylindrical coordinates (r, θ, z) . We will be talking about a third coordinate system today.

For a point P in space:

- ρ is the distance to the origin.
- ϕ is the angle that OP makes with the positive z -axis.
- θ is the same θ as in cylindrical coordinates.

The relations between these coordinates are listed below:

$$r = \rho \sin \phi, \quad x^2 + y^2 + z^2 = \rho^2, \quad z = \rho \cos \phi, \quad x = \underbrace{\rho \sin \phi}_{r} \cos \theta, \quad y = \underbrace{\rho \sin \phi}_{r} \sin \theta, .$$

Example 22.1. Identify each of the following shapes in space:

- $\rho = 5$
- $\phi = \pi/3$
- $\theta = \pi/4$
- $\theta = \pi/2$ and $\phi = \pi/3$

When switching to spherical coordinates we use $dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$.

Example 22.2. Find the volume of a sphere of radius a .

Example 22.3. Find the volume of the solid that lies outside the sphere $\rho = 1$, inside the sphere $\rho = 2$, and above the cone $z = \sqrt{x^2 + y^2}$.

Example 22.4. Evaluate the total mass of a solid whose mass density at any point is equal to the reciprocal of the distance to the xz -plane and is positioned inside the sphere $x^2 + y^2 + z^2 = 2y$.

22.1 Summary

- $x^2 + y^2 + z^2 = \rho^2$, $r = \rho \sin \phi$, $z = \rho \cos \phi$, $x = \rho \sin \phi \cos \theta$, and $y = \rho \sin \phi \sin \theta$.
- When switching to spherical coordinates:
 - Graph the solid.
 - Write down the integrand in terms of ρ, ϕ, θ .
 - Substitute dV by $\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$.
 - For limits of ρ , by looking at an arbitrary ray through the origin, use the closest and farthest points from the origin. Your limits may depend on θ and ϕ .
 - Find the maximum and minimum values of ϕ by looking at the half-planes through the origin. These limits may depend on θ .
 - Limits of θ would be constants and can be obtained by looking at the projection of the solid onto the xy -plane.
- Spherical coordinates are usually used when dealing with cones and spheres.

23 Lecture 23

Change of Variables

When we learned double integrals we learned how to find double integrals in Cartesian coordinates as well as polar coordinates. These coordinate systems allow us to evaluate double integrals over some -but not all- regions. Here we will learn how to find double integrals over more complicated regions.

When we switch from (x, y) to a new coordinate system (u, v) , we will have to use $dA = |\text{Jacobian}| \, du \, dv$, where Jacobian is given by the following 2×2 determinant:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

For example, when we switch from (x, y) to (r, θ) , the Jacobian will be

$$\begin{vmatrix} \frac{\partial(r \cos \theta)}{\partial r} & \frac{\partial(r \cos \theta)}{\partial \theta} \\ \frac{\partial(r \sin \theta)}{\partial r} & \frac{\partial(r \sin \theta)}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r,$$

which is precisely the polar factor that we used when we switched from Cartesian to polar.

Example 23.1. Evaluate $\iint_R x \, dA$, where R is the parallelogram with vertices $(0, 0)$, $(1, 2)$, $(2, 1)$, and $(3, 3)$.

Example 23.2. Find the volume of the solid that lies below the graph $z = x^2 + y^2$, and above the region in the xy -plane bounded by the curves $xy = 1$, $xy = 2$ and the lines $y = x$ and $y = 2x$.

Example 23.3. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a, b > 0$ are two constants.

23.1 Summary

- When switching from (x, y) to (u, v) coordinates follow the following steps for evaluating $\iint_R f(x, y) \, dA$.
 - Draw the region R in the xy -plane.
 - Choose (u, v) in a way that it turns R into a horizontally simple, vertically simple or polar region in the uv -plane.
 - Write the integrand in terms of u and v .
 - Draw the new region S in the uv -plane, by re-writing the boundary of R in terms of u and v .
 - Use $dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du \, dv$ to set up the new iterated double integral.
 - Note that the fact that $\frac{\partial(x, y)}{\partial(u, v)}$ is the reciprocal of $\frac{\partial(u, v)}{\partial(x, y)}$ could sometimes help.

- $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

24 Lecture 24

Parametrized Surfaces

To parametrize a surface we need two parameters.

Example 24.1. Parametrize each surface:

- (a) The surface given by $z = x^2 + 2xy$ that lies above the square in the xy -plane with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$.
- (b) The surface given by $z = 2xy$ above the disk $x^2 + y^2 \leq 1$ in the xy -plane.
- (c) The sphere centered at the origin with radius a , where a is a positive constant.

Example 24.2. Parametrize each surface:

- (a) The cylinder given by $x^2 + y^2 = a^2$, where a is a positive constant.
- (b) Part of the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 2$.

Example 24.3. Parametrize the surface obtained by revolving the graph of $y = x^2$ in the xy -plane about the x -axis.

Tangent Planes

If $\mathbf{r}(u, v)$ is a parametrization of a surface Σ , then \mathbf{r}_u and \mathbf{r}_v are both tangent to the surface. Thus, $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the surface.

Example 24.4. Find an equation of the plane tangent to the surface of revolution in the previous problem at point $(1, \frac{1}{2}, \frac{\sqrt{3}}{2})$.

24.1 Summary

- The graph of a function $f(x, y)$ can be parametrized by $\mathbf{r}(x, y) = (x, y, f(x, y))$.
- Spheres are often parametrized by spherical coordinates.
- Cylinders are often parametrized using cylindrical coordinates.
- Cones can often be parametrized using either cylindrical or spherical coordinates.
- By rotating the graph $y = f(x)$ in the xy -plane about the x -axis, we obtain a surface that can be parametrized as $\mathbf{r}(x, \theta) = (x, f(x) \cos \theta, f(x) \sin \theta)$.
- A complete parametrization must include the limits of its parameters.
- $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the surface given by $\mathbf{r}(u, v)$.

25 Lecture 25

Vector Fields

A vector field $\mathbf{F}(x, y)$ or $\mathbf{F}(x, y, z)$ is a function that assigns vectors to points on the plane or in the 3D space. Examples of vector fields include electric fields, fluid flows (i.e. the velocity of a fluid), and forces.

Example 25.1. Find a formula for the gravitational force for an object of mass m on Earth.

Example 25.2. Let $f(x, y, z)$ be a multivariable function. Then ∇f is a vector field.

The **divergence** of a vector field $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$ denoted by $\text{div } \mathbf{F}$ is defined as $P_x + Q_y + R_z$. This is often memorized using the following shortcut:

Let $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, and $\mathbf{F} = (P, Q, R)$. We can think of $\text{div } \mathbf{F}$ as a dot product $\nabla \cdot \mathbf{F}$.

$\text{div } \mathbf{F}$ is some form of derivative of a vector field. If \mathbf{F} is the fluid flow, then $\text{div } \mathbf{F}$ measures the rate of mass flow per unit volume from a given point.

Another kind of derivative for vector fields is called curl and is denoted by $\text{curl } \mathbf{F}$. Using the notation that we used above $\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$.

Suppose \mathbf{F} is a fluid flow. The particles in the fluid rotate about the vector $\text{curl } \mathbf{F}$. The magnitude of $\text{curl } \mathbf{F}$ measures the rate of rotation.

Example 25.3. Find $\text{div } \mathbf{F}$ and $\text{curl } \mathbf{F}$, where $\mathbf{F}(x, y, z) = (x^2 + y)\mathbf{i} - yz\mathbf{j} + z^2\mathbf{k}$.

Two identities that relate curl, divergence and gradient are very useful: $\text{curl } (\nabla f) = \mathbf{0}$ and $\text{div } (\text{curl } \mathbf{F}) = 0$.

A vector field \mathbf{F} is called **conservative** if $\mathbf{F} = \nabla f$ for some function f . The function f is called a **potential function** of \mathbf{F} .

If the domain of \mathbf{F} is the entire plane or the entire 3D space, and $\text{curl } \mathbf{F} = \mathbf{0}$, then \mathbf{F} is conservative.

Example 25.4. Determine if each vector field is conservative. If it is, find a potential function.

(a) $\mathbf{F}(x, y, z) = ye^{xy}\mathbf{i} + (xe^{xy} + \cos y)\mathbf{j} + \mathbf{k}$.

(b) $\mathbf{F}(x, y) = xe^x\mathbf{i} + (y^2 - 1)\mathbf{j}$.

25.1 Summary

- $\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$, where $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$.
- $\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$.
- $\text{curl } (\nabla f) = \mathbf{0}$ and $\text{div } (\text{curl } \mathbf{F}) = 0$
- \mathbf{F} is conservative if $\mathbf{F} = \nabla f$ for some f . We call f a potential function of \mathbf{F} . A vector field \mathbf{F} that is defined over the entire plane or the 3 dimensional space is conservative precisely when $\text{curl } \mathbf{F} = \mathbf{0}$.
- To find a potential function f for $\mathbf{F} = (P, Q, R)$, start with solving $f_x = P$, by integrating with respect to x . Note that instead of “ C ” you would need to include $g(y, z)$, a function of y and z . Then differentiate f with respect to y , use $f_y = Q$ and find f , including $h(z)$ instead of “ C ”, a function of z . Repeat that to find a function f that satisfies all three equations $f_x = P, f_y = Q$, and $f_z = R$. Note that you often only need one potential functions, and not all of them. In that case, you can skip the last “ $+C$ ”.

26 Lecture 26

Line Integrals

Assume the mass density of a thin wire C at any point (x, y, z) is given by $f(x, y, z)$. Note that since we are assuming the wire is thin, its density is given by density = $\frac{\text{mass}}{\text{length}}$. The line integral $\int_C f(x, y, z) ds$ gives us the total mass of this wire. This is called the line integral of f over C with respect to the arc length s . Recall that the arc length satisfies $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$. Thus, to evaluate the line integral $\int_C f(x, y, z) ds$, we substitute ds by $\|\mathbf{r}'(t)\| dt$ and use the limits of t that determine the curve C .

Example 26.1. Evaluate $\int_C (x + y) ds$, where C is the helix given by $(\cos t, \sin t, t)$ with $0 \leq t \leq \pi/2$.

Example 26.2. Evaluate $\int_C x^2 y ds$, where C is the segment joining $(0, 1, -1)$ and $(-2, 3, 0)$.

Example 26.3. Evaluate $\int_C (x + y) ds$, where C is part of the unit circle in the xy -plane centered at the origin in the first quadrant, followed by the segment from $(1, 0)$ to $(0, 0)$ and the segment from $(0, 0)$ to $(0, 1)$.

Recall that the work done by a force \mathbf{F} with displacement \mathbf{D} is given by $\mathbf{F} \cdot \mathbf{D}$.

The work done by a vector field \mathbf{F} along a curve C is given by the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$. This line integral can be evaluated by parametrizing C and using $d\mathbf{r} = \mathbf{r}'(t) dt$. Note that the orientation that C is traversed is important. The curve obtained by reversing the orientation of C is denoted by $-C$. Reversing the orientation of C negates the line integral of a vector field, but it does not change the line integral of a scalar function.

Example 26.4. Let C be the unit circle in the xy -plane centered at the origin oriented clockwise. Evaluate $\int_C (y\mathbf{i} - x\mathbf{j}) \cdot d\mathbf{r}$.

If C is a curve parametrized by $\mathbf{r}(t) = (x(t), y(t), z(t))$ with t from a to b , and $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, then $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b (M\mathbf{i} + N\mathbf{j} + P\mathbf{k}) \cdot (x', y', z') dt = \int_a^b (Mx' + Ny' + Pz') dt$. This is often denoted by

$$\int_C M dx + N dy + P dz.$$

In other words $\int_C \mathbf{F} \cdot d\mathbf{r}$ and $\int_C M dx + N dy + P dz$ are the same.

Example 26.5. Evaluate $\int_C (x^2 + y) dx + (x - y) dy$, where C is the segment from $(1, 0, 2)$ to $(3, -1, 1)$.

26.1 Summary

- $\int_C f(x, y, z) ds$ evaluates the total mass of a thin wire with density f .
- To evaluate $\int_C f(x, y, z) ds$
 - Find a parametrization $\mathbf{r}(t)$ with $a \leq t \leq b$ for C .
 - Replace ds by $\|\mathbf{r}'(t)\| dt$ and evaluate the integral: $\int_C f(x, y, z) ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\| dt$.
- $\int_C \mathbf{F} \cdot d\mathbf{r}$ evaluates the work done by \mathbf{F} along C .
- To evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$
 - Find a parametrization $\mathbf{r}(t)$ with t from a to b for C . Note that it may be the case that $a > b$.
 - Replace $d\mathbf{r}$ by $\mathbf{r}'(t) dt$ and evaluate the integral: $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b f(\mathbf{r}(t))\mathbf{r}'(t) dt$.
- Note that $\int_C f ds$ does not change if we change the orientation of C but $\int_C \mathbf{F} \cdot d\mathbf{r}$ changes sign.
- When $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, we often write $\int_C M dx + N dy + P dz$ instead of $\int_C \mathbf{F} \cdot d\mathbf{r}$.

27 Lecture 27

The Fundamental Theorem of Line Integrals

When \mathbf{F} is a conservative vector field, the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ can be evaluated using the Fundamental Theorem of Line Integrals. This theorem states that if $\mathbf{F} = \nabla f$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\text{terminal point of } C) - f(\text{initial point of } C)$$

Example 27.1. Let C be the curve from $(1, 0, 0)$ to $(0, 1, 1)$ given by $\mathbf{r}(t) = \cos^3 t \mathbf{i} + \frac{2t}{\pi} \mathbf{j} + \sin^4 t \mathbf{k}$. Find $\int_C ye^{xy} dx + xe^{xy} dy + \sin z dz$.

The Fundamental Theorem of Line Integrals shows that when \mathbf{F} is conservative, the line integral only depends on the endpoints of C . That fact is stated as “ $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path**”.

Example 27.2. Let $\mathbf{F}(x, y, z) = \frac{x \mathbf{i} + y \mathbf{j} + z \mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$.

(a) Show that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.

(b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is an unknown curve from $(1, 2, 2)$ to $(-3, 4, 0)$.

Green's Theorem

Let R be a plane region with boundary C . An orientation of C is called **positive** if when C is traversed in that direction, R lies on our left.

Green's Theorem relates a line integral over a closed curve C with a double integral over the region R whose boundary is C .

Green's Theorem. If R is a region in the xy -plane whose boundary C is positively oriented, then

$$\int_C M dx + N dy = \iint_R (N_x - M_y) dA$$

27.1 Summary

- Fundamental Theorem of Line Integrals states: $\int_C \nabla f \cdot d\mathbf{r} = f(\text{terminal}) - f(\text{initial})$.
- Saying a vector field is independent of path is the same as saying it is conservative.

28 Lecture 28

Example 28.1. Let C be the triangle in the xy -plane whose vertices are $(0, 0)$, $(2, 0)$, and $(1, 1)$ oriented counterclockwise. Evaluate $\int_C xy dx + e^{y^2} dy$.

Recall that the area of a region R in the xy -plane can be evaluated using the double integral $\iint_R 1 \, dA$. The Green's Theorem relates a double integral with a line integral. We can use that to find the area of the region R if we choose M and N in a way that $N_x - M_y = 1$. Here are three ways of evaluating the area of R :

$$\text{Area of } R = \int_C x \, dy = \int_C -y \, dx = \frac{1}{2} \int_C x \, dy - y \, dx$$

Example 28.2. Find the area of the region enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are two positive constants.

Surface Integrals

Let Σ be a surface parametrized by $\mathbf{r}(u, v)$ where (u, v) is in a region R in the uv -plane. Assume a thin metal surface located at Σ has mass density $f(x, y, z)$. Then the surface integral $\iint_{\Sigma} f(x, y, z) \, dS$ evaluates the total mass of this metal surface. Note that in this situation, density = $\frac{\text{mass}}{\text{area}}$, since we are dealing with a thin surface.

To evaluate this surface integral, we use $dS = \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA$ and turn this surface integral into a double integral over R .

Example 28.3. Evaluate $\iint_{\Sigma} z^2 \, dS$, where Σ is the sphere $x^2 + y^2 + z^2 = 1$.

28.1 Summary

- Green's Theorem: $\int_C M \, dx + N \, dy = \iint_R (N_x - M_y) \, dA$, where C is the boundary of R and is positively oriented.
- Area of $R = \int_C x \, dy = \int_C -y \, dx = \frac{1}{2} \int_C x \, dy - y \, dx$.
- $\iint_{\Sigma} f(x, y, z) \, dS$ evaluates the total mass of the surface Σ , when the density is $f(x, y, z)$.
- $\iint_{\Sigma} f(x, y, z) \, dS = \iint_R \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA$.
- Area of $\Sigma = \iint_{\Sigma} 1 \, dS$.

29 Lecture 29

Example 29.1. Evaluate $\iint_{\Sigma} (x + y) \, dS$, where Σ is part of the plane $x + y + z = 5$ that lies in the first octant.

Note that $\iint_{\Sigma} 1 \, dS$ evaluates the area of surface Σ , since when the density is 1, the area and mass are the same.

Example 29.2. Find the area of a sphere of radius a .

Integrals over Oriented Surfaces

An orientation of a surface Σ is a selection of unit normal vectors \mathbf{n} for each point on the surface in which \mathbf{n} changes continuously as the point changes.

Assume $\mathbf{v}(x, y, z)$ is the velocity of a fluid with density $\delta(x, y, z)$. After 1 second the distance that a particle travels in the direction of the unit normal vector is $\|\mathbf{v}\| \cos \theta$, where θ is the angle between \mathbf{v} and \mathbf{n} . Thus, the volume of fluid that passes through Σ after 1 second is $\underbrace{\|\mathbf{v}\| \cos \theta}_{\text{distance}}(\text{area}) = (\mathbf{n} \cdot \mathbf{v})(\text{area})$. The total mass that passes through the surface per second is $\mathbf{n} \cdot \mathbf{v}(\text{area})\delta$.

Therefore, the total amount of fluid passing through Σ in the direction of \mathbf{n} per second is $\boxed{\iint_{\Sigma} \delta \mathbf{v} \cdot \mathbf{n} dS}$. We often denote $\mathbf{n} dS$ by $d\mathbf{S}$

Since $dS = \|\mathbf{r}_u \times \mathbf{r}_v\| dA$, and $\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$ we have

$$\mathbf{n} dS = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \|\mathbf{r}_u \times \mathbf{r}_v\| dA = (\mathbf{r}_u \times \mathbf{r}_v) dA.$$

This can be used to evaluate $\iint_{\Sigma} \mathbf{F} \cdot \mathbf{n} \cdot dS$.

29.1 Summary

- To evaluate $\iint_{\Sigma} \mathbf{F} \cdot d\mathbf{S} = \iint_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS$:
 - Find a parametrization $\mathbf{r}(u, v)$ for the surface Σ .
 - Evaluate $\mathbf{r}_u \times \mathbf{r}_v$ and make sure the orientation matches the one given in the problem. If it does not it must be negated.
 - Use $\mathbf{n} dS = d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v) dA$.
 - Change dA to $du dv$, $dv du$ or $r dr d\theta$ using the parameter region in the uv -plane.
- When the surface is given by $z = f(x, y)$, a normal vector is $\mathbf{r}_x \times \mathbf{r}_y = (-f_x, -f_y, 1)$.

30 Lecture 30

Example 30.1. Let Σ be the sphere $x^2 + y^2 + z^2 = 1$ oriented inward. Evaluate $\iint_{\Sigma} z^2 \mathbf{k} \cdot d\mathbf{S}$.

Example 30.2. Let Σ be part of the paraboloid $z = x^2 + y^2 - 4$ that lies below the xy -plane oriented upward. Find $\iint_{\Sigma} (x^2 \mathbf{i} + y \mathbf{k}) \cdot d\mathbf{S}$.

When the surface is given by $z = f(x, y)$, a parametrization is given by $\mathbf{r}(x, y) = (x, y, f(x, y))$. Thus, $\mathbf{r}_x = (1, 0, f_x), \mathbf{r}_y = (0, 1, f_y)$, which yields $\mathbf{r}_x \times \mathbf{r}_y = (-f_x, -f_y, 1)$.

Example 30.3. Evaluate $\iint_{\Sigma} z \mathbf{k} \cdot d\mathbf{S}$, where Σ is the cylinder $x^2 + y^2 = 1$ between the planes $z = 0$ and $z = 1$ oriented outward.

Stokes' Theorem

Stokes' Theorem relates a surface integral of an oriented surface with a line integral over its boundary.

Let Σ be a surface with a closed boundary C . An orientation of Σ is called **positive** relative to an orientation of C , if the two orientations match using the right-hand rule. For example if Σ is the upper hemisphere $\rho = 1$, with $z \geq 0$, then its boundary is the unit circle in the xy -plane. An upward orientation of Σ matches an orientation of C that is counterclockwise when viewed from above.

The **Stokes' Theorem** states that if Σ is a positively oriented surface with boundary C , then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{\Sigma} (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS.$$

30.1 Summary

- The Stokes' Theorem states that if Σ is a positively oriented surface with boundary C , then $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{\Sigma} (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS$.
- If two surfaces Σ_1 and Σ_2 have the same boundary and the same orientation, then $\iint_{\Sigma_1} (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS = \iint_{\Sigma_2} (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS$.

31 Lecture 31

Examples on Stokes' Theorem

Example 31.1. Let C be the triangle with vertices $(0, 0, 0), (1, 0, 0)$, and $(1, 1, 1)$ oriented clockwise when viewed from above. Evaluate $\int_C (y + x^3) dx + y^3 dy + (z^3 + 1) dz$.

Example 31.2. Let Σ be part of the ellipsoid $x^2 + 4y^2 + z^2 = 1$ above the xy -plane oriented upward. Evaluate $\iint_{\Sigma} (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS$, where $\mathbf{F} = x \mathbf{i} + y \mathbf{j} + z \sin(xy) \mathbf{k}$.

32 Lecture 32

More Examples

Example 32.1. Let Σ be part of the cylinder $x^2 + z^2 = 1$ between $y = 0$ and $y = 1$ oriented outward along with the disk $x^2 + z^2 \leq 1$ on the xz -plane oriented to the left. Evaluate $\iint_{\Sigma} (\text{curl } \mathbf{F}) \cdot \mathbf{ndS}$, where $\mathbf{F} = (z + 1)y^2 \mathbf{i} - (z + 1)xy \mathbf{j} + e^{x^2y^2} \mathbf{k}$.

Example 32.2. Let C be the triangle whose vertices are $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ oriented clockwise when viewed from the origin. Evaluate $\int_C 5x^4y^3 dx + 3y^2x^5 dy + x dz$.

33 Lecture 33

The Divergence Theorem

Let E be a solid with boundary Σ oriented outward from E , and let \mathbf{F} be a vector field. The Divergence Theorem states that

$$\boxed{\iint_{\Sigma} \mathbf{F} \cdot \mathbf{ndS} = \iiint_E \text{div } \mathbf{F} dV}$$

Example 33.1. Suppose Σ is the sphere given by $x^2 + y^2 + z^2 = 1$ oriented outward and $\mathbf{F}(x, y, z) = 5x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Evaluate $\iint_{\Sigma} \mathbf{F} \cdot \mathbf{ndS}$.

Example 33.2. Let Σ be the unit cube with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, and $(1, 1, 1)$ oriented inward. Evaluate $\iint_{\Sigma} (x^2\mathbf{i} + (z^3 \sin x - 2xy)\mathbf{j} + (z^2 + e^x)\mathbf{k}) \cdot \mathbf{ndS}$.

33.1 Chapter Summary

Line Integrals

1. Given a thin wire located at curve C whose density (i.e. mass/length) at point (x, y, z) is given by $f(x, y, z)$, its total mass is evaluated by $\int_C f ds$. To evaluate this integral:

- Parametrize C by $\mathbf{r}(t)$ with $a \leq t \leq b$.
- Use $ds = \|\mathbf{r}'(t)\| dt$ to get $\int_C f ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt$.
- Note that $\int_C f ds$ does not depend on the orientation of C . (Mass is independent of orientation!)

2. $\int_C \mathbf{F} \cdot d\mathbf{r}$ gives the work done by \mathbf{F} over C . Given $\mathbf{F} = (M, N, P)$, this integral may also be denoted by $\int_C Mdx + Ndy + Pdz$. Note that changing the orientation of C , changes the sign of the line integral. There are four different methods that may help in evaluating this line integral:

- **The definition:** Given a parametrization $\mathbf{r}(t)$ with $t = a$ to $t = b$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

Note that a corresponds to the initial point and b corresponds to the terminal point of C . In other words, the orientation matters and a may be larger than b .

- **The Fundamental Theorem of Line Integrals:** If $\mathbf{F} = \nabla f$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\text{terminal point of } C) - f(\text{initial point of } C).$$

- **The Green's Theorem:** Assume D is a plane region with C as its boundary. An orientation for C is called positive if D lies on the left when walking along C in that direction. Assuming C is positively oriented, we have

$$\int_C Mdx + Ndy = \iint_D (N_x - M_y) dA.$$

This is very useful if M and N are too complex but $N_x - M_y$ is simple.

- **The Stokes' Theorem:** This is a 3D version of the Green's Theorem. Given an oriented surface Σ with boundary C , an orientation of C is called positive (or induced) if orientation of C matches the orientation of Σ using the right hand rule. In which case we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{\Sigma} (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS.$$

Surface Integrals

3. Given a thin sheet located at the surface Σ whose density (i.e. mass/area) at point (x, y, z) is given by $f(x, y, z)$, its total mass is evaluated by $\iint_{\Sigma} f dS$. To evaluate this integral:

- Parametrize Σ as $\mathbf{r}(u, v)$ with (u, v) in a region R .
- Use $dS = \|\mathbf{r}_u \times \mathbf{r}_v\| dA$ to obtain: $\iint_{\Sigma} f dS = \iint_R f(\mathbf{r}(u, v)) \|\mathbf{r}_u \times \mathbf{r}_v\| dA$.
- Note that $\iint_{\Sigma} f dS$ does not depend on the orientation of Σ . (Mass is independent of orientation!)

4. Given a vector field \mathbf{F} over a surface Σ , $\iint_{\Sigma} \mathbf{F} \cdot d\mathbf{S}$ (or $\iint_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS$) gives the total flow of \mathbf{F} through Σ in the direction of \mathbf{n} . There are three ways of evaluating this flux integral:

- **The definition:** Given a parametrization $\mathbf{r}(u, v)$ of Σ , with (u, v) in R , $\iint_{\Sigma} \mathbf{F} \cdot d\mathbf{S} = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$. Make sure the orientation of Σ matches $\mathbf{r}_u \times \mathbf{r}_v$. If it does not, multiply by a negative sign.
- **The Stokes' Theorem:** If the vector field is of form $\text{curl } \mathbf{F}$, you may use the Stokes' Theorem using appropriate orientations of C and Σ : $\iint_{\Sigma} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$. Note that if Σ and Σ_1 have the same boundary with matching orientations, $\iint_{\Sigma} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_{\Sigma_1} \text{curl } \mathbf{F} \cdot d\mathbf{S}$.
- **The Divergence Theorem:** If the surface Σ is closed you may use the Divergence Theorem: With outward orientation $\iint_{\Sigma} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} dV$, where Σ is the boundary of the solid E .

Some tips for simplifying the computation.

- For surfaces given by $z = f(x, y)$, we may use $\mathbf{r}(x, y) = (x, y, f(x, y))$, which gives $\mathbf{r}_x \times \mathbf{r}_y = (-f_x, -f_y, 1)$ and $\|\mathbf{r}_x \times \mathbf{r}_y\| = \sqrt{(f_x)^2 + (f_y)^2 + 1}$.
- For spheres centered at the origin, we often use $\mathbf{r}(\phi, \theta) = (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)$. This gives $\mathbf{r}_\phi \times \mathbf{r}_\theta = \rho \sin \phi \mathbf{r}(\phi, \theta)$, and $\|\mathbf{r}_\phi \times \mathbf{r}_\theta\| = \rho^2 \sin \phi$.