Putnam Guide

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Introduction

I have been training the University of Maryland Putnam Team since 2016. One of the main obstacles for students is lack of one single appropriate, self-contained, accessible source for them to use to prepare for Putnam Competition as well as the Virginia Tech Regional Math Competition. There are several goals that we are achieving in these notes:

- Filling the knowledge gap for those students who may not have taken all the necessary courses.
- Getting students familiar with the thought process behind solving each problem.
- Making sure students are able to write solutions that maximize their chance of getting the credit that they deserve.

This book assumes familiarity with multivariable calculus, but it is self-contained otherwise. The book is divided into 14 chapters. Each chapter covers one of the essential topics that would appear on Putnam. Each chapter is divided into six sections.

- **Basics**: This section covers all basics such as definitions for each topic.
- **Important Theorems**: This section covers all theorems that are often used in competitions.
- **Classical Examples**: This section covers examples that are often expected for those who compete in math competitions to be familiar with.
- **Further Examples**: These are actual competition problems with in-depth solutions. Each solution starts with a discussion of some ideas that *might* work, followed by a discussion of which ideas work and which ones may not work. Some of these examples have video solutions on YouTube.
- **General Strategies**: These are relevant strategies that would make a good problem-solver.
- **Exercises**: These are problems from past Putnam and Virginia Tech tests along with some others from various sources.

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These note may contain occasional typos or errors. Feel free to email me at ebrahimi@umd.edu if you notice a typo or an error.

Chapter 1

Communication Tools

1.1 Basics

Proof by Contradiction: To prove a statement q given assumption p, we assume q is false but p is true and get a contradiction.

Proof by Induction: To prove a statement $P(n)$ depending on a natural number n, we prove

- $P(1)$ (basis step); and
- Assume $P(n)$ for some $n \geq 1$, and prove $P(n+1)$ (inductive step).

If you need to use $P(n-1)$ in your proof of $P(n+1)$, then the basis step must involve two consecutive integers, e.g. $P(1)$ and $P(2)$.
Often times we use what is called **strong induction** which involves assuming $P(1), \ldots, P(n)$ and proving $P(n+1)$ in addition to proving the basis step.

The basis of many proofs is a good understanding of logic along with set theory.

Definition 1.1. A set S is said to be **countable** if there is a bijection $f : S \to \mathbb{N}$. In other words, an infinite set S is countable if all of its elements can be listed as $s_1, s_2, \ldots$. We say S is **uncountable** if S is infinite and not countable.

1.2 Important Theorems

Theorem 1.1 (Useful Identities). *(a)* $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$.

(b) If n is odd, then $a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \cdots - ab^{n-2} + b^{n-1})$.

(c) $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)$.

(d) $x^2 + y^2 + z^2 - xy - yz - zx = \frac{1}{2}((x-y)^2 + (y-z)^2 + (z-x)^2)$.

(e) $x^4+4y^4 = (x^2+2y^2+2xy)(x^2+2y^2-2xy)$.

Theorem 1.2. *The sets* $\mathbb{N},\mathbb{Z}$, *and* $\mathbb{Q}$ *are countable, and* $\mathbb{R}$ *is uncountable.*

Theorem 1.3. *Suppose* $A_1, A_2, A_3, \ldots$ *is a sequence of countable sets. Then*

(a) $\bigcup_{n=1}^{\infty} A_n$ *is countable.*

(b) $A_1 \times A_2 \times \cdots \times A_n$ *is countable.*

Theorem 1.4. *Suppose* $A_1, A_2, \ldots$ *is a sequence of sets for which each* A_i *has at least 2 elements. Then* $A_1 \times A_2 \times A_3 \times \cdots$ *(the set consisting of all sequences* $a_1, a_2, \ldots$ *with* $a_i \in A_i$ *for all* i*) is uncountable.*

Proof. Suppose on the contrary $A_1 \times A_2 \times A_3 \times \cdots$ is countable, and let $s_i : a_{i1}, a_{i2}, a_{i3}, \ldots$ where $i \in \mathbb{N}$ be the list of all elements of $A_1 \times A_2 \times A_3 \times \cdots$. We will create an element in $A_1 \times A_2 \times A_3 \times \cdots$ that is not listed above. For every i, let $b_i \in A_i$ be an element for which $b_i \neq a_{ii}$. This element exists since $|A_i| \geq 2$. By assumption the sequence $b : b_1, b_2, \ldots$ is different from s_i since $b_i \neq a_{ii}$ for all i. Thus b is an element in $A_1 \times A_2 \times A_3 \times \cdots$ that is not listed, a contradiction. $\square$

The argument presented above is called the **Cantor's diagonal argument** and is illustrated below:

$$\begin{aligned}
s_1 &: \boxed{a_{11}}, a_{12}, a_{13}, a_{14}, \ldots \\
s_2 &: a_{21}, \boxed{a_{22}}, a_{23}, a_{24}, \ldots \\
s_3 &: a_{31}, a_{32}, \boxed{a_{33}}, a_{34}, \ldots \\
s_4 &: a_{41}, a_{42}, a_{43}, \boxed{a_{44}}, \ldots \\
&\vdots
\end{aligned}$$

A similar argument can be used to show $\mathbb{R}$ is uncountable.

Theorem 1.5. *If* X *is a countable set, then the set consisting of all finite subsets of* X *is also countable.*

Video Proof

1.3 Classical Examples

Example 1.1. *Prove that if* X *is an infinite set, then the power set* $\mathscr{P}(X)$ *is uncountable.*

Scratch: In order to show $\mathscr{P}(X)$ is uncountable we need to show there is no bijection between the power set and $\mathbb{N}$. We clearly see that there is an injection from X to its power set by mapping any element x to the singleton $\{x\}$. So, what we really need to show is that if $f : X \to \mathscr{P}(X)$ is a function, then it cannot be onto. So, we should look for a subset of X that cannot be in the image of f. In other words we are looking for a subset S of X that from the fact that $f(x) = S$ for some $x \in X$ we deduce a contradiction. Let's see how the element x and the set S should be related. Should we assume $x \in S$ or $x \notin S$? In fact both of these must lead to a contradiction. So, how can we define S in a way that $x \in S$ and $x \notin S$ yield contradictions? From $x \in S$ we want to deduce that $x \notin S$ and from $x \notin S$, we want to conclude that $x \in S$. This along with the fact that $S = f(x)$ tells us what we need S to be. We need to choose S to be the set of all

x's for which $x \notin f(x)$. This can be written as follows:

Solution.(Video Solution) Since X is infinite, it is enough to show $|X| < |\mathscr{P}(X)|$, which is equivalent to showing no function $f : X \to \mathscr{P}(X)$ is onto. Suppose on the contrary that there is an onto function $f : X \to \mathscr{P}(X)$. Let $S = \{x \in X \mid x \notin f(x)\}$. Since f is onto, there is $a \in X$ for which $f(a) = S$. There are two possibilities:
Case I: $a \in S$, which by definition of S it means $a \notin f(a) = S$, which is a contradiction.
Case II: $a \notin S$, which by definition of S it means $a \in f(a) = S$, which is also a contradiction.
Therefore, f cannot be onto, as desired. $\square$

Example 1.2. *Let $k \geq 2$ be an integer. Every positive integer n has a unique base k representation of the form $n = a_0 + a_1 k + \cdots + a_r k^r$, where $a_j \in \{0, 1, \ldots, k-1\}$, and $a_r \neq 0$.*

Scratch: We note that a_0 is the remainder when n is divided by k. So, we can find a_0 and then reduce n by considering the quotient $\frac{n-a_0}{k}$. Thus, induction is a good tool to use in solving this problem.

Solution. We will prove the statement of the problem by induction on n.
Basis step: If $n < k$, then $a_0 = n$ and $r = 0$ work. Furthermore, since $n < k$, in any representation of n of the given form we must have $r = 0$ and thus $a_0 = n$, which proves the uniqueness as well.
Inductive step: Now, suppose $n \geq k$. By division algorithm there are integers a_0 and q for which $n = a_0 + kq$ with $0 \leq a_0 < k$. Note that $q > 0$, since $n \geq k$. Also we see that $q < kq \leq n$. By applying the inductive hypothesis to q we conclude that there are integers $a_1, \ldots, a_r \in \{0, \ldots, k-1\}$ with $a_r \neq 0$ for which $q = a_1 + a_2 k + \cdots + a_r k^{r-1}$. This implies $n = a_0 + a_1 k + \cdots + a_r k^r$, which proves the existence part of the claim.
Now, suppose $n = \sum_{i=0}^{r} a_i k^i = \sum_{i=0}^{s} b_i k^i$ are two representations of n in the given form. Since the two sides are congruent to a_0 and $b_0 \mod k$, we must have $a_0 = b_0$. This implies $q = \sum_{i=1}^{r} a_i k^{i-1} = \sum_{i=1}^{s} b_i k^{i-1}$. By inductive hypothesis $a_i = b_i$ for all i, as desired. $\square$

Example 1.3. *Prove that*

(a) there are infinitely many primes.

(b) there are infinitely many primes of the form $4k-1$.

Scratch: The first part is a well-known theorem due to Euclid. To show there are infinitely many of an object we often use proof by contradiction. So, we assume $p_1, p_2, \ldots, p_n$ are all primes and then we produce a new prime by looking at the integer $p_1 \cdots p_n + 1$. Even though this integer may not be a prime, it certainly has a *new* prime factor.

For the second part we could use a similar strategy, however even though $p_1 \cdots p_n + 1$ has a new prime factor, it is possible that this prime factor is not of the form $4k-1$. We know every prime is either 2 or of the form $4k \pm 1$. We could eliminate the possibility of 2 by considering an integer that is odd: $2p_1 \cdots p_n + 1$ is a good one. Is it possible that all factors of this number are of the form $4k+1$? In that case the number must itself be also $1 \mod 4$, so let's change this number to get something that is not $1 \mod 4$. We could consider $4p_1 \cdots p_n - 1$. Putting these together we obtain the following solution:

Solution.(Video Solution) We will use proof by contradiction for both parts.

(a) On the contrary assume $p_1 = 2, p_2 = 3, \ldots, p_n$ is the list of all primes. Note that the integer $p_1 p_2 \cdots p_n + 1$ is more than one and thus has a prime divisor. Since the list $p_1 = 2, p_2 = 3, \ldots, p_n$ consists of all primes, p_i must divide $p_1 \cdots p_n + 1$ for some i, however p_i divides $p_1 \cdots p_n$, which means p_i must divide their difference of 1, which is impossible. This contradiction proves the claim.

(b) On the contrary assume $p_1 = 3, p_2 = 7, \ldots, p_n$ is the list of all primes of the form $4k-1$. Note that the integer $d = 4p_1 p_2 \cdots p_n - 1$ is $-1 \mod p_i$ and thus not divisible by any of the p_i's. In addition to that d is odd. Therefore, all of its prime factors must be $1 \mod 4$. This means $d \equiv 1 \mod 4$, which is a contradiction. □

Example 1.4. *Find a formula for each of the following:*

$$\sum_{j=1}^{n} j, \quad \sum_{j=1}^{n} j^2, \quad \sum_{j=1}^{n} j^3.$$

Sketch: The idea that works for all three series is to use telescoping sums. For example $(j+1)^2 - j^2 = 2j+1$ gives us the first sum and produces a telescoping sum. Similarly $(j+1)^3 - j^3 = 3j^2 + 3j + 1$ produces the second sum.

Solution. We will denote the sums by S_1, S_2, S_3, respectively. We obtain the following:

$$\sum_{j=1}^{n}((j+1)^2 - j^2) = \sum_{j=1}^{n} 2j+1 = 2S_1 + n \Rightarrow (n+1)^2 - 1 = 2S_1 + n \Rightarrow n^2 + 2n = 2S_1 + n \Rightarrow S_1 = \frac{n(n+1)}{2}.$$

Similarly we have

$$\sum_{j=1}^{n}((j+1)^3 - j^3) = \sum_{j=1}^{n}(3j^2 + 3j + 1) = 3S_2 + 3S_1 + n.$$

Therefore,

$$(n+1)^3 - 1 = 3S_2 + 3\frac{n(n+1)}{2} + n \Rightarrow S_2 = \frac{n(n+1)(2n+1)}{6}.$$

Using the equality $(j+1)^4 = j^4 + 4j^3 + 6j^2 + 4j + 1$ we obtain the following:

$$\sum_{j=1}^{n}((j+1)^4 - j^4) = \sum_{j=1}^{n}(4j^3 + 6j^2 + 4j + 1) = 4S_3 + 6S_2 + 4S_1 + n.$$

Therefore,

$$(n+1)^4 - 1 = 4S_3 + 6\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} + n \Rightarrow S_3 = \left(\frac{n(n+1)}{2}\right)^2.$$

□

Example 1.5. *Prove that e is irrational.*

Solution.(Video Solution) On the contrary assume $e = \frac{a}{b}$ is rational, where a,b are positive integers.

We will use the Taylor series $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$. Substituting $x = 1$ we obtain

$$\frac{a}{b} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Multiplying both sides by $b!$ we obtain

$$a(b-1)! = b! + \frac{b!}{1!} + \cdots + \frac{b!}{b!} + \frac{b!}{(b+1)!} + \frac{b!}{(b+2)!} + \cdots.$$

Since $a(b-1)!$ and $\frac{b!}{k!}$ are both integers for $k = 1, \ldots, b$, we must have

$$\frac{b!}{(b+1)!} + \frac{b!}{(b+2)!} + \cdots \in \mathbb{Z}$$

This can be written as

$$0 < \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \cdots < \frac{1}{b+1} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \cdots = \frac{1}{b} \leq 1$$

This is a contradiction, since there are no integers in the interval $(0,1)$. □

1.4 Further Examples

Example 1.6. *Suppose a,b,c are positive integers for which $a+b+c$ divides abc. Prove that $a+b+c$ is composite.*

Scratch: We notice that to show $a+b+c$ is composite we need to show it is not prime, however if it were prime, dividing the product abc would mean dividing at least one of a,b or c. This yields the following solution:

Solution. We will prove the claim by contradiction. Suppose $a+b+c$ is not composite. Since it is more than 1, it must be a prime. By assumption $a+b+c$ divides abc. Thus, it must divide a,b, or c. WLOG assume $a+b+c$ divides a. We know $a+b+c > a > 0$. This contradicts the fact that $a+b+c$ divides a, which shows that $a+b+c$ must be composite. □

Example 1.7 (Putnam 1996, A4)**.** *Let S be the set of ordered triples (a,b,c) of distinct elements of a finite set A. Suppose that*

1. *$(a,b,c) \in S$ if and only if $(b,c,a) \in S$;*
2. *$(a,b,c) \in S$ if and only if $(c,b,a) \notin S$;*
3. *(a,b,c) and (c,d,a) are both in S if and only if (b,c,d) and (d,a,b) are both in S.*

Prove that there exists a one-to-one function g from A to $\mathbb{R}$ such that $g(a) < g(b) < g(c)$ implies $(a,b,c) \in S$.

Scratch: Here are my initial thoughts:

- The first condition is cyclic. In other words if $(a,b,c) \in S$, then all of its cyclic permutations (c,a,b), and (b,c,a) are also in S. The second condition tells us only those three permutations of a,b,c are in S and not the other three!
- If we were to think of these as permutations, the third condition tells us that the composition of permutations (a,b,c) and (c,d,a) which are (b,c,d) and (d,a,b) are in S. At this point we may not be sure how that could help but it *may* be helpful to keep that in mind.
- Can we solve the problem when A has 3, 4, or 5 elements?

We will start placing (a,b,c) in counterclockwise order around a circle if $(a,b,c) \in S$. We would like to define g in such a way that $g(a) < g(b) < g(c)$. Now for a new element d, by assumption we want to know where we need to place this new element. For example if $(a,d,b) \in S$, then we just define $g(d)$ some number between $g(a)$ and $g(b)$. In that case we need to make sure the condition is satisfied. We know $g(a) < g(d) < g(b) < g(c)$. We only need to show (a,d,c) and (d,b,c) are in S. This follows from the fact that (b,c,a) and (a,d,b) are in S. Similar reasoning works when $(b,d,c) \in S$. If neither of these are in S, then it makes sense to define $g(d)$ something either more than $g(c)$ or less than $g(a)$, both of which would work, but of course need to be checked. This logic seems to work for any number of elements in the set A. To simplify the proof we will use induction on the size of A.

Solution. First note that by the first property if $(a,b,c) \in S$, then (c,a,b), and (b,c,a) are both in S. We will prove the claim by induction on $|A|$.

Basis step: Suppose $|A| = 3$, and $A = \{a,b,c\}$, and assume $(a,b,c) \in S$. We will define $g(a) = 1, g(b) = 2, g(c) = 3$. The result is thus clear.

Inductive step: Suppose $|A| = n+1 > 3$ and let S be a set with the given conditions. Fix an element $a_0 \in S$ and let T be the set of all $(a,b,c) \in S$ for which neither of the elements a,b,c is a_0. Clearly T also satisfies all of the given conditions. By inductive hypothesis there is a function $g : A\backslash\{a_0\} \to \mathbb{R}$ for which whenever $g(a) < g(b) < g(c)$, we have $(a,b,c) \in T$ and thus $(a,b,c) \in S$. Suppose $A = \{a_0, a_1, \ldots, a_n\}$ is such a way that $g(a_1) < g(a_2) < \cdots < g(a_n)$. We will consider two cases:

Case I: There is some $j > 0$ for which $(a_j, a_0, a_{j+1}) \in S$. Define $f : A \to \mathbb{R}$ by $f(a_0) = \dfrac{g(a_j) + g(a_{j+1})}{2}$ and $f(a_i) = g(a_i)$ for all $i > 0$. Suppose $f(a_i) < f(a_k) < f(a_\ell)$. If none of i,k or ℓ is zero, then by inductive hypothesis $(a_i, a_k, a_\ell) \in T \subseteq S$. If $f(a_0) < f(a_i) < f(a_k)$, then $j < i$, since $f(a_j) < f(a_0) < f(a_i)$. We know $(a_j, a_{j+1}, a_k) \in S$ and $(a_j, a_0, a_{j+1}) \in S$. By the first property $(a_{j+1}, a_k, a_j) \in S$. By the second property $(a_0, a_{j+1}, a_k) \in S$ which also means $(a_k, a_0, a_{j+1}) \in S$. Since $j+1 \leq i$, either $j+1 = i$ or $(a_{j+1}, a_i, a_k) \in S$, both of which imply $(a_i, a_k, a_0) \in S$, which implies $(a_0, a_i, a_k) \in S$, as desired.

The case when $f(a_i) < f(a_k) < f(a_0)$ is similar.

If $f(a_i) < g(a_0) < f(a_k)$, then $i \leq j < k$. If $i = j$, then $(a_i, a_0, a_{j+1}) \in S$. If $i < j$, then since (a_i, a_j, a_{j+1}) and (a_j, a_0, a_{j+1}) are in S, by property 1 we obtain $(a_{j+1}, a_i, a_j) \in S$ and thus $(a_0, a_{j+1}, a_i) \in S$. This implies $(a_i, a_0, a_{j+1}) \in$

S. If $j+1=k$, then $(a_i,a_0,a_k)\in S$, as desired. Otherwise $j+1<k$ and thus $(a_i,a_{j+1},a_k)\in S$ which implies $(a_{j+1},a_k,a_i)\in S$. Combining this with the fact that $(a_i,a_0,a_{j+1})\in S$, using the second property we obtain $(a_k,a_i,a_0)\in S$. Applying the first property we obtain $(a_i,a_0,a_k)\in S$, as desired.

Case II: For all $i>0$, $(a_i,a_0,a_{i+1})\notin S$. Define $f:A\to\mathbb{R}$ by $f(a_0)=g(a_1)-1$, and $f(a_j)=g(a_j)$ for all $j>0$. Similar to above assume $f(a_j)<f(a_k)<f(a_0)$. We know $(a_j,a_0,a_{j+1})\notin S$, and thus the first property implies $(a_j,a_{j+1},a_0)\in S$. If $j+1=k$, then we are done. Otherwise, note that $(a_{j+1},a_{j+2},a_0)\in S$, which can be combined with $(a_0,a_j,a_{j+1})\in S$, to obtain $(a_{j+2},a_0,a_j)\in S$, or $(a_0,a_j,a_{j+2})\in S$. If $k=j+2$, then we are done. Repeating this argument we obtain $(a_0,a_j,a_k)\in S$, as desired. $\square$

Here is a somewhat simpler solution which is perhaps more difficult to obtain.

Second Solution. Note that by the first property, if $(a,b,c)\in S$, then both of its cyclic permutations (c,a,b) and (b,c,a) are in S.

Let a_0 be an element in A. We will define an order on A by setting $a_0\preceq x$ for all $x\in A$, and $x\preceq y$ for $x,y\neq a_0$ whenever $(a_0,x,y)\in S$ or $x=y$. We will show $\preceq$ is a total order.
Reflexive: By definition $x\preceq x$ for all $x\in A$.
Anti symmetry: Suppose $x\preceq y$ and $y\preceq x$ but $x\neq y$. If neither x nor y is a_0, then (a_0,x,y) and (a_0,y,x) are both in S which contradicts the second property. If $x=a_0\neq y$, then since $y\preceq a_0$ we must have $(a_0,y,a_0)\in S$, which is not true. Therefore, in all cases we must have $x=y$.
Transitive: Suppose $x\preceq y$ and $y\preceq z$. If $x=a_0$, $x=y$ or $y=z$, then $x\preceq z$.
If $y=a_0$, then $y\preceq x$, and thus $x=y$, which implies $x\preceq z$.
If $z=a_0$, then since $y\preceq z$ and $z\preceq y$, we have $z=y$ and thus $x\preceq z$.
If $x=z$, then by definition $x\preceq z$.
Suppose x,y,z,a_0 are distinct. By assumption $(a_0,x,y),(a_0,y,z)$ are elements of S. By the first property $(y,z,a_0)\in S$, which by the second property implies $(z,a_0,x)\in S$, or $(a_0,x,z)\in S$. This means $x\preceq z$, as desired.
Comparability: We know $a_0\preceq x$ for all x and also $x\preceq x$ for all $x\in A$. Suppose x,y,a_0 are distinct. By the first property one of (a_0,x,y) or (a_0,y,x) must be in S. Therefore, $x\preceq y$ or $y\preceq x$, as desired.

Suppose $A=\{x_1,\dots,x_n\}$, where $x_1\preceq\cdots\preceq x_n$. Define $g:A\to\mathbb{R}$ by $g(x_i)=i$ for all i. Then, if $g(x_i)<g(x_j)<g(x_k)$, then $i<j<k$, and thus $x_i\preceq x_j\preceq x_k$. Therefore, by definition of $\preceq$, we have $(a_0,x_i,x_j)\in S$, and $(x_0,x_j,x_k)\in S$. By the first property $(x_j,x_k,x_0)\in S$. Thus, by the third property we have $(x_i,x_j,x_k)\in S$, as desired. $\square$

Example 1.8 (VTRMC 2017)**.** *Determine the number of real solutions to the equation* $\sqrt{2-x^2}=\sqrt[3]{3-x^3}$.

Solution.(Video Solution) We show the equation has no solutions.
On the contrary assume $x\in\mathbb{R}$ satisfies the given equation, ane let $y=\sqrt{2-x^2}=\sqrt[3]{3-x^3}$. This yields the following

system of equations:

$$\begin{cases} x^2+y^2=2 \\ x^3+y^3=3 \end{cases}$$

We will show this system has no solutions. Since $x^2+y^2=2$, we have $x^2, y^2 \leq 2$. Therefore, $|x|, |y| \leq \sqrt{2}$. Thus, $x^3 \leq \sqrt{2}x^2$, and $x^3 \leq \sqrt{2}x^2$. Adding up the two equations we obtain the following:

$$x^3+y^3 \leq \sqrt{2}(x^2+y^2) = 2\sqrt{2} < 3.$$

This contradiction shows the system above does not have a solution. □

Example 1.9. *Each term in the sequence* $1,0,1,0,1,0,\ldots$*, starting with the seventh is the sum of the previous 6 terms mod 10. Prove that* $0,1,0,1,0,1$ *never occurs in this sequence.*

Scratch: Let's denote the n-th term of the given sequence by x_n. The first thing I tried was to list the first few terms and see if there is a pattern. If we can find a pattern then we should be done. Listing the first thirty terms we cannot find a repeatition, but is there one?

$$1,0,1,0,1,0,3,5,0,9,8,5,0,7,9,8,7,6,7,4,1,3,8,9,2,7,0,9,5,2,\ldots$$

Since each term is between 0 and 9 there will eventually be a repetition. In other words if we look at 10^6+1 tuples of the form $(x_n, x_{n+1}, \ldots, x_{n+5})$, we will be able to find a repetition, but let's not do that. I also note that since we can backtrack, i.e. $x_{n-1} = x_{n+5} - x_{n+4} - x_{n+3} - \cdots - x_n$, we know the pattern starts from the first term. I tried the same thing with the sequence $0,1,0,1,0,1$ to see if it is easier to find a pattern in that sequence, but that did not help either. Next, I thought of breaking down the sequence mod 10 to two sequences mod 2 and mod 5. That would hopefully make the job of finding a pattern easier. Taking the sequence mod 2 we get the following:

$$1,0,1,0,1,0,1,1,0,1,0,1,0,1,\ldots$$

This sequence does contain six consecutive terms $0,1,0,1,0,1$, so perhaps we can show the sequence modulo 5 does not. Taking the sequence mod 5 did not help, I was unable to find a repetition after writing down 30 terms.

$$x_n \mod 5: 1,0,1,0,1,0,3,0,0,4,3,0,0,2,4,3,2,1,2,4,1,3,3,4,2,2,0,4,0,2,\ldots$$

In fact a similar calculation shows we might have to write around 5^6 terms in order to see a pattern. That does not seem like a good idea! At this point I realized this idea does not work.

For simplicity, let's focus on the sequence mod 5 from now on. We will look for an invariant. What it means is to find a property that all six consecutive terms of this sequence share but the sequence $0,1,0,1,0,1$ does not share that property. My initial thought was to exploit the fact that the alternating sums are different. In other words of

$$1-0+1-0+1-0=3 \neq -3 = 0-1+0-1+0-1.$$

But after evaluating the alternating sum for sequences, I realized we can in fact obtain -3. So this is not a good choice of an invariant, even if it were one. Since this did not work, I tried to find a something similar, but this time I tried to create a linear invariant. In other words, would like to associate something of the following form to each six consecutive terms:

$$I(y_1,\dots,y_6) = a_1y_1 + a_2y_2 + \cdots + a_6y_6.$$

We want this to be an invariant, which means we would need to have

$$I(y_1,y_2,\dots,y_6) = I(y_2,\dots,y_6,y_1+\cdots+y_6).$$

This gives us the following:

$$a_1y_1 + \cdots + a_6y_6 = a_1y_2 + \cdots + a_5y_6 + a_6(y_1+\cdots+y_6).$$

Setting the coefficients equal we obtain the following system:

$$\begin{cases} a_1 = a_6 \\ a_2 = a_1 + a_6 \Rightarrow a_2 = 2a_6 \\ a_3 = a_2 + a_6 \Rightarrow a_3 = 3a_6 \\ a_4 = a_3 + a_6 \Rightarrow a_4 = 4a_6 \\ a_5 = a_4 + a_6 \Rightarrow a_5 = 5a_6 = 0 \\ a_6 = a_5 + a_6 \Rightarrow a_5 = 0 \end{cases}$$

Setting $a_6 = 1$, we can see that $I(y_1,\dots,y_6) = y_1 + 2y_2 + 3y_3 + 4y_4 + y_6$ is an invariant. Observe that

$$I(1,0,1,0,1,0) = 4 \neq I(0,1,0,1,0,1) = 2.$$

This yields the following solution:

Solution. (Video Solution)

□

Example 1.10 (IMO 2021, Shortlisted Problem, A1)**.** *Let n be an integer, and let A be a subset of $\{0,1,2,3,...,5^n\}$ consisting of $4n+2$ numbers. Prove that there exist $a,b,c \in A$ such that $a < b < c$ and $c + 2a > 3b$.*

Scratch: Here are a few ideas that come to mind:

- Induction might help.
- We can make the given inequality easier to understand if we re-write it as: $c - b > 2(b-a)$.
- The inequality $c - b > 2(b-a)$ only depends on the gaps between a,b,c, so the problem would not change if we were to consider A as a subset of integers between k and $5^n + k$.
- We may assume this inequality fails for every $a < b < c$ to get limitations on the elements of A and then obtain a contradiction.

We will assume the statement is true for n and try to prove it for $n+1$. If $x_{4n+2}-x_1 \leq 5^n$, then we can apply the inductive hypothesis and obtain the result. Otherwise, $x_{4n+2}-x_1 > 5^n$. Using proof by contradiction, we know $x_{4n+3}-x_{4n+2} \leq 2(x_{4n+2}-x_1)$, but this does not give us anything meaningful. At this point, I realized we need to start from the larger elements to get meaningful inequalities. This led me to the following solution:

Solution.(Video Solution) The inequality $c+2a > 3b$ is equivalent to $c-b > 2(b-a)$. Since only the gaps of a,b,c show up in this inequality, we can replace the set $\{0,1,2,3,\ldots,5^n\}$ by any subset of 5^n+1 consecutive integers.

We will use induction. For $n=1$, the set A is a subset of $\{0,1,\ldots,5\}$ with six elements. Therefore, all six integers must be in A. Thus, $0<1<4$ satisfy the given condition.

Assume the given statement is true for some positive integer n.

Now, assume $x_1 < \ldots < x_{4n+6}$ are $4(n+1)+2$ elements from the set $\{0,1,\ldots,5^{n+1}\}$. Assume on the contrary the given condition does not hold for these $4n+6$ integers. Therefore, for every $1 \leq i < j < k \leq 4n+6$ we have $x_k - x_j \leq 2(x_j - x_i)$ or equivalently,

$$x_j - x_i \geq \frac{x_k - x_j}{2} \qquad (*)$$

If $x_{4n+6}-x_5 \leq 5^n$, then by inductive hypothesis, we are done. Otherwise, $x_{4n+6}-x_5 > 5^n$. Therefore, by $(*)$,

$$x_5 - x_4 \geq \frac{x_{4n+6}-x_5}{2} > \frac{5^n}{2}.$$

Similarly,

$$x_4 - x_3 \geq \frac{x_{4n+6}-x_4}{2} = \frac{x_{4n+6}-x_5}{2} + \frac{x_5-x_4}{2} > \frac{5^n}{2} + \frac{5^n}{4} = \frac{3\cdot 5^n}{4}$$

$$x_3 - x_2 \geq \frac{x_{4n+6}-x_3}{2} = \frac{x_{4n+6}-x_4}{2} + \frac{x_4-x_3}{2} > \frac{3\cdot 5^n}{4} + \frac{3\cdot 5^n}{8} = \frac{9\cdot 5^n}{8}$$

$$x_2 - x_1 \geq \frac{x_{4n+6}-x_2}{2} = \frac{x_{4n+6}-x_3}{2} + \frac{x_3-x_2}{2} > \frac{9\cdot 5^n}{8} + \frac{9\cdot 5^n}{16} = \frac{27\cdot 5^n}{16}$$

Therefore,

$$\begin{aligned}
x_{4n+6}-x_1 &= (x_{4n+6}-x_5)+(x_5-x_4)+(x_4-x_3)+(x_3-x_2)+(x_2-x_1) \\
&> 5^n + \frac{5^n}{2} + \frac{3\cdot 5^n}{4} + \frac{9\cdot 5^n}{8} + \frac{27\cdot 5^n}{16} \\
&= 5^n\left(1+\frac{1}{2}+\frac{3}{4}+\frac{9}{8}+\frac{27}{16}\right) \\
&= 5^n\left(\frac{81}{16}\right) > 5^{n+1}
\end{aligned}$$

This contradiction completes the proof. □

Example 1.11 (Putnam 1989, B4)**.** *Does there exist an uncountable set of subsets of the positive integers such that any two distinct subsets have finite intersection?*

Solution.(Video Solution)

□

1.5 General Strategies

Things to keep in mind when solving problems, especially competition problems:

- Competition problems are solvable! They often have a tricky aspect but they *can* be solved. Somebody knows the solution. So, maintain a positive and hopeful attitude.
- Know your strength. I am personally not a fond of problems that have a lot of computation, but I am good at Number Theory, so that is where I would focus on. If I am left with problems 5 and 6 on a Putnam competition, with 5 being a multivariable analysis problem and 6 being a number theory problem, I would personally choose to work on number 6.
- Try small cases. This is perhaps the simplest and most under-appreciated approach in problem solving.
- Read all of the problems first. In most competitions (e.g. Putnam) problems are written in order of difficulty, but "difficulty" is subjective. If you haven't solved problem 2 on a Putnam exam you may want to try problem 3 if you think that is your strength, but you don't want to work on problem 6 without having done 1, 2 and 3.
- One problem at a time, but use your time efficiently. You don't want to use all of your time on one problem.

Things to keep in mind when writing a solution during a competition:

- Do NOT bury the lead! Write down your final claim at the very beginning.
- Reflect on your solution before writing it. Make sure your solution uses all of the assumptions. If you have not used one of the assumptions, that should raise a red flag. It is true that in some rare cases a given assumption is unnecessary, but that is rare. If you don't use an assumption that means either you have made a mistake or you have done something better than what they asked you to do. Which is more likely?!
- If your solution is messy and difficult to write. Spend a few minutes and see if there are things that you could do to make writing it easier and less time consuming.
- Don't make mathematical claims that you know to be false. If I were to grade your exam and saw a false claim or two I would likely stop reading the rest of your solution. Graders in these competitions have hundreds if not thousands of papers to grade. They are busy and they don't have the time to decipher your solution. Partial credits are rare.

Tips on what method of proof to use:

- The idea of induction is used when things can be reduced to smaller cases.
- When dealing with recursive sequences, using induction is often a good option.
- To prove negatives we often use proof by contradiction.
- To prove infiniteness we often assume finiteness and get a contradiction.
- To proved non-existence we often assume the existence and get a contradiction.
- In general to prove negatives we often use proof by contradiction.

1.6 Exercises

Exercise 1.1. Let $f_1(x) = (2x-1)/(x+1)$, define $f_{n+1}(x) = f_1(f_n(x))$ for every positive integer n. Determine constants A, B, C, and D so that $f_{1000}(x) = (Ax+B)/(Cx+D)$.

Exercise 1.2 (VTRMC 1980, modified)**.** Let S be the set of all ordered pairs of integers (m,n) satisfying $m > 0$ and $n < 0$. Let $\langle$ be a partial ordering on S defined by the statement: $(m,n)\langle(m',n')$ if and only if $m < m'$ and $n < n'$. An example is $(5,-10)\langle(8,-2)$. Now, let O be a completely ordered subset of S, i.e. if $(a,b) \in O$ and $(c,d) \in O$, then $(a,b)\langle(c,d)$ or $(c,d)\langle(a,b)$ or $(a,b) = (c,d)$. Also let $\mathscr{O}$ denote the collection of all such completely ordered sets.

(a) Determine whether every $O \in \mathscr{O}$ must be finite.

(b) Determine whether there is $n \in \mathbb{N}$ for which $|O| \leq n$ for all finite sets $O \in \mathscr{O}$.

(c) Determine whether $\mathscr{O}$ is countable or uncountable.

Exercise 1.3. Let $a < b$ be two real numbers. Prove that there is an integer n and a prime number p for which $a < \dfrac{n}{p} < b$.

Exercise 1.4. Show that we cannot place an uncountable number of letters of X on a given plane in such a way that no two X's have a point in common. Note that the letters of X may be of different sizes.

Exercise 1.5. Suppose $g_0, g_1, g_2, \ldots$ is a strictly increasing sequence of positive integers for which $g_0 = 1$, and g_{i-1} divides g_i for each $i \geq 1$. Let $g_i = d_i g_{i-1}$. Prove that each positive integer n can uniquely be written as

$$n = \sum_{i=0}^{r} a_i g_i, \qquad \text{where } a_i \in \{0, 1, \ldots, d_{i+1} - 1\}, \text{ for all } i, \text{ and } a_r \neq 0.$$

Exercise 1.6 (VTRMC 1982)**.** Let S be a set of positive integers and let E be the operation on the set of subsets of S defined by $EA = \{x \in A \mid x \text{ is even }\}$, where $A \subseteq S$. Let CA denote the complement of A in S . $ECEA$ will denote $E(C(EA))$ etc.

(a) Show that $ECECEA = EA$.

(b) Find the maximum number of distinct subsets of S that can be generated by applying the operations E and C to a subset A of S an arbitrary number of times in any order.

Exercise 1.7 (VTRMC 1982)**.** A box contains marbles, each of which is red, white or blue. The number of blue marbles is at least half the number of white marbles and at most one third the number of red marbles. The number which are white or blue is at least 55. Find the minimum possible number of red marbles.

Exercise 1.8 (VTRMC 1985)**.** (a) Find an expression for $3/5$ as a finite sum of distinct reciprocals of positive integers. (For example: $2/7 = 1/7 + 1/8 + 1/56$.)

(b) Prove that any positive rational number can be so expressed.

Exercise 1.9 (VTRMC 1986)**.** Express $\sinh 3x$ as a polynomial in $\sinh x$. As an example, the identity $\cos 2x = 2\cos^2 x - 1$ shows that $\cos 2x$ can be expressed as a polynomial in $\cos x$. (Recall that sinh denotes the hyperbolic sine defined by $\sinh x = (e^x - e^{-x})/2$.

Exercise 1.10 (VTRMC 1988)**.** For any set S of real numbers define a new set $f(S)$ by

$$f(S) = \{x/3 \mid x \in S\} \cup \{(x+2)/3 \mid x \in S\}.$$

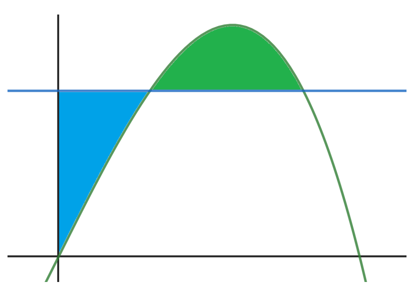
(a) Sketch, carefully, the set $f(f(f(I)))$, where I is the interval $[0,1]$.

(b) If T is a bounded set such that $f(T) = T$, determine, with proof, whether T can contain 1/2.

Exercise 1.11 (VTRMC 1989)**.** Three farmers sell chickens at a market. One has 10 chickens, another has 16, and the third has 26. Each farmer sells at least one, but not all, of his chickens before noon, all farmers selling at the same price per chicken. Later in the day each sells his remaining chickens, all again selling at the same reduced price. If each farmer received a total of \$35 from the sale of his chickens, what was the selling price before noon and the selling price after noon? (From ”Math Can Be Fun” by Ya Perelman.)

Exercise 1.12 (VTRMC 1990)**.** Three pasture fields have areas of $10/3, 10$ and 24 acres, respectively. The fields initially are covered with grass of the same thickness and new grass grows on each at the same rate per acre. If 12 cows eat the first field bare in 4 weeks and 21 cows eat the second field bare in 9 weeks, how many cows will eat the third field bare in 18 weeks? Assume that all cows eat at the same rate. (From Math Can be Fun by Ya Perelman.)

Exercise 1.13 (Putnam 1993, A1)**.** The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal.



Exercise 1.14 (Putnam 1993, B6)**.** Let S be a set of three, not necessarily distinct, positive integers. Show that one can transform S into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say x and y, where $x \le y$ and replace them with $2x$ and $y - x$.

Exercise 1.15 (VTRMC 1995)**.** Let $\tau = (1+\sqrt{5})/2$. Show that $\left[\tau^2 n\right] = [\tau[\tau n]+1]$ for every positive integer n. Here $[r]$ denotes the largest integer that is not larger than r.

Exercise 1.16 (Putnam 1995, B5)**.** A game starts with four heaps of beans, containing 3,4,5 and 6 beans. The two players move alternately. A move consists of taking **either**

a) one bean from a heap, provided at least two beans are left behind in that heap, **or**

b) a complete heap of two or three beans.

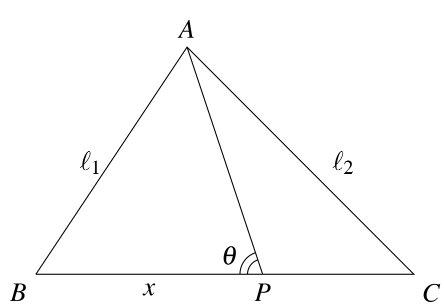
The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

Exercise 1.17 (VTRMC 1997)**.** The VTRC bus company serves cities in the USA. A subset S of the cities is called well-served if it has at least three cities and from every city A in S, one can take a nonstop VTRC bus to at least two different other cities B and C in S (though there is not necessarily a nonstop VTRC bus from B to A or from C to A). Suppose there is a well-served subset S. Prove that there is a well-served subset T such that for any two cities A, B in T , one can travel by VTRC bus from A to B, stopping only at cities in T.

Exercise 1.18 (Putnam 1997, A2)**.** Players $1, 2, 3, \ldots, n$ are seated around a table, and each has a single penny. Player 1 passes a penny to player 2, who then passes two pennies to player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

Exercise 1.19 (Putnam 1998, A2)**.** Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A+B$ depends only on the arc length, and not on the position, of s.

Exercise 1.20 (VTRMC 2000)**.** In the following diagram, $\ell_1 = \overline{AB}$, $\ell_2 = \overline{AC}$, $x = \overline{BP}$, and $\ell = \overline{BC}$, where $\overline{AB}$ indicates the length of AB. Prove that $\ell_2 - \ell_1 = \displaystyle\int_0^\ell \cos(\theta(x))\,\mathrm{d}x$, where $\theta(x) = \angle CPA$.



Exercise 1.21 (VTRMC 2001)**.** Find a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(f(x)) = \dfrac{3x+1}{x+3}$, for all positive real numbers x (here $\mathbb{R}^+$ denotes the positive (nonzero) real numbers).

Exercise 1.22 (Putnam 2001, B2)**.** Find all pairs of real numbers (x,y) satisfying the system of equations

$$\frac{1}{x}+\frac{1}{2y}=(x^2+3y^2)(3x^2+y^2)$$
$$\frac{1}{x}-\frac{1}{2y}=2(y^4-x^4).$$

Exercise 1.23 (VTRMC 2002)**.** Find rational numbers a,b,c,d,e such that

$$\sqrt{7+\sqrt{40}}=a+b\sqrt{2}+c\sqrt{5}+d\sqrt{7}+e\sqrt{10}$$

Exercise 1.24 (Putnam 2002, A2)**.** Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

Exercise 1.25 (Putnam 2002, B5)**.** A palindrome in base b is a positive integer whose base-b digits read the same backwards and forwards; for example, 2002 is a 4-digit palindrome in base 10. Note that 200 is not a palindrome in base 10, but it is the 3-digit palindrome 242 in base 9, and 404 in base 7. Prove that there is an integer which is a 3-digit palindrome in base b for at least 2002 different values of b.

Exercise 1.26 (VTRMC 2003)**.** It is known that $2\cos^3\frac{\pi}{7}-\cos^2\frac{\pi}{7}-\cos\frac{\pi}{7}$ is a rational number. Write this rational number in the form p/q, where p and q are integers with q positive.

Exercise 1.27 (Putnam 2002, A5)**.** Define a sequence by $a_0=1$, together with the rules $a_{2n+1}=a_n$ and $a_{2n+2}=a_n+a_{n+1}$ for each integer $n\geq 0$. Prove that every positive rational number appears in the set

$$\left\{\frac{a_{n-1}}{a_n}:n\geq 1\right\}=\left\{\frac{1}{1},\frac{1}{2},\frac{2}{1},\frac{1}{3},\frac{3}{2},\dots\right\}.$$

Exercise 1.28 (Putnam 2004, B6)**.** Let $\mathscr{A}$ be a non-empty set of positive integers, and let $N(x)$ denote the number of elements of $\mathscr{A}$ not exceeding x. Let $\mathscr{B}$ denote the set of positive integers b that can be written in the form $b=a-a'$ with $a\in\mathscr{A}$ and $a'\in\mathscr{A}$. Let $b_1<b_2<\cdots$ be the members of $\mathscr{B}$, listed in increasing order. Show that if the sequence $b_{i+1}-b_i$ is unbounded, then

$$\lim_{x\to\infty}N(x)/x=0.$$

Exercise 1.29 (VTRMC 2005, modified)**.** Prove that for every positive integer n there is a permutation $a_1,a_2,\dots,a_n$ of $1,2,\dots,n$ for which $j+a_j$ is a power of 2 for every $j=1,2,\dots,n$. (To illustrate, a permutation of $(1,2,3,4,5)$ such that $k+p(k)$ is a power of 2 for $k=1,2,\dots,5$ is clearly $(1,2,5,4,3)$, because $1+1=2, 2+2=4, 3+5=8, 4+4=8$, and $5+3=8$.)

Exercise 1.30 (Putnam 2005, A1)**.** Show that every positive integer is a sum of one or more numbers of the form 2^r3^s, where r and s are nonnegative integers and no summand divides another. (For example, 23 = 9 + 8 + 6.)

Exercise 1.31 (Putnam 2006, B1)**.** Show that the curve $x^3+3xy+y^3=1$ contains only one set of three distinct points, A, B, and C, which are vertices of an equilateral triangle, and find its area.

Exercise 1.32 (VTRMC 2009)**.** A walker and a jogger travel along the same straight line in the same direction. The walker walks at one meter per second, while the jogger runs at two meters per second. The jogger starts one meter in front of the walker. A dog starts with the walker, and then runs back and forth between the walker and the jogger with constant speed of three meters per second. Let $f(n)$ meters denote the total distance travelled by the dog when it has returned to the walker for the n-th time (so $f(0) = 0$). Find a formula for $f(n)$.

Exercise 1.33 (Putnam 2009, A1)**.** Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A)+f(B)+f(C)+f(D) = 0$. Does it follow that $f(P) = 0$ for all points P in the plane?

Exercise 1.34 (Putnam 2009, A4)**.** Let S be a set of rational numbers such that

(a) $0 \in S$;

(b) If $x \in S$ then $x+1 \in S$ and $x-1 \in S$; and

(c) If $x \in S$ and $x \notin \{0,1\}$, then $\frac{1}{x(x-1)} \in S$.

Must S contain all rational numbers?

Exercise 1.35 (Putnam 2009, B1)**.** Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$

Exercise 1.36 (Putnam 2011, B1)**.** Let h and k be positive integers. Prove that for every $\varepsilon > 0$, there are positive integers m and n such that

$$\varepsilon < |h\sqrt{m} - k\sqrt{n}| < 2\varepsilon.$$

Exercise 1.37 (Putnam 2012, A1)**.** Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1,12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

Exercise 1.38 (Putnam 2012, A4)**.** Let q and r be integers with $q > 0$, and let A and B be intervals on the real line. Let T be the set of all $b+mq$ where b and m are integers with b in B, and let S be the set of all integers a in A such that ra is in T. Show that if the product of the lengths of A and B is less than q, then S is either empty or all elements of S form an arithmetic progression.

Exercise 1.39. Prove that for every integer $n \geq 2$, at least one of the coefficients of the expansion of $(1+x+x^2)^n$ is even.

Exercise 1.40 (Putnam 2013, A2)**.** Let S be the set of all positive integers that are *not* perfect squares. For n in S, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minumum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3$, $2 \cdot 4$, $2 \cdot 5$, $2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5$, $2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.

Exercise 1.41 (Putnam 2013, B3)**.** Let $\mathscr{P}$ be a nonempty collection of subsets of $\{1, \ldots, n\}$ such that:

(i) if $S, S' \in \mathscr{P}$, then $S \cup S' \in \mathscr{P}$ and $S \cap S' \in \mathscr{P}$, and

(ii) if $S \in \mathscr{P}$ and $S \neq \emptyset$, then there is a subset $T \subset S$ such that $T \in \mathscr{P}$ and T contains exactly one fewer element than S.

Suppose that $f : \mathscr{P} \to \mathbb{R}$ is a function such that $f(\emptyset) = 0$ and

$$f(S \cup S') = f(S) + f(S') - f(S \cap S') \text{ for all } S, S' \in \mathscr{P}.$$

Must there exist real numbers $f_1, \ldots, f_n$ such that

$$f(S) = \sum_{i \in S} f_i$$

for every $S \in \mathscr{P}$?

Exercise 1.42 (Putnam 2014, B1)**.** A *base* 10 *over-expansion* of a positive integer N is an expression of the form

$$N = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_0 10^0$$

with $d_k \neq 0$ and $d_i \in \{0, 1, 2, \ldots, 10\}$ for all i. For instance, the integer $N = 10$ has two base 10 over-expansions: $10 = 10 \cdot 10^0$ and the usual base 10 expansion $10 = 1 \cdot 10^1 + 0 \cdot 10^0$. Which positive integers have a unique base 10 over-expansion?

Exercise 1.43 (Putnam 2014, B5)**.** In the 75th annual Putnam Games, participants compete at mathematical games. Patniss and Keeta play a game in which they take turns choosing an element from the group of invertible $n \times n$ matrices with entries in the field $\mathbb{Z}/p\mathbb{Z}$ of integers modulo p, where n is a fixed positive integer and p is a fixed prime number. The rules of the game are:

(1) A player cannot choose an element that has been chosen by either player on any previous turn.

(2) A player can only choose an element that commutes with all previously chosen elements.

(3) A player who cannot choose an element on his/her turn loses the game.

Patniss takes the first turn. Which player has a winning strategy? (Your answer may depend on n and p.)

Exercise 1.44 (VTRMC 2017)**.** Let $f(x,y) = \dfrac{x+y}{2}, g(x,y) = \sqrt{xy}, h(x,y) = \dfrac{2xy}{x+y}$, and let

$$S = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid a \neq b \text{ and } f(a,b), g(a,b), h(a,b) \in \mathbb{N}\}$$

where $\mathbb{N}$ denotes the positive integers. Find the minimum of f over S.

Exercise 1.45 (Putnam 2017, A1)**.** Let S be the smallest set of positive integers such that

(a) 2 is in S,

(b) n is in S whenever n^2 is in S, and

(c) $(n+5)^2$ is in S whenever n is in S.

Which positive integers are not in S?

(The set S is "smallest" in the sense that S is contained in any other such set.)

Exercise 1.46 (VTRMC 2019)**.** Let S be a subset of $\mathbb{R}$ with the property that for every $s \in S$, there exists $\varepsilon > 0$ such that $(s-\varepsilon, s+\varepsilon) \cap S = \{s\}$. Prove there exists a function $f : S \to \mathbb{N}$, the positive integers, such that for all $s, t \in S$, if $s \neq t$ then $f(s) \neq f(t)$.

Exercise 1.47. Prove that there are no four consecutive binomial coefficients that form an arithmetic sequence in this order:

$$\binom{n}{k}, \binom{n}{k+1}, \binom{n}{k+2}, \binom{n}{k+3}.$$

Chapter 2

Abstract Algebra and Functional Equations

2.1 Basics

Definition 2.1. A **group** is a set G along with a binary operation $\star$ that satisfies the following properties:

- $\forall a, b \in G, a \star b \in G$. [$G$ is closed under $\star$.]
- $\forall a, b, c \in G, (a \star b) \star c = a \star (b \star c)$. [$\star$ is associative.]
- $\exists e \in G$ such that $\forall a \in G, a \star e = e \star a = a$. [$e$ is called the identity element.]
- $\forall a \in G\ \exists b \in G$ for which $a \star b = b \star a = e$. [$b$ is called the inverse of a and is denoted by a^{-1}.]

When in addition to above $\forall a, b \in G$, $a \star b = b \star a$, then we say G is an **Abelian group**. The binary operation of Abelian groups is generally denoted by $+$, their identity is denoted by 0, and the additive inverse of a is denoted by $-a$.

You can find more about the definition of group in this YouTube video: https://youtu.be/65iaguYB0Jc.

Example 2.1. *Each of the following along with the given operation is a group:* $(\mathbb{Z}, +), (\mathbb{R}, +)$, $(\mathbb{R}\backslash\{0\}, \cdot)$, $(\mathbb{Q}\backslash\{0\}, \cdot)$, $(M_n(\mathbb{R}), +)$, $(\mathbb{Z}_n, +)$. *However,* $(\mathbb{Z}, \cdot)$ *and* $(\mathbb{R}, \cdot)$ *are not groups.*

Definition 2.2. A **ring** is a set R along with two binary operations $+$ and $\cdot$ that satisfy the following properties:

- $(R, +)$ is an Abelian group,
- $(R, \cdot)$ is closed and associative, and
- $\forall a, b, c \in R$, we have $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$. [Multiplication distributes over addition.]

A ring R is called **commutative** if $a \cdot b = b \cdot a$ for all $a, b \in R$.
A ring is called a **ring with unity** if $(R, \cdot)$ has a nonzero identity element. The multiplicative identity is denoted by 1.
If in addition $(R\backslash\{0\}, \cdot)$ forms an Abelian group, then we say R is a **field**.

Example 2.2. *Here are some examples of rings and fields:*

- $(2\mathbb{Z},+,\cdot)$ *is a commutative ring without a unity.* $(M_n(\mathbb{R}),+,\cdot)$ *is a non-commutative ring with a unity.*
- $(\mathbb{Z},+,\cdot)$ *and* $(\mathbb{R}[x],+,\cdot)$ *are both rings with unity. (*$\mathbb{R}[x]$ *is the set of all polynomials with real coefficients.)*
- $(\mathbb{R},+,\cdot)$*,* $(\mathbb{Q},+,\cdot)$*,* $(\mathbb{R}(x),+,\cdot)$ *are all fields. (*$\mathbb{R}(x)$ *is the set of all rational functions with real coefficients. i.e.* $p(x)/q(x)$ *with* $p(x),q(x)\in\mathbb{R}[x]$ *and* $q(x)\neq 0$*)*

Definition 2.3. A subset H of a group G is called a **subgroup** if H along with the operation of G forms a group.

Definition 2.4. For a subgroup H of a group G and an element $a\in G$, the set $aH=\{ah \mid h\in H\}$ is called a left **coset** of H in G. Right cosets are defined similarly. The number of left cosets of H in G is denoted by $[G:H]$.

Definition 2.5. Let a be a group element. The smallest positive integer n for which $a^n=e$ is called the **order** of a and is denoted by $|a|$. If no such positive integer exists then we say the order of a is infinity and we write $|a|=\infty$.

Definition 2.6. Two groups G and H are called **isomorphic** if there is a bijection $\phi: G\to H$ for which $\phi(ab)=\phi(a)\phi(b)$ for all $a,b\in G$. In that case we say ϕ is an **isomorphism**.

Definition 2.7. Two rings R and S are said to be **isomorphic** if there is a bijection $\phi: R\to S$ for which $\phi(a+b)=\phi(a)+\phi(b)$ and $\phi(ab)=\phi(a)\phi(b)$ for all $a,b\in R$.

Definition 2.8. Let a be an element of a group G and H be a subgroup of G. The **centralizer** of a and the **normalizer** of H are defined as $C(a)=\{x\in G \mid xa=ax\}$, $N(H)=\{x\in G \mid xHx^{-1}=H\}$.

2.2 Important Theorems

Theorem 2.1 (Subgroup Test)**.** *Let H be a nonempty subset of a group G. Then H is a subgroup of G if and only if for every $a,b\in H$, $ab^{-1}\in H$.*

Theorem 2.2 (Intersection of Subgroups)**.** *The intersection of every collection of subgroups of a group is itself a subgroup.*

Theorem 2.3 (Important subgroups)**.** *If a is an element of a group G and H is a subgroup of G, then $C(a)$ and $N(H)$ are subgroups of G.*

Theorem 2.4 (Lagrange's Theorem)**.** *If H is a subgroup of a group G, then left cosets of H in G partition G. This implies $|G|=|H|[G:H]$, which implies if G is finite, then $|H|$ divides $|G|$. Same is true for right cosets.*

Theorem 2.5 (Orders)**.** *Suppose a is an element of finite order in a group, and n is an integer. $a^n=e$ if and only if $|a|$ divides n.*

Theorem 2.6 (Cyclic Groups)**.** *Cyclic groups up to isomorphism are $\mathbb{Z}_n$ or $\mathbb{Z}$.*

Theorem 2.7 (Fundamental Theorem of Finite Abelian Groups)**.** *Any finite Abelian group is isomorphic to a unique Cartesian product of groups of form $\mathbb{Z}_{p^k}$, where p is a prime and k is a positive integer.*

Theorem 2.8 (Sylow Theorems)**.** *Let G be a finite group, p be a prime and m a positive integer for which p^m divides $|G|$, and p^{m+1} does not divide $|G|$. Then*

- *G has a subgroup of order p^m. Every such subgroup is called a Sylow p-subgroup.*
- *Each two subgroups H and K are conjugates, i.e. $H = aKa^{-1}$ for some $a \in G$.*
- *The number of Sylow p-subgroups of G is $1 \mod p$ and divides $|G|$.*
- *If L is a p-subgroup of G, then there is a Sylow p-subgroup of G containing L.*

Theorem 2.9 (Multiplicative Groups in Fields)**.** *Let F be a finite field. The multiplicative group $F^* = F\backslash\{0\}$ is cyclic. Furthermore, every finite subgroup of the multiplicative group of any field is cyclic.*

2.3 Classical Examples

Example 2.3 (Cauchy's Functional Equation)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Then $f(r) = rf(1)$ for every $r \in \mathbb{Q}$. Furthermore if f satisfies one of the following:*

(a) f is monotone.

(b) f is continuous at zero.

Then there is a constant c for which $f(x) = cx$ for every $x \in \mathbb{R}$.

Scratch. We start by substituting x and y by positive integer values.

$$f(2) = f(1) + f(1) = 2f(1); f(3) = f(2) + f(1) = 3f(1); f(4) = f(3) + f(1) = 4f(1).$$

At this point it is clear, we can prove $f(n) = nf(1)$ for all positive integers n. How about when n is negative or zero? We see $f(0) = f(0) + f(0)$, which implies $f(0) = 0$. How can we find $f(-1)$? We know $f(0)$ and $f(1)$. So we can get $f(-1)$ by substituting $x = -1$ and $y = 1$ to obtain $f(0) = f(-1) + f(1)$, which means $f(-1) = -1$. We now notice this can be done for *any* real number x. In other words, $f(0) = f(x) + f(-x)$, which means $f(-x) = -f(x)$. This means $f(n) = nf(1)$ for every $n \in \mathbb{Z}$. Now, let's try evaluating $f(1/2)$. This isn't difficult to evaluate noticing that $f(1/2 + 1/2) = f(1/2) + f(1/2)$, which means $2f(1/2) = f(1)$, or $f(1/2) = \frac{1}{2}f(1)$. Let's try to find $f(2/3)$. We know we can create $f(2)$ by adding three copies of $f(2/3)$. So that way, we can get $3f(2/3) = f(2) = 2f(1)$ or $f(2/3) = \frac{2}{3}f(1)$. We can now turn this into a complete proof of $f(r) = rf(1)$ for every $r \in \mathbb{Q}$. When f is monotone or continuous, we will use the fact that real numbers can be approximated by rationals. Here is a complete solution:

Solution.(Video Solution) First, we will prove by induction on n that $f(nx) = nf(x)$ for every integer $n \geq 0$ and every $x \in \mathbb{R}$.

Basis step. $f(0) = f(0+0) = f(0) + f(0)$, and thus $f(0) = 0$. Therefore, $f(0x) = 0f(x)$.

Inductive step. Suppose $f(nx) = nf(x)$ for some integer $n \geq 0$ and all $x \in \mathbb{R}$. We have $f((n+1)x) = f(nx) + f(x) =$

$nf(x)+f(x)=(n+1)f(x)$, as desired.

Now, note that $f(0)=f(-x+x)=f(-x)+f(x)$, and hence $f(-x)=-f(x)$. Therefore, $f(nx)=nf(x)$ for every $n\in\mathbb{Z}$ and $x\in\mathbb{R}$.

Next, let $r=n/m$ be a rational number with $n,m\in\mathbb{Z}$. By what we showed above $f(mr)=mf(r)$, however, $f(mr)=f(n)=nf(1)$. This implies $mf(r)=nf(1)$, which implies $f(r)=rf(1)$, as desired.

(a) Assume f is increasing. Let $x\in\mathbb{R}$, and let r_n,s_n be two sequences of rationals that converge to x and that $r_n<x<s_n$ for all n. Since f is increasing we have $f(r_n)\le f(x)\le f(s_n)$. Applying $f(r)=rf(1)$ proved above, we obtain the following:

$$r_nf(1)\le f(x)\le s_nf(1)\Rightarrow \lim_{n\to\infty} r_nf(1)\le f(x)\le \lim_{n\to\infty} s_nf(1)\Rightarrow xf(1)\le f(x)\le xf(1)\Rightarrow f(x)=xf(1).$$

The proof for when f is decreasing is similar.

(b) Suppose f is continuous at zero, and let $x\in\mathbb{R}$. Suppose r_n is a sequence of rationals that converges to x. Since $r_n\to x$, we have $x-r_n\to 0$. On the other hand, since f is continuous, $f(x-r_n)$ converges to $f(0)=0$. This yields the following:

$$\lim_{n\to\infty} f(x-r_n)=0\Rightarrow \lim_{n\to\infty} f(x)+f(-r_n)=0\Rightarrow \lim_{n\to\infty} f(x)-r_nf(1)=0\Rightarrow f(x)=xf(1).$$

This completes the proof. □

Example 2.4 (Cyclic Functions)**.** *Solve each of the following functional equations:*

(a) $f:\mathbb{R}\backslash\{0\}\to\mathbb{R}$ *for which* $f(x)+2f(1/x)=x^2+1$.

(b) $f:\mathbb{R}\backslash\{0,1\}\to\mathbb{R}$ *with* $f(x)+3f((x-1)/x)=x+1$.

Scratch. As usual, we start examining different functional values. Substituting $x=2$, we obtain $f(2)+2f(1/2)=5$, which means in order to find $f(2)$ we need to find $f(1/2)$. So, let's try $x=1/2$. This yields $f(1/2)+2f(2)=5/4$, which means in order to find $f(1/2)$ we need to know what $f(2)$ is! This appears to be circular, however we in fact obtained a system of equations that we can solve.

$$\begin{cases} f(2)+2f(1/2)=5 \\ f(1/2)+2f(2)=5/4 \end{cases}$$

After solving we find $f(2)=-5/6$ and $f(1/2)=35/12$. Let's try $x=3$. This yields $f(3)+2f(1/3)=10$. So, again, we cannot find $f(3)$ without knowing $f(1/3)$. Let's try $x=1/3$. This yields $f(1/3)+2f(3)=10/9$. Solving the system we can find $f(3)$ and $f(1/3)$. This can be replicated for all values of x.

For part (b) we will try something similar. Substituting $x = 2$ we obtain $f(2)+3f(1/2) = 3$. Substituting $x = 1/2$ we obtain $f(1/2)+3(-1) = 3/2$. Before giving up, let's try $x = -1$. This gives us $f(-1)+3f(2) = 0$. There, we have a system of three equations and three unknowns!

$$\begin{cases} f(2)+3f(1/2) = 3 \\ f(1/2)+3(-1) = 3/2 \\ f(-1)+3f(2) = 0 \end{cases}$$

Solution.(Video Solution) (a) Let $x \neq 0$ be a real number. $1/x$ is also not zero. Substituting x by $1/x$ we obtain the following system:

$$\begin{cases} f(x)+2f(1/x) = x^2+1 \\ f(1/x)+2f(x) = \dfrac{1}{x^2}+1 \end{cases}$$

Multiplying the second equation by 2 and subtracting the first equation we obtain $3f(x) = \dfrac{2}{x^2}+2-x^2-1$, which means $f(x) = \dfrac{2+x^2-3x^4}{3x^2}$. We also see that this function does satisfy the given functional equation. Therefore, this is the only solution to this part of the problem.

(b) Suppose $x \neq 0, 1$ is a real number. $(x-1)/x$ is also neither zero nor 1, because $(x-1)/x$ implies $x = 1$ and $(x-1)/x = 1$ implies $x-1 = x$, neither of which is possible. Substituting x by $(x-1)/x$ twice we obtain the following system:

$$\begin{cases} f(x)+3f\left(\dfrac{x-1}{x}\right) = x+1 \\ f\left(\dfrac{x-1}{x}\right)+3f\left(\dfrac{1}{1-x}\right) = \dfrac{x-1}{x}+1 = \dfrac{2x-1}{x} \\ f\left(\dfrac{1}{1-x}\right)+3f(x) = \dfrac{1}{1-x}+1 = \dfrac{2-x}{1-x} \end{cases}$$

If we multiply the second equation by -3, the third equation by 9 and add them to the first equation we obtain

$$f(x) = \frac{1}{28}\left(x+1-\frac{6x-3}{x}+\frac{18-9x}{1-x}\right)$$

After checking the following function $f(x)$ satisfies the original functional equation we conclude it is the only such function. □

2.4 Further Examples

Example 2.5 (VTRMC 1979)**.** *Let S be a set which is closed under the binary operation $\circ$, with the following properties:*

(i) there is an element $e \in S$ such that $a \circ e = e \circ a = a$, for each $a \in S$,

(ii) $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$*, for all* $a, b, c, d \in S$.

Prove or disprove:

(a) $\circ$ *is associative on S.*

(b) $\circ$ *is commutative on S.*

Scratch: For associativity we need to show $a \circ (b \circ c) = (a \circ b) \circ c$. To make one of the two parentheses in the second condition be a single element we need to replace one of the elements with e. For commutativity, we see that in the second condition c and b are swapped, so we will take advantage of that.

Solution. We will prove both parts.
(a) Substituting $b = e$ in the second condition, we obtain $(a \circ e) \circ (c \circ d) = (a \circ c) \circ (e \circ d)$. Since $e \circ d = d$ and $a \circ e = a$, we obtain $a \circ (c \circ d) = (a \circ c) \circ d$, as desired.
(b) Substituting $a = d = e$ in the second condition, we obtain $(e \circ b) \circ (c \circ e) = (e \circ c) \circ (b \circ e)$. Using the first condition we obtain $b \circ c = c \circ b$, as desired. □

Example 2.6 (Putnam 1992, A1)**.** *Prove that* $f(n) = 1 - n$ *is the only integer-valued function defined on the integers that satisfies the following conditions.*

(i) $f(f(n)) = n$*, for all integers n;*

(ii) $f(f(n+2)+2) = n$ *for all integers n;*

(iii) $f(0) = 1$.

Scratch: First note that since f is its own inverse, f is one-to-one, which means $f(n) = f(n+2)+2$. Using $f(0) = 1$, we can obtain the value of f at all even integers. Using (iii) in (i) we obtain $f(1) = 0$. This along with what we discussed above gives us all odd values.

Solution.(Video Solution) Applying f to both sides of (ii) we obtain $f(f(f(n+2)+2)) = f(n)$, which implies $f(n+2)+2 = f(n)$ $(*)$, by (i).
We will prove by induction on m that $f(2m) = 1-2m, f(-2m) = 1-(-2m), f(2m-1) = 1-(2m-1)$, and $f(-2m-1) = 1-(-2m-1)$ for all nonnegative integers m.
Basis step: For $m = 0$, we know $f(0) = 1 = 1-0$, $f(1) = f(f(0)) = 0 = 1-1$, and $f(-1) = f(-1+2)+2 = 0+2 = 1-(-1)$, as desired.
Inductive step: Suppose the statement above is valid for some nonnegative integer m, by $(*)$, we have $f(2(m+1)) = f(2m+2) = f(2m)-2 = 1-2m-2 = 1-(2m+2)$, $f(-2(m+1)) = f(-2m-2) = f(-2m-2+2)+2 = f(-2m)+2 = 1-(-2m)+2 = 1-(-2m-2)$, $f(2(m+1)-1) = f(2m-1)-2 = 1-(2m-1)-2 = 1-(2(m+1)-1)$, and $f(-2(m+1)-1) = f(-2m-1-2) = f(-2m-1-2+2)+2 = 1-(-2m-1)+2 = 1-(-2(m+1)-1)$, as desired.
Note that every even integer n is of form $n = \pm 2m$ for some nonnegative integer m, and every odd integer n is of form $n = \pm 2m-1$ for some nonnegative integer m. Therefore, $f(n) = 1-n$ for all integers n. □

Example 2.7 (IMC 2018, Problem 2)**.** *Does there exist a field such that its multiplicative group is isomorphic to its additive group?*

Scratch: We start with trying to construct an example and see if it is possible. So, start with an isomorphism $\phi : (F,+) \to (F^*,\cdot)$. We know ϕ is bijective, $\phi(x+y) = \phi(x)+\phi(y)$, and $\phi(xy) = \phi(x)\phi(y)$. Other properties of ϕ that may be helpful are $\phi(0) = 1$, and $\phi(-x) = x^{-1}$. So, this is pretty much like solving a functional equation. We start plugging in different values of x and y and see where this leads us to. Eventually we get the following solution:

Solution. The answer is no.

On the contrary suppose F is a field with an isomorphism $\phi : (F,+) \to (F^*,\cdot)$. Since ϕ is an isomorphism we must have $\phi(0) = 1$. Suppose $x \in F$ for which $\phi(x) = -1$. By properties of homomorphisms we have $\phi(2x) = (-1)^2 = 1$, and thus $2x = 0$. Thus, either $x = 0$ or $2 = 0$. If $x = 0$, then $\phi(0) = -1$ and thus $1 = -1$, which means both cases imply that $2 = 0$.
$1 = \phi(0) = \phi(2) = \phi(1)^2$, which implies $1-\phi(1)^2 = 0$. Since $1 = -1$, the latter implies $(1-\phi(1))^2 = 0$, or $\phi(1) = 1$. This shows $\phi(1) = \phi(0)$, which violated the fact that ϕ is a bijection. This contradiction shows there is no such isomorphism. $\square$

Example 2.8 (IMC 2018, Problem 4)**.** *Find all differentiable functions* $f : (0,\infty) \to \mathbb{R}$ *such that*

$$f(b)-f(a) = (b-a)f'\left(\sqrt{ab}\right) \quad \textit{for all} \quad a,b>0.$$

Scratch: Here are my first thoughts:

- This equation is linear. In other words any linear combination of solutions is also a solution.
- 1 and x and thus all linear functions $ax+b$ are solutions. Quadratics do not work and it looks like these are the only polynomials.
- I think the answer is $ax+b$. To prove that we need to prove $f''(x) = 0$, but we are not told f is twice differentiable. However we could see that f' is in terms of f and thus it is differentiable.

At this point I took the derivative of both sides to get $f'(x) = f'(\sqrt{ax}) + (x-a)f''(\sqrt{ax})\dfrac{\sqrt{a}}{2\sqrt{x}}$. I played with this some more but couldn't find a way to show $f''(x) = 0$. So, maybe my guess is wrong?! Could there be other solutions? I have already eliminated polynomials of higher degree. By checking x^n we realize that x^{-1} is in fact a solution! So, we have more solutions: $\dfrac{a}{x}+bx+c$. This allows me to make sure $f'(1) = f''(1) = 0$ by using the linearity, which gives us the following solution:

Solution. The answer is all functions of form $f(x) = \dfrac{a}{x}+bx+c$, where $a,b,c \in \mathbb{R}$ are constants.

Note that $1/x, x$, and 1 are all solutions and since both sides are linear $f(x) = \dfrac{a}{x}+bx+c$ is also a solution.

By substituting $2x$ and $x/2$ in the given equation we obtain $f(2x)-f(x/2)=3x/2f'(x)$ and thus $f'(x)=\dfrac{2(f(2x)-f(x/2))}{3x}$, which is differentiable. Thus, f is twice differentiable. Differentiating $f(x)-f(a)=(x-a)f'(\sqrt{ax})$ we obtain $f'(x)=f'(\sqrt{ax})+(x-a)f''(\sqrt{ax})\dfrac{\sqrt{a}}{2\sqrt{x}}$. Substituting $a=1/x$, we obtain $f'(x)=f'(1)+(x-1/x)\dfrac{f''(1)}{2x}=f'(1)+\dfrac{f''(1)}{2}-\dfrac{f''(1)}{2x^2}=b-\dfrac{c}{x^2}$, where b and c are constants. Integrating we obtain $f(x)=bx+\dfrac{c}{x}+d$, for some constant d, as desired. □

Example 2.9 (Putnam 2016, A3)**.** *Suppose that f is a function from $\mathbb{R}$ to $\mathbb{R}$ such that*

$$f(x)+f\left(1-\frac{1}{x}\right)=\arctan x$$

for all real $x\neq 0$. (As usual, $y=\arctan x$ means $-\pi/2<y<\pi/2$ and $\tan y=x$.) Find

$$\int_0^1 f(x)\,dx.$$

Scratch. Similar to Example 2.4, we realize $\varphi(x)=1-1/x$ is a cyclic function with $\varphi(\varphi(x))=\dfrac{1}{1-x}$, and $\varphi(\varphi(\varphi(x)))=x$. From there, we know we can find $f(x)$. Evaluating the resulting integral is not easy, however. We use the technique of swapping the limits of integration!

Solution. (Video Solution)

□

Example 2.10. *Find all functions $f:\mathbb{R}^2\to\mathbb{R}$ for which*

$$3f(x,y)+2f(y,x)=x+y^2,\ \textit{for all } x,y\in\mathbb{R}.$$

Solution. (Video Solution) $f(x,y)=\dfrac{3x+3y^2-2y-2x^2}{5}$ is the only function satisfying the given functional equation.

Substituting (x,y) by (y,x) we obtain the following:

$$3f(y,x)+2f(x,y)=y+x^2.$$

This yields a system of linear equations:

$$\begin{cases} 3f(x,y)+2f(y,x)=x+y^2 \\ 3f(y,x)+2f(x,y)=y+x^2 \end{cases}$$

Multipying the first equation by 3 and the second one by -2 and adding them up we obtain the following:

$$5f(x,y)=3x+3y^2-2y-2x^2\Rightarrow f(x,y)=\frac{3x+3y^2-2y-2x^2}{5}.$$

To finish the solution we need to show the above function does in fact satisfy the given functional equation:

$$3f(x,y)+2f(y,x)=\frac{9x+9y^2-6y-6x^2}{5}+\frac{6y+6x^2-4x-4y^2}{5}=x+y^2.$$

□

Example 2.11 (USAMO 2023, Problem 2). *Find all functions* $f:(0,\infty)\to(0,\infty)$ *for which,*

$$f(xy+f(x)) = xf(y)+2 \text{ for all } x,y\in(0,\infty)$$

Solution. (Video Solution) We claim that $f(x)=x+1$ is the only such function.

First, note that $f(x)=x+1$ satisfies the given functional equation:
$f(xy+f(x)) = f(xy+x+1) = xy+x+2 = x(y+1)+2 = xf(y)+2.$

Now, we will show this is the only function satisfying the given functional equation. For simplicity let $P(x,y)$ be the given assertion.

$P(f(x),y)$ yields $f(f(x)y+f(f(x))) = f(x)f(y)+2.$

$P(f(y),x)$ yields $f(f(y)x+f(f(y))) = f(y)f(x)+2$

Therefore,

$$f(f(x)y+f(f(x))) = f(f(y)x+f(f(y))) \qquad (*)$$

We will now show that f is one-to-one. Suppose $f(a)=f(b)$.

$P(a,b)$ yields $f(ab+f(a)) = af(b)+2.$
$P(b,a)$ yields $f(ba+f(b)) = bf(a)+2.$
Since $f(a)=f(b)$, $ab+f(a)=ba+f(b)$ and thus $af(b)+2=bf(a)+2$, and thus $a=b$ since $f(a)=f(b)\neq 0$.

By $(*)$ we conclude

$$f(x)y+f(f(x)) = f(y)x+f(f(y)) \qquad (**)$$

$P(1,y)$ yields $f(y+f(1)) = f(y)+2$. Therefore, $f(f(1)+1) = f(1)+2.$ □

Example 2.12 (IMO 2022, Problem 2). *Let* $\mathbb{R}^+$ *denote the set of positive real numbers. Find all functions* $f:\mathbb{R}^+\to\mathbb{R}^+$ *such that for each* $x\in\mathbb{R}^+$*, there is exactly one* $y\in\mathbb{R}^+$ *satisfying*

$$xf(y)+yf(x)\leq 2.$$

Solution.(Video Solution) Let's call the unique y satisfying $xf(y)+yf(x)\leq 2$ the twin of x. By symmetry the twin of y would be x. We claim that every $x\in\mathbb{R}^+$ is its own twin. Suppose x is not its own twin. This means $xf(x)+xf(x)>2$, i.e. $f(x)>1/x$. Suppose y is the twin of x. We have $f(x)>1/x$ and $f(y)>1/y$. By assumption we have the following:

$$2\geq xf(y)+yf(x) > \frac{x}{y}+\frac{y}{x}\geq 2,$$

where the last inequality is a consequence of the AM-GM Inequality. This contradiction show every $x \in \mathbb{R}^+$ is its own twin, or $xf(x) + xf(x) \le 2$. Thus $f(x) \le 1/x$ for all $x \in \mathbb{R}^+$. By uniqueness for every $z \ne x$ we have

$$xf(z) + zf(x) > 2 \Rightarrow zf(x) > 2 - xf(z) \ge 2 - \frac{x}{z}.$$

Now, allowing z to approach x we conclude that $xf(x) \ge 2 - 1 = 1$. Thus $f(x) \ge 1/x$. Therefore, $f(x) = 1/x$, as desired. □

Example 2.13 (IMC 2023, Problem 1)**.** *Find all functions $f : \mathbb{R} \to \mathbb{R}$ that have a continuous second derivative and for which the equality $f(7x+1) = 49f(x)$ holds for all $x \in \mathbb{R}$.*

Solution.(Video Solution)

□

Example 2.14 (Putnam 1989, B2)**.** *Let S be a non-empty set with a binary operation $*$ such that all of the following are satisfied:*

(a) $*$ is associative;

(b) $a*b = a*c$ implies $b = c$;

(c) $b*a = c*a$ implies $b = c$; and

(d) For each element $a \in S$, the set $\{a, a^2, a^3, a^n, \dots\}$ is finite.

Is S necessarily a group?

Note: *a^n is defined inductively by $a^1 = a$, and $a^{n+1} = a^n * a$ for every $n \ge 1$.*

Solution.(Video Solution)

□

Example 2.15 (IMO 2018, Shortlisted Problem, A1)**.** *Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ that satisfy $f(x^2 f(y)^2) = f(x)^2 f(y)$ for all $x, y \in \mathbb{Q}^+$. (Note: $\mathbb{Q}^+$ is the set of all positive rational numbers.)*

Solution.(Video Solution)

□

Example 2.16 (German National Olympiad, 2022, Problem 6)**.** *Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies all of the following:*

1. *$f(xy) = f(x)f(y)$ for every $x, y \in \mathbb{R}$,*
2. *$f(x+y) \le 2(f(x) + f(y))$ for every $x, y \in \mathbb{R}$, and*
3. *$f(2) = 4$.*

Prove that $f(3) \le 9$.

Solution.(Video Solution) □

2.5 General Strategies

To solve functional equations follow the steps below:

- Keep your work organized and keep all of the results that you obtain on a separate sheet of paper. You will end up referring to this sheet later, sometimes over and over.
- Start with evaluating the function at different values. Can you find $f(0)$? $f(1)$? $f(-1)$? $f(\text{integers})$? $f(\text{rationals})$? Keep in mind that at times finding those values is impossible or may be as difficult as solving the problem.
- Find some functions that satisfy the functional equation. You could try constant, linear, or quadratic functions and see if you can find any candidates.
- Can you prove f is odd or even?
- Can you prove f is decreasing or increasing?
- Can you prove f is one-to-one or onto?
- See if there are any cyclic functions involved.
- Can you prove the function satisfies the Cauchy Functional Equation?
- In every step come back to the list of results that you have obtained along with the initial functional equation and re-write them using the new information.
- To find polynomial solutions to functional equations you should compare the corresponding coefficients one by one.
- If you are stuck it may be because you are trying to prove something that is wrong! So, go back and look for more example.
- Eventually practice is the most important key to success! So, do lots of functional equation problems!

2.6 Exercises

Exercise 2.1 (Putnam 1968, B2)**.** A is a subset of a finite group G, and A contains more than one half of the elements of G. Prove that each element of G is the product of two, not necessarily distinct, elements of A.

Exercise 2.2 (VTRMC 1980)**.** Let $\star$ denote a binary operation on a non-empty set S with the property that $(w \star x) \star (y \star z) = w \star z$ for all $w, x, y, z \in S$. Show

(a) If $a \star b = c$, then $c \star c = c$.

(b) If $a \star b = c$, then $a \star x = c \star x$ for all $x \in S$.

Exercise 2.3 (VTRMC 1981)**.** Let A be non-zero square matrix with the property that $A^3 = 0$, where 0 is the zero matrix, but with A being otherwise arbitrary.

(a) Express $(I-A)^{-1}$ as a polynomial in A, where I is the identity matrix.

(b) Find a 3×3 matrix satisfying $B^2 \neq 0, B^3 = 0$.

Exercise 2.4 (VTRMC 1981)**.** Two elements A, B in a group G have the property $ABA^{-1}B = 1$, where 1 denotes the identity element in G.

(a) Show that $AB^2 = B^{-2}A$.

(b) Show that $AB^n = B^{-n}A$ for any integer n.

(c) Find u and v so that $\left(B^aA^b\right)\left(B^cA^d\right) = B^uA^v$.

Exercise 2.5 (VTRMC 1986)**.** A function f from the positive integers to the positive integers has the properties:

- $f(1) = 1$,
- $f(n) = 2$ if $n \geq 100$,
- $f(n) = f(n/2)$ if n is even and $n < 100$,
- $f(n) = f\left(n^2+7\right)$ if n is odd and $n > 1$

(a) Find all positive integers n for which the stated properties require that $f(n) = 1$.

(b) Find all positive integers n for which the stated properties do not determine $f(n)$.

Exercise 2.6 (VTRMC 1990)**.** Let f be defined on the natural numbers as follows: $f(1) = 1$ and for $n > 1$, $f(n) = f(f(n-1)) + f(n - f(n-1))$. Find, with proof, a simple explicit expression for $f(n)$ which is valid for all $n = 1, 2, \ldots$.

Exercise 2.7 (Putnam 1990, B3)**.** Let S be a set of 2×2 integer matrices whose entries a_{ij}

(1) are all squares of integers, and

(2) satisfy $a_{ij} \leq 200$.

Show that if S has more than 50387 $(= 15^4 - 15^2 - 15 + 2)$ elements, then it has two elements that commute.

Exercise 2.8 (Putnam 1990, B4)**.** Let G be a finite group of order n generated by a and b. Prove or disprove: there is a sequence

$$g_1, g_2, g_3, \cdots, g_{2n}$$

such that

(1) every element of G occurs exactly twice, and

(2) g_{i+1} equals g_ia or g_ib for $i = 1, 2, \ldots, 2n$. (Interpret g_{2n+1} as g_1.)

Exercise 2.9 (Putnam 1995, A1)**.** Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

Exercise 2.10 (Putnam 1997, A4)**.** Let G be a group with identity e and $\phi : G \to G$ a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1g_2g_3 = e = h_1h_2h_3$. Prove that there exists an element $a \in G$ such that $\psi(x) = a\phi(x)$ is a homomorphism (i.e. $\psi(xy) = \psi(x)\psi(y)$ for all $x,y \in G$).

Exercise 2.11 (Putnam 2000, B5)**.** Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \dots$ of positive integers as follows: the integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n. Show that there exist infinitely many integers N for which $S_N = S_0 \cup \{N + a : a \in S_0\}$.

Exercise 2.12 (VTRMC 2001)**.** Let G denote a set of invertible 2×2 matrices (matrices with complex numbers as entries and determinant nonzero) with the property that if a, b are in G, then so are ab and a^{-1}. Suppose there exists a function $f : G \to \mathbb{R}$ with the property that either $f(ga) > f(a)$ or $f(g^{-1}a) > f(a)$ for all a, g in G with $g \neq I$ (here I denotes the identity matrix, $\mathbb{R}$ denotes the real numbers, and the inequality signs are strict inequality). Prove that given finite nonempty subsets A, B of G, there is a matrix in G which can be written in exactly one way in the form xy with x in A and y in B.

Exercise 2.13 (Putnam 2001, A1)**.** Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

Exercise 2.14 (Putnam 2001, B5)**.** Let a and b be real numbers in the interval $(0, 1/2)$, and let g be a continuous real-valued function such that $g(g(x)) = ag(x) + bx$ for all real x. Prove that $g(x) = cx$ for some constant c.

Exercise 2.15 (Putnam 2002, B6)**.** Let p be a prime number. Prove that the determinant of the matrix

$$\begin{pmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{pmatrix}$$

is congruent modulo p to a product of polynomials of the form $ax + by + cz$, where a, b, c are integers. (We say two integer polynomials are congruent modulo p if corresponding coefficients are congruent modulo p.)

Exercise 2.16 (VTRMC 2003)**.** Let $f : [0,1] \to [0,1]$ be a continuous function such that $f(f(f(x))) = x$ for all $x \in [0,1]$. Prove that $f(x) = x$ for all $x \in [0,1]$.

Exercise 2.17 (Putnam 2007, A5)**.** Suppose that a finite group has exactly n elements of order p, where p is a prime. Prove that either $n = 0$ or p divides $n + 1$.

Exercise 2.18 (Putnam 2008, A1)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x,y) + f(y,z) + f(z,x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = g(x) - g(y)$ for all real numbers x and y.

Exercise 2.19 (Putnam 2008, A6)**.** Prove that there exists a constant $c > 0$ such that in every nontrivial finite group G there exists a sequence of length at most $c \log |G|$ with the property that each element of G equals the product of some subsequence. (The elements of G in the sequence are not required to be distinct. A *subsequence* of a sequence is obtained by selecting some of the terms, not necessarily consecutive, without reordering them; for example, $4,4,2$ is a subsequence of $2,4,6,4,2$, but $2,2,4$ is not.)

Exercise 2.20 (Putnam 2008, B5)**.** Find all continuously differentiable functions $f:\mathbb{R}\to\mathbb{R}$ such that for every rational number q, the number $f(q)$ is rational and has the same denominator as q. (The denominator of a rational number q is the unique positive integer b such that $q=a/b$ for some integer a with $\gcd(a,b)=1$.) (Note: gcd means greatest common divisor.)

Exercise 2.21 (Putnam 2009, A5)**.** Is there a finite abelian group G such that the product of the orders of all its elements is 2^{2009}?

Exercise 2.22 (Putnam 2010, A5)**.** Let G be a group, with operation $*$. Suppose that

(i) G is a subset of $\mathbb{R}^3$ (but $*$ need not be related to addition of vectors);

(ii) For each $\mathbf{a},\mathbf{b}\in G$, either $\mathbf{a}\times\mathbf{b}=\mathbf{a}*\mathbf{b}$ or $\mathbf{a}\times\mathbf{b}=0$ (or both), where $\times$ is the usual cross product in $\mathbb{R}^3$.

Prove that $\mathbf{a}\times\mathbf{b}=0$ for all $\mathbf{a},\mathbf{b}\in G$.

Exercise 2.23 (Putnam 2010, B4)**.** Find all pairs of polynomials $p(x)$ and $q(x)$ with real coefficients for which

$$p(x)q(x+1)-p(x+1)q(x)=1.$$

Exercise 2.24 (Putnam 2011, A6)**.** Let G be an abelian group with n elements, and let

$$\{g_1=e,g_2,\ldots,g_k\}\subsetneqq G$$

be a (not necessarily minimal) set of distinct generators of G. A special die, which randomly selects one of the elements $g_1,g_2,...,g_k$ with equal probability, is rolled m times and the selected elements are multiplied to produce an element $g\in G$. Prove that there exists a real number $b\in(0,1)$ such that

$$\lim_{m\to\infty}\frac{1}{b^{2m}}\sum_{x\in G}\left(\mathrm{Prob}(g=x)-\frac{1}{n}\right)^2$$

is positive and finite.

Exercise 2.25 (Putnam 2012, A2)**.** Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S, there exists z in S such that $x*z=y$. (This z may depend on x and y.) Show that if a,b,c are in S and $a*c=b*c$, then $a=b$.

Exercise 2.26 (Putnam 2012, A3)**.** Let $f:[-1,1]\to\mathbb{R}$ be a continuous function such that

(i) $f(x)=\frac{2-x^2}{2}f\left(\frac{x^2}{2-x^2}\right)$ for every x in $[-1,1]$,

(ii) $f(0)=1$, and

(iii) $\lim_{x\to1^-}\frac{f(x)}{\sqrt{1-x}}$ exists and is finite.

Prove that f is unique, and express $f(x)$ in closed form.

Exercise 2.27 (Putnam 2012, B1)**.** Let S be a class of functions from $[0,\infty)$ to $[0,\infty)$ that satisfies:

(i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x+1)$ are in S;

(ii) If $f(x)$ and $g(x)$ are in S, the functions $f(x)+g(x)$ and $f(g(x))$ are in S;

(iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x)-g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

Exercise 2.28 (Putnam 2012, B6)**.** Let p be an odd prime number such that $p \equiv 2 \mod 3$. Define a permutation π of the residue classes modulo p by $\pi(x) \equiv x^3 \mod p$. Show that π is an even permutation if and only if $p \equiv 3 \mod 4$.

Exercise 2.29 (Putnam 2013, A6)**.** Define a function $w : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ as follows. For $|a|,|b| \leq 2$, let $w(a,b)$ be as in the table shown; otherwise, let $w(a,b) = 0$.

$w(a,b)$		b				
		-2	-1	0	1	2
	-2	-1	-2	2	-2	-1
	-1	-2	4	-4	4	-2
a	0	2	-4	12	-4	2
	1	-2	4	-4	4	-2
	2	-1	-2	2	-2	-1

For every finite subset S of $\mathbb{Z}\times\mathbb{Z}$, define

$$A(S) = \sum_{(\mathbf{s},\mathbf{s}')\in S\times S} w(\mathbf{s}-\mathbf{s}').$$

Prove that if S is any finite nonempty subset of $\mathbb{Z}\times\mathbb{Z}$, then $A(S) > 0$.

For example, if $S = \{(0,1),(0,2),(2,0),(3,1)\}$, then the terms in $A(S)$ are

$$12,12,12,12,4,4,0,0,0,0,-1,-1,-2,-2,-4,-4.$$

Exercise 2.30 (VTRMC 2016)**.** Let q be a real number with $|q| \neq 1$ and let k be a positive integer. Define a Laurent polynomial $f_k(X)$ in the variable X, depending on q and k, by

$$f_k(X) = \prod_{i=0}^{k-1}(1-q^iX)(1-q^{i+1}X^{-1}).$$

(Here $\prod$ denotes product.) Show that the constant term of $f_k(X)$, i.e. the coefficient of X^0 in $f_k(X)$, is equal to

$$\frac{(1-q^{k+1})(1-q^{k+2})\cdots(1-q^{2k})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

Exercise 2.31 (Putnam 2016, A5)**.** Suppose that G is a finite group generated by the two elements g and h, where the order of g is odd. Show that every element of G can be written in the form

$$g^{m_1}h^{n_1}g^{m_2}h^{n_2}\cdots g^{m_r}h^{n_r}$$

with $1 \leq r \leq |G|$ and $m_1,n_1,m_2,n_2,\ldots,m_r,n_r \in \{-1,1\}$. (Here $|G|$ is the number of elements of G.)

Exercise 2.32 (Putnam 2016, B5)**.** Find all functions f from the interval $(1,\infty)$ to $(1,\infty)$ with the following property: if $x,y \in (1,\infty)$ and $x^2 \leq y \leq x^3$, then $(f(x))^2 \leq f(y) \leq (f(x))^3$.

Exercise 2.33. Let G be a group and m,n be two relatively prime integers. Suppose $(ab)^m = a^m b^m$ and $(ab)^n = a^n b^n$, for all $a,b \in G$. Prove that G is Abelian.

Exercise 2.34. Suppose G is a group, and m,n are relatively prime integers for which $a^n b^n = b^n a^n$, and $a^m b^m = b^m a^m$ for all $a,b \in G$. Prove that G is Abelian.

Exercise 2.35 (VTRMC 2018)**.** Prove that there is no function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = n+1$. Here $\mathbb{N}$ is the positive integers $\{1,2,3,\ldots\}$.

Exercise 2.36 (Putnam 2018, A4)**.** Let m and n be positive integers with $\gcd(m,n) = 1$, and let

$$a_k = \left\lfloor \frac{mk}{n} \right\rfloor - \left\lfloor \frac{m(k-1)}{n} \right\rfloor$$

for $k = 1,2,\ldots,n$. Suppose that g and h are elements in a group G and that

$$gh^{a_1} gh^{a_2} \cdots gh^{a_n} = e,$$

where e is the identity element. Show that $gh = hg$. (As usual, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

Exercise 2.37 (Putnam 2022, B6)**.** Find all continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(xf(y)) + f(yf(x)) = 1 + f(x+y)$$

for all $x,y > 0$.

Exercise 2.38. Find all binary operations $* : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $a,b,c \in \mathbb{R}^+$,

- $a*(b*c) = (a*b)c$; and
- if $a \geq 1$, then $a*a \geq 1$.

Exercise 2.39 (Putnam 2023, B5)**.** Determine which positive integers n have the following property: For all integers m that are relatively prime to n, there exists a permutation $\pi : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ such that $\pi(\pi(k)) \equiv mk \pmod{n}$ for all $k \in \{1,2,\ldots,n\}$.

Chapter 3

Calculus and Differential Equations

3.1 Basics

Definition 3.1. For a sequence of real (or complex numbers) a_n and a number a we say $\lim_{n\to\infty} a_n = a$ if the following is satisfied:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ for which if } n \geq N, \text{ then } |a_n - a| < \varepsilon.$$

Definition 3.2. For a function f over real numbers and a real number a we say $\lim_{x\to a} f(x) = L$ if the following is satisfied:

$$\forall\ \varepsilon > 0\ \exists\ \delta > 0 \text{ for which if } 0 < |x-a| < \delta, \text{ and } x \text{ is in the domain of } f, \text{ then } |f(x) - L| < \varepsilon.$$

Definition 3.3. The derivative is defined as

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

It is helpful to remember Taylor series of some well-known functions listed below:

- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \cdots$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots$

Linear Differential Equations are the ones of the form

$$y^{(n)} + a_n(t)y^{(n-1)} + \cdots + a_2(t)y' + a_1(t)y = f(t).$$

3.2 Important Theorems

There are too many theorems to state all of them, but here are some of the important ones. Some are more well-known and some are less well-known.

Theorem 3.1 (Intermediate Value Theorem). *If $f : [a,b] \to \mathbb{R}$ is continuous, and c is a value between $f(a)$ and $f(b)$, then there is $x \in [a,b]$ for which $f(x) = c$.*

Theorem 3.2 (Mean Value Theorem). *Suppose $f : [a,b] \to \mathbb{R}$ is continuous and its restriction to (a,b) is differentiable. Then, there is $c \in (a,b)$ for which $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.*

Theorem 3.3 (Binomial Theorem). *For every real number α and every real number $x \in (-1,1)$ we have*

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n,$$

where $\binom{\alpha}{0} = 1$, and $\binom{\alpha}{n} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$, for all $n \geq 1$.

Theorem 3.4. *If $f : [a,b] \to \mathbb{R}$ is a continuous, one-to-one function, then f is monotone.*

Theorem 3.5 (Mean Value Theorem for Integrals). *If f is a continous function over $[a,b]$, then there is $c \in (a,b)$ for which $f(c) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,\mathrm{d}x$.*

Theorem 3.6 (Lagrange Remainder Theorem). *Let I be an open interval containing x_0 and $f : I \to \mathbb{R}$ be $n+1$ times differentiable. Then, for every $x_0 \neq x \in I$, there is a real number c strictly between x_0 and x for which,*

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

3.3 Further Examples

Example 3.1 (Putnam 1992, A2). *Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series about $x = 0$ of $(1+x)^\alpha$. Evaluate*

$$\int_0^1 \left(C(-y-1) \sum_{k=1}^{1992} \frac{1}{y+k} \right) \mathrm{d}y.$$

Scratch: Here are my initial thoughts

- We could try this for some small cases instead of 1992.
- When know $C(-y-1)$ from the Binomial Theorem.

$\binom{-y-1}{1992} = \dfrac{(-y-1)(-y-2)\cdots(-y-1992)}{1992!} = \dfrac{(y+1)(y+2)\cdots(y+1992)}{1992!}$.

After trying this for $1,2,3$ instead of 1992 we see the integral becomes $1,2,3$, respectively. This is a clear pattern, but how do we prove it? We will go back and see if we can come up with not-too-computational solutions for the small cases. $n = 3$ is a good one to look at as it is not too large that we cannot do the computation and also not too small that we cannot see the pattern. For that we will need to show

$$\int_0^1 (y+1)(y+2) + (y+1)(y+3) + (y+2)(y+3)\,\mathrm{d}y = 3 \cdot 3!$$

Expanding the integrand we get $3y^2 + 2(1+2+3)y + (1\cdot 2 + 1\cdot 3 + 2\cdot 3)$. The coefficients 3 and 2 are very convenient as they make the integration easier for those of us who hate fractions! So, the integral becomes $y^3 + (1+2+3)y^2 +$

$(1\cdot 2+1\cdot 3+2\cdot 3)y+C$, but this is identical to the product $(y+1)(y+2)(y+3)$, if we choose $C=1\cdot 2\cdot 3$. Aha! This gives us a very neat solution:

Solution. The answer is $\boxed{1992}$.

For simplicity we let $n=1992$. By the Binomial Theorem we know

$$\binom{-y-1}{n}=\frac{(-y-1)(-y-2)\cdots(-y-n)}{n!}=\frac{(y+1)(y+2)\cdots(y+n)}{n!}.$$

Therefore, we are evaluating $\frac{1}{n!}\int_0^1 (y+1)(y+2)\cdots(y+n)\left(\sum_{k=1}^{n}\frac{1}{y+k}\right)$ dy. Note that by the product rule the derivative of $P(y)=(y+1)(y+2)\cdots(y+n)$ equals

$$\begin{aligned}&1\cdot(y+2)\cdots(y+n)+(y+1)\cdot 1\cdot(y+3)\cdots(y+n)+\cdots+(y+1)(y+2)\cdots(y+n-1)\cdot 1\\&=(y+1)(y+2)\cdots(y+n)\left(\sum_{k=1}^{n}\frac{1}{y+k}\right),\end{aligned}$$

which is the integrand. Therefore, the desired integral is $\frac{1}{n!}[P(1)-P(0)]$. We see that $P(1)=2\cdot 3\cdots(n+1)=(n+1)!$ and $P(0)=1\cdot 2\cdots n=n!$, which implies $P(1)-P(0)=n!(n+1-1)=n!\cdot n$. Therefore, the answer is n, as desired. $\square$

Example 3.2 (Putnam 1992, A4). *Let f be an infinitely differentiable real-valued function defined on the real numbers. If*

$$f\left(\frac{1}{n}\right)=\frac{n^2}{n^2+1},\qquad n=1,2,3,\ldots,$$

compute the values of the derivatives $f^{(k)}(0), k=1,2,3,\ldots$.

Scratch: Here are my initial thoughts:

- With only a countably many values, we, of course, won't be able to find f, but can we find at least an example of such a function to get an idea of what the answer might be?
- As usual, can we at least find $f'(0)$, and $f''(0)$?

Setting $x=1/n$ and thus substituting $n=1/x$, we see that $f(x)=\frac{1}{1+x^2}$ is one such function. After evaluating a few terms I get $f'(0)=0, f''(0)=-2, f'''(0)=0$, and $f^{(4)}(0)=24$. Okay, we can keep going, but we do notice that finding these values is the same as finding the Taylor series of this function, which we know how to find. So, let's consider this problem done for the function that we guessed. What if there are other functions satisfying these conditions? If two functions both satisfy the given conditions their difference is zero at $1/n$, for every positive integer n. Let their difference be g. Thus, $g(0)=0$, since $g(1/n)=0$ and g is continuous. $g'(0)$ is the limit of $\frac{g(1/n)-g(0)}{1/n}$ which is also zero, since $g(1/n)=g(0)=0$. However for the next derivative I will have a more difficult time because we don't know $g'(1/n)$. Can we find *any* sequence whose terms are roots of g'? can be done by applying the Rolle's Theorem to $1/n$ and $1/(n+1)$, for instance. So, putting these together we obtain the following solution:

Solution. (Video Solution) The answer is $\boxed{f^{(k)}(0)=0 \text{ if } k \text{ is odd , and } f^{(k)}(0)=(-1)^{k/2}k! \text{ if } k \text{ is even.}}$

First we will prove the following claim:

Claim: If g is an infinitely differentiable real-valued function defined on the real numbers for which $g(1/n) = 0$, then for every k there is a strictly decreasing sequence x_n for which $g^{(k)}(x_n) = 0$ for all n and $\lim_{n\to\infty} x_n = 0$.

We will prove this by induction on k.

Basis step: We know $g(1/n) = 0$. Since the sequence $\frac{1}{n}$ is a strictly decreasing sequence approaching zero, the claim is true for $k = 0$.

Inductive step: Suppose $g^{(k)}(x_n) = 0$ for a sequence x_n that decreases to zero. Applying Rolle's Theorem to $g^{(k)}(x_n) = g^{(k)}(x_{n+1})$, we obtain $y_n \in (x_{n+1}, x_n)$ for which $g^{(k+1)}(y_n) = 0$. Applying the Squeeze Theorem to $y_n \in (x_{n+1}, x_n)$ we obtain $\lim_{n\to\infty} y_n = 0$. We also see that $y_n > x_{n+1} > y_{n+1}$, which means y_n is strictly decreasing. This completes the proof of the inductive step, which completes the proof of the claim.

Suppose $f(x)$ is a function satisfying the assumptions, we notice that if we set $h(x) = \frac{1}{1+x^2}$, then $h(1/n) = f(1/n)$ and thus $g(x) = f(x) - h(x)$ satisfies the assumptions of the claim. Therefore, for each k there is a sequence x_n decreasing to zero such that $g^{(k)}(x_0) = 0$. Since g is infinitely times differentiable, we have $g^{(k)}(0) = \lim_{n\to\infty} g^{(k)}(x_n) = 0$, which proves $g^{(k)}(0) = 0$. Thus, $f^{(k)}(0) = h^{(k)}(0)$. Using geometric sum we obtain

$$h(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$$

Therefore by the Taylor series formula we see that $h^{(k)}(0) = 0$ when k is odd, and $\frac{h^{(k)}(0)}{k!} = (-1)^{k/2}$, when k is even. This completes the proof. $\square$

Example 3.3 (IMC 2019, Problem 6). *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that g is differentiable. Assume that $(f(0) - g'(0))(g'(1) - f(1)) > 0$. Show that there exists a point $c \in (0,1)$ such that $f(c) = g'(c)$.*

Scratch: We are trying to show $f(x) - g'(x) = 0$ has a root. Typically we use the Intermediate Value Theorem to show the existence of roots, however $g'(x)$ may not be continuous, so that is not an option! BUT, we know that IVP is valid for derivatives of functions even when the derivative is not continuous. This yields the following solution.

Solution. Let $F(x) = \int_0^x f(t)\,\mathrm{d}t - g(x)$. By Fundamental Theorem of Calculus, $F'(x) = f(x) - g'(x)$. By assumption $F'(0)F'(1) < 0$. Since derivatives satisfy the Intermediate Value Property, we conclude that there is some $c \in (0,1)$ such that $F'(c) = 0$, or $f(c) = g'(c)$, as desired. $\square$

Example 3.4 (Putnam 2022, A1). *Determine all ordered pairs of real numbers (a,b) such that the line $y = ax + b$ intersects the curve $y = \ln(1+x^2)$ in exactly one point.*

Scratch: Here are my initial thoughts:

- This looks like a standard single variable calculus problem.
- Given the nature of the problem and the fact that we are dealing with problem A1, it is probably not that difficult.
- Typically to find the number of solutions of an equation, we graph the function. For that we need to find the critical points, determine the endpoint behavior of the function, and where it is monotone.
- We also notice that to determine the long term behavior of the function we need to know the sign of a, so we can take cases for that.

When we start working on the problem, you may find it easy to get frustrated with all the computation and case work. The key is to be persistent and not worry about all the details in the first attempt. If you need to consider cases, do so. You can always come back and fill in the details.

We note that if we let $f(x) = \ln(1+x^2) - ax$, then $f'(x) = \dfrac{2x}{1+x^2} - a = \dfrac{2x-a-ax^2}{1+x^2}$. We then need to find all critical points, which means we need to solve the equation $-ax^2+2x-a=0$. Understanding this equation requires separate cases, based on its discriminant and the value of a.

Considering all of the above, we end up with the following solution:

Solution.(Video Solution) We claim the line $y = ax+b$ intersects the graph $y = \ln(1+x^2)$ at precisely one point if and only if one of the following holds:

- $a = b = 0$,
- $|a| \geq 1, b \in \mathbb{R}$,
- $0 < |a| < 1$, and $b > \ln(1+s^2) - as$, where $s = \dfrac{1+\sqrt{1-a^2}}{a}$,
- $0 < |a| < 1$, and $b < \ln(1+r^2) - ar$, where $r = \dfrac{1-\sqrt{1-a^2}}{a}$.

Consider the function $f(x) = \ln(1+x^2) - ax$. We have $f'(x) = \dfrac{2x}{1+x^2} - a = \dfrac{2x-a-ax^2}{1+x^2}$.

Case I. $a = 0$. The only critical point is $x = 0$. The function is strictly decreasing over $(-\infty, 0]$ and strictly increasing over $[0, \infty)$. Also note that $\lim\limits_{x\to\pm\infty} \ln(1+x^2) = \infty$, and that $f(0) = \ln 1 = 0$. Therefore, by the Intermediate Value Theorem, if $b > 0$, the equation $f(x) = b$ has two solutions. If $b = 0$, the equation $f(x) = 0$ has only one solution $x = 0$, and the equation $f(x) = b$ has no solutions when $b < 0$.

Note that when $a \neq 0$, $\lim\limits_{x\to\infty}(\ln(1+x^2) - ax) = \pm\infty$, depending on if a is negative or positive, and $\lim\limits_{x\to-\infty}(\ln(1+x^2) - ax) = \pm\infty$, depending on if a is negative or positive. Therefore, by the Intermediate Value Theorem, the equation $f(x) = b$ has at least one solution.

Case II. $|a| \geq 1$. The discriminant of the quadratic equation $-ax^2+2x-a=0$ is $4-4a^2 \leq 0$. Thus, $f'(x)$ does not change signs. Therefore, f is strictly monotone, which means, $f(x)=b$ has a unique solution for all $b \in \mathbb{R}$.

Case III. $0<a<1$. Write $-ax^2+2x-a=-a(x-r)(x-s)$, where $r=\dfrac{1-\sqrt{1-a^2}}{a}$, and $s=\dfrac{1+\sqrt{1-a^2}}{a}$. We see that $f'(x)<0$ over $(-\infty,r)\cup(s,\infty)$ and positive over (r,s). Therefore, $f(x)=b$ has a unique solution if and only if $b>f(s)$ or $b<f(r)$.

Case IV. $-1<a<0$. Similar to above $f'(x)$ is positive over $(-\infty,s)\cup(r,\infty)$ and negative over (s,r). Therefore, $f(x)=b$ has a unique solution if and only if $b<f(r)$ or $b>f(s)$. $\square$

Example 3.5 (Putnam 2022, B1)**.** *Suppose that $P(x)=a_1x+a_2x^2+\cdots+a_nx^n$ is a polynomial with integer coefficients, with a_1 odd. Suppose that $e^{P(x)}=b_0+b_1x+b_2x^2+\cdots$ for all x. Prove that b_k is nonzero for all $k \geq 0$.*

Solution.(Video Solution) Let $f(x)=e^{P(x)}$. By induction on n we show for every $n \geq 0$ there is a polynomials $Q_n(x)$ with integer coefficients for which $f^{(n)}(x)=e^{P(x)}Q_n(x)$, with $Q_n(0)$ odd and $Q_n'(0)$ even.
Basis step. $f(x)=e^{P(x)}Q_0(x)$, where $Q_0(x)=1$, whose derivative $Q_0'(0)=0$ is even.
Inductive step. Suppose the claim is true for n. We have $f^{(n+1)}(x)=e^{P(x)}(P'(x)Q_n(x)+Q_n'(x))$. We see that $Q_{n+1}(x)=P'(x)Q_n(x)+Q_n'(x)$ is a polynomial, $P'(0)Q_n(0)+Q_n'(0)=a_1Q_n(0)+Q_n'(0)$ is odd since a_1 and $Q_n(0)$ are odd and $Q_n'(0)$ is even. Furthermore, $Q_{n+1}'(0)=P''(0)Q_n(0)+P'(0)Q_n'(0)+Q_n''(0)=2a_2Q_n(0)+a_1Q_n'(0)+2q_2$, where q_2 is the coefficient of x^2 in $Q_n(x)$. Since $Q_n'(0)$ is even, $Q_{n+1}'(0)$ is even. This proves the claim.

Note that $b_k=\dfrac{f^{(k)}(0)}{k!}=\dfrac{Q_k(0)}{k!}$ is nonzero, since $Q_k(0)$ is an odd integer. $\square$

Example 3.6 (Putnam 1980, A3)**.** *Evaluate*

$$\int_0^{\pi/2} \frac{1}{1+(\tan x)^{\sqrt{2}}}\,dx$$

Scratch. It doesn't seem possible to find an anti-derivative of the integrand. So, we should probably take advantage of the fact that this is a definite integral and not an indefinite one. One common technique is the Limit Swapping technique. In this method, we use a u-substitution that swaps the limits of integral. In other words, we will choose a function u for which $u(0)=\pi/2$, while $u(\pi/2)=0$. The easiest such function is the linear function $u=\pi/2-x$. This yields the following solution:

Solution.(Video Solution) We will show the integral is equal to $\pi/4$.

Let's call the given definite integral I. We write $\tan x$ in terms of $\sin x$ and $\cos x$ to obtain:

$$I=\int_0^{\pi/2} \frac{(\cos x)^{\sqrt{2}}}{(\cos x)^{\sqrt{2}}+(\sin x)^{\sqrt{2}}}\,dx \qquad (*)$$

Using $u = \pi/2 - x$ we can convert the given integral to

$$I = \int\limits_{\pi/2}^{0} \frac{(\cos(\pi/2-u))^{\sqrt{2}}}{(\cos(\pi/2-u))^{\sqrt{2}} + (\sin(\pi/2-u))^{\sqrt{2}}}(-\,du) = \int\limits_{0}^{\pi/2} \frac{(\sin u)^{\sqrt{2}}}{(\sin u)^{\sqrt{2}} + (\cos u)^{\sqrt{2}}}\,du$$

Therefore,

$$I = \int\limits_{0}^{\pi/2} \frac{(\sin x)^{\sqrt{2}}}{(\sin x)^{\sqrt{2}} + (\cos x)^{\sqrt{2}}}\,dx \qquad (**)$$

Adding $(*)$ and $(**)$, we obtain

$$2I = \int\limits_{0}^{\pi/2} \frac{(\cos x)^{\sqrt{2}} + (\sin x)^{\sqrt{2}}}{(\sin x)^{\sqrt{2}} + (\cos x)^{\sqrt{2}}}\,dx = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}.$$

Therefore, $I = \dfrac{\pi}{4}$. □

Example 3.7 (VTRMC 2011)**.** *Evaluate*

$$\int_1^4 \frac{x-2}{(x^2+4)\sqrt{x}}dx.$$

Solution. (Video Solution) The answer is zero.

Let $I = \displaystyle\int_1^4 \frac{x-2}{(x^2+4)\sqrt{x}}dx$. We will use the substitution $u = 4/x$.

$$x = \frac{4}{u} \Rightarrow dx = -\frac{4}{u^2}du$$

Therefore,

$$I = \int_4^1 \frac{\frac{4}{u}-2}{\left(\frac{16}{u^2}+4\right)\sqrt{\frac{4}{u}}}\frac{-4}{u^2}du = \int_1^4 \frac{4-2u}{u\,(16+4u^2)\,\frac{2}{\sqrt{u}}}4du = \int_1^4 \frac{2-u}{(4+u^2)\sqrt{u}}du = -I$$

Therefore, $I = -I$, or $I = 0$. □

Example 3.8 (VTRMC 2016)**.** *Evaluate* $\displaystyle\int_1^2 \frac{\ln x}{2-2x+x^2}\,\mathrm{d}x$.

Solution. (Video Solution) The answer is $\dfrac{\pi\ln 2}{8}$.

Let $I = \displaystyle\int_1^2 \frac{\ln x}{2-2x+x^2}\,\mathrm{d}x$. We will use the substitution $u = 2/x$.

$$x = \frac{2}{u} \Rightarrow dx = -\frac{2}{u^2}du$$

Therefore,

$$I = \int_2^1 \frac{\ln(2/u)}{2-\frac{4}{u}+\frac{4}{u^2}}\cdot\frac{-2}{u^2}du = \int_1^2 \frac{\ln 2 - \ln u}{2u^2-4u+4}2du = \int_1^2 \frac{\ln 2-\ln u}{u^2-2u+2}du = \int_1^2 \frac{\ln 2}{2-2u+u^2}du - I$$

This implies

$$I = \frac{1}{2}\int_1^2 \frac{\ln 2}{2-2u+u^2}du = \frac{1}{2}\int_1^2 \frac{\ln 2}{(u-1)^2+1}du = \frac{\ln 2}{2}\tan^{-1}(u-1)]_1^2 = \frac{\pi\ln 2}{8}.$$

□

YouTube Video on Integration Techniques.

Example 3.9 (IMC 2022, Problem 1)**.** *Let $f : [0,1] \to (0,\infty)$ be an integrable function such that $f(x)f(1-x) = 1$ for all $x \in [0,1]$. Prove that*

$$\int_0^1 f(x)\,dx \geq 1.$$

Solution. (Video Solution) First note that using the limit swapping substitution $u = 1-x$ we obtain

$$I = \int_0^1 f(x)\,dx = \int_1^0 f(1-u)\,(-du) = \int_0^1 f(1-u)\,du.$$

Adding the two we obtain

$$2I = \int_0^1 (f(x) + f(1-x))\,dx.$$

By the AM-GM inequality we see $f(x) + f(1-x) \geq 2\sqrt{f(x)f(1-x)} = 2$. Therefore, $2I \geq \int_0^1 2\,dx = 2$, hence $I \geq 1$, as desired. □

Example 3.10 (IMC 2023, Problem 7)**.** *Let V be the set of all continuous functions $f : [0,1] \to \mathbb{R}$, differentiable on $(0,1)$, with the property that $f(0) = 0$ and $f(1) = 1$. Determine all $\alpha \in \mathbb{R}$ such that for every $f \in V$, there exists some $\xi \in (0,1)$ such that*

$$f(\xi) + \alpha = f'(\xi).$$

Solution.(Video Solution)

□

Example 3.11. *Evaluate $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$.*

Solution. (Video Solution) □

3.4 General Strategies

- This point has been repeated several times but it is worth repeating it again: simplify the problem if possible. If the question is asking for the n-th derivative, find the first derivative first. Try to see what kinds of functions might satisfy the assumptions given.

- Draw a graph of the function, and see why what they asked us to prove must be true.

- Some differential equations of form $u(x,y) + v(x,y)\dfrac{\mathrm{dy}}{\mathrm{dx}} = w(x,y)$ can be solved by finding an integrating factor μ. This integrating factor can be found by solving $\dfrac{\partial(\mu u)}{\partial y} = \dfrac{\partial(\mu v)}{\partial x}$.

3.5 Exercises

Exercise 3.1 (VTRMC 1980)**.** Let $P(x)$ be any polynomial of degree at most 3. It can be shown that there are numbers x_1 and x_2 such that $\int_{-1}^{1} P(x)\,\mathrm{d}x = P(x_1)+P(x_2)$, where x_1 and x_2 are independent of the polynomial P.

(a) Show that $x_1 = -x_2$.

(b) Find x_1 and x_2.

Exercise 3.2 (VTRMC 1980)**.** For $x > 0$, show that $e^x < (1+x)^{1+x}$.

Exercise 3.3. Let a, b be two positive real numbers. Prove that there is $c \in (-1,1)$ for which

$$\frac{a}{c^3+2c^2-1}+\frac{b}{c^3+c-2}=0.$$

Exercise 3.4 (VTRMC 1981)**.** For which real numbers b does the function $f(x)$, defined by the conditions $f(0) = b$ and $f' = 2f - x$, satisfy $f(x) > 0$ for all $x \geq 0$?

Exercise 3.5 (VTRMC 1983)**.** Find the function $f(x)$ such that for all $L \geq 0$, the area under the graph of $y = f(x)$ and above the x-axis from $x = 0$ to $x = L$ equals the arc length of the graph from $x = 0$ to $x = L$. (Hint: recall that $\frac{d}{dx}\cosh^{-1} x = 1/\sqrt{x^2-1}$.)

Exercise 3.6 (VTRMC 1984)**.** Let $f(x)$ satisfy the conditions for Rolle's theorem on $[a,b]$ with $f(a) = f(b) = 0$. Prove that for each real number k the function $g(x) = f'(x)+kf(x)$ has at least one zero in (a,b).

Exercise 3.7 (VTRMC 1984)**.** Find the greatest real r such that some normal line to the graph of $y = x^3 + rx$ passes through the origin, where the point of normality is not the origin.

Exercise 3.8 (VTRMC 1984)**.** Let $f = f(x)$ be an arbitrary differentiable function on $I = [x_0-h, x_0+h]$ with $|f'(x)| \leq M$ on I where $M \geq \sqrt{3}$. Let $f(x_0-h) \leq f(x_0)$ and $f(x_0+h) \leq f(x_0)$. Find the smallest positive number r such that at least one local maximum of f lies inside or on the circle of radius r centered at $(x_0, f(x_0))$. Express your answer in terms of h, M and $d = \min\{f(x_0)-f(x_0-h), f(x_0)-f(x_0+h)\}$.

Exercise 3.9 (VTRMC 1985, Modified)**.** Find all functions $f:\mathbb{R}\to\mathbb{R}$, that satisfy $f(x+1) = f(x)+x$ for all $x \in \mathbb{R}$ and $f(1) = 0$.

Exercise 3.10 (VTRMC 1985)**.** Let $f = f(x)$ be a real function of a real variable which has continuous third derivative and which satisfies, for a given c and all real $x, x \neq c$,

$$\frac{f(x)-f(c)}{x-c} = \left(f'(x)+f'(c)\right)/2$$

Show that $f''(x) = (f'(x)-f'(c))/(x-c)$.

Exercise 3.11 (VTRMC 1986)**.** Find the quadratic polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ such that $\int_0^1 t^n p(t)dt = n$ for $n = 0,1,2$.

Exercise 3.12 (VTRMC 1986)**.** Verify that, for $f(x) = x+1$

$$\lim_{r\to 0^+}\left(\int_0^1 (f(x))^r dx\right)^{1/r} = e^{\int_0^1 \ln f(x) dx}$$

Exercise 3.13 (VTRMC 1988)**.** Find the general solution of $y(x) + \int_1^x y(t)\, dt = x^2$.

Exercise 3.14 (VTRMC 1988)**.** Let f be differentiable on $[0,1]$ and let $f(\alpha) = 0$ and $f(x_0) = -.0001$ for some α and $x_0 \in (0,1)$. Also let $|f'(x)| \geq 2$ on $[0,1]$. Find the smallest upper bound on $|\alpha - x_0|$ for all such functions.

Exercise 3.15 (Putnam 1990, B1)**.** Find all real-valued continuously differentiable functions f on the real line such that for all x,

$$(f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2]\, dt + 1990.$$

Exercise 3.16 (VTRMC 1990)**.** Let the following conditions be satisfied:

(i) $f = f(x)$ and $g = g(x)$ are continuous functions on $[0,1]$,

(ii) there exists a number a such that $0 < f(x) \leq a < 1$ on $[0,1]$,

(iii) there exists a number u such that $\max_{0\leq x\leq 1}(g(x) + uf(x)) = u$.

Find constants A and B such that $F(x) = \frac{Ag(x)}{f(x)+B}$ is a continuous function on $[0,1]$ satisfying $\max_{0<x\leq 1} F(x) = u$, and prove that your function has the required properties.

Exercise 3.17 (VTRMC 1991)**.** Find all differentiable functions f which satisfy $f(x)^3 = \int_0^x f(t)^2 dt$ for all real x.

Exercise 3.18 (Putnam 1991, B2)**.** Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty,\infty)$. Furthermore, suppose that for each pair of real numbers x and y,

$$f(x+y) = f(x)f(y) - g(x)g(y),$$
$$g(x+y) = f(x)g(y) + g(x)f(y).$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

Exercise 3.19 (VTRMC 1992)**.** Find all inflection points of the graph of $F(x) = \int_0^{x^3} e^{t^2}\, dt$, for $x \in \mathbb{R}$.

Exercise 3.20 (VTRMC 1993)**.** Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) = \int_0^x f(t)\, dt$, then $f(x)$ is identically zero.

Exercise 3.21 (VTRMC 1993)**.** Let $f_1(x) = x$ and $f_{n+1}(x) = x^{f_n(x)}$, for $n = 1,2,\ldots$. Prove that $f_n'(1) = 1$ and $f_n''(1) = 2$, for all $n \geq 2$.

Exercise 3.22 (Putnam 1993, A5)**.** Show that

$$\int_{-100}^{-10}\left(\frac{x^2-x}{x^3-3x+1}\right)^2 dx + \int_{\frac{1}{101}}^{\frac{1}{11}}\left(\frac{x^2-x}{x^3-3x+1}\right)^2 dx + \int_{\frac{101}{100}}^{\frac{11}{10}}\left(\frac{x^2-x}{x^3-3x+1}\right)^2 dx$$

is a rational number.

Exercise 3.23 (VTRMC 1994)**.** Find all continuously differentiable solutions $f(x)$ for

$$f(x)^2 = \int_0^x \left(f(t)^2 - f(t)^4 + \left(f'(t)\right)^2\right) dt + 100$$

where $f(0)^2 = 100$.

Exercise 3.24 (Putnam 1994, A2)**.** Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

Exercise 3.25 (Putnam 1994, B3)**.** Find the set of all real numbers k with the following property: For any positive, differentiable function f that satisfies $f'(x) > f(x)$ for all x, there is some number N such that $f(x) > e^{kx}$ for all $x > N$.

Exercise 3.26 (Putnam 1995, A2)**.** For what pairs (a,b) of positive real numbers does the improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}}\right) dx$$

converge?

Exercise 3.27 (VTRMC 1996)**.** For each rational number r, define $f(r)$ to be the smallest positive integer n such that $r = m/n$ for some integer m, and denote by $P(r)$ the point in the (x,y) plane with coordinates $P(r) = (r, 1/f(r))$. Find a necessary and sufficient condition that, given two rational numbers r_1 and r_2 such that $0 < r_1 < r_2 < 1$,

$$P\left(\frac{r_1 f(r_1) + r_2 f(r_2)}{f(r_1) + f(r_2)}\right)$$

will be the point of intersection of the line joining $(r_1, 0)$ and $P(r_2)$ with the line joining $P(r_1)$ and $(r_2, 0)$.

Exercise 3.28 (VTRMC 1996)**.** Solve the differential equation $y^y = e^{dy/dx}$ with the initial condition $y = e$ when $x = 1$.

Exercise 3.29 (VTRMC 1996)**.** Let $f(x)$ be a twice continuously differentiable function over the interval $(0,\infty)$. If

$$\lim_{x\to\infty} (x^2 f''(x) + 4x f'(x) + 2f(x)) = 1$$

find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} x f'(x)$. Do not assume any special form of $f(x)$.

Exercise 3.30 (Putnam 1997, A3)**.** Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \cdots\right)$$
$$\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right) dx.$$

Exercise 3.31 (Putnam 1997, B2)**.** Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -x g(x) f'(x),$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

Exercise 3.32 (Putnam 1998, A3)**.** Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a)\cdot f'(a)\cdot f''(a)\cdot f'''(a) \geq 0.$$

Exercise 3.33 (VTRMC 1999)**.** Suppose that $f:\mathbb{R}\to\mathbb{R}$ is infinitely differentiable and satisfies both of the following properties.

(i) $f(1)=2$

(ii) If α,β are real numbers satisfying $\alpha^2+\beta^2=1$, then $f(\alpha x)f(\beta x)=f(x)$ for all x.

Find $f(x)$. Guesswork will not be accepted.

Exercise 3.34 (VTRMC 1999)**.** Let $f:\mathbb{R}_+\to\mathbb{R}_+$be a function from the set of positive real numbers to the same set satisfying $f(f(x))=x$ for all positive x. Suppose that f is infinitely differentiable for all positive X, and that $f(a)\neq a$ for some positive a. Prove that $\lim_{x\to\infty}f(x)=0$.

Exercise 3.35 (Putnam 1999, B4)**.** Let $f:\mathbb{R}\to\mathbb{R}$ has continuous third derivative such that $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$ are positive for all x. Suppose that $f'''(x)\le f(x)$ for all x. Show that $f'(x)<2f(x)$ for all x.

Exercise 3.36 (VTRMC 2000)**.** Evaluate $\displaystyle\int_0^\alpha \frac{\mathrm{d}\theta}{5-4\cos\theta}$.
(Your answer will involve inverse trig functions; you may assume that $0\le\alpha<\pi$). Use your answer to show that $\displaystyle\int_0^{\pi/3}\frac{\mathrm{d}\theta}{5-4\cos\theta}=\frac{2\pi}{9}$.

Exercise 3.37 (Putnam 2000, B4)**.** Let $f(x)$ be a continuous function such that $f(2x^2-1)=2xf(x)$ for all x. Show that $f(x)=0$ for $-1\le x\le 1$.

Exercise 3.38 (VTRMC 2001)**.** Determine the interval of convergence of the power series $\sum\limits_{n=1}^{\infty}\dfrac{n^nx^n}{n!}$. (That is, determine the real numbers x for which the above power series converges; you must determine correctly whether the series is convergent at the end points of the interval.)

Exercise 3.39 (VTRMC 2005)**.** Define $f(x,y)=\dfrac{xy}{x^2+(y\ln(x^2))^2}$ if $x\neq 0$, and $f(0,y)=0$ if $y\neq 0$. Determine whether $\lim\limits_{(x,y)\to(0,0)}f(x,y)$ exists, and what its value is if the limit does exist.

Exercise 3.40 (VTRMC 2005)**.** Compute $\displaystyle\int_0^1((e-1)\sqrt{\ln(1+ex-x)}+e^{x^2})dx$.

Exercise 3.41 (Putnam 2005, A5)**.** Evaluate $\int_0^1\frac{\ln(x+1)}{x^2+1}\,dx$.

Exercise 3.42 (VTRMC 2006)**.** We want to find functions $p(t),q(t),f(t)$ such that

(a) p and q are continuous functions on the open interval $(0,\pi)$.

(b) f is an infinitely differentiable nonzero function on the whole real line $(-\infty,\infty)$ (i.e. f is not identically zero over $\mathbb{R}$), such that $f(0)=f'(0)=f''(0)$.

(c) $y=\sin t$ and $y=f(t)$ are solutions of the differential equation $y''+p(t)y'+q(t)y=0$ on $(0,\pi)$

Is this possible? Either prove this is not possible, or show this is possible by providing an explicit example of such f,p,q.

Exercise 3.43 (VTRMC 2007)**.** Solve the initial value problem $\frac{dy}{dx}=y\ln y+ye^x, y(0)=1$ (i.e. find y in terms of x).

Exercise 3.44 (Putnam 2007, A1)**.** Find all values of α for which the curves $y=\alpha x^2+\alpha x+\frac{1}{24}$ and $x=\alpha y^2+\alpha y+\frac{1}{24}$ are tangent to each other.

Exercise 3.45 (Putnam 2007, B2)**.** Suppose that $f:[0,1]\to\mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\,dx=0$. Prove that for every $\alpha\in(0,1)$,

$$\left|\int_0^\alpha f(x)\,dx\right|\le\frac{1}{8}\max_{0\le x\le 1}|f'(x)|.$$

Exercise 3.46 (VTRMC 2009)**.** Define $f(x)=\int_0^x\int_0^x e^{u^2v^2}dudv$. Calculate $2f''(2)+f'(2)$ (here $f'(x)=\dfrac{df}{dx}$).

Exercise 3.47 (VTRMC 2009)**.** Does there exist a twice differentiable function $f:\mathbb{R}\to\mathbb{R}$ such that $f'(x)=f(x+1)-f(x)$ for all x and $f''(0)\neq 0$? Justify your answer. (Here $\mathbb{R}$ denotes the real numbers and f' denotes the derivative of f.)

Exercise 3.48 (Putnam 2009, B5)**.** Let $f:(1,\infty)\to\mathbb{R}$ be a differentiable function such that

$$f'(x)=\frac{x^2-f(x)^2}{x^2(f(x)^2+1)}\qquad\text{for all } x>1.$$

Prove that $\lim_{x\to\infty}f(x)=\infty$.

Exercise 3.49 (Putnam 2010, A2)**.** Find all differentiable functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f'(x)=\frac{f(x+n)-f(x)}{n}$$

for all real numbers x and all positive integers n.

Exercise 3.50 (Putnam 2010, A6)**.** Let $f:[0,\infty)\to\mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty}f(x)=0$. Prove that $\int_0^\infty\frac{f(x)-f(x+1)}{f(x)}\,dx$ diverges.

Exercise 3.51 (Putnam 2010, B5)**.** Is there a strictly increasing function $f:\mathbb{R}\to\mathbb{R}$ such that $f'(x)=f(f(x))$ for all x?

Exercise 3.52 (VTRMC 2011)**.** Find $\lim\limits_{x\to\infty}\left((2x)^{1+\frac{1}{2x}}-x^{1+\frac{1}{x}}-x\right)$.

Exercise 3.53 (Putnam 2011, A3)**.** Find a real number c and a positive number L for which

$$\lim_{r\to\infty}\frac{r^c\int_0^{\pi/2}x^r\sin x\,dx}{\int_0^{\pi/2}x^r\cos x\,dx}=L.$$

Exercise 3.54 (Putnam 2011, B3)**.** Let f and g be (real-valued) functions defined on an open interval containing 0, with g nonzero and continuous at 0. If fg and f/g are differentiable at 0, must f be differentiable at 0?

Exercise 3.55 (Putnam 2012, B5)**.** Prove that, for any two bounded functions $g_1,g_2:\mathbb{R}\to[1,\infty)$, there exist functions $h_1,h_2:\mathbb{R}\to\mathbb{R}$ such that, for every $x\in\mathbb{R}$,

$$\sup_{s\in\mathbb{R}}(g_1(s)^x g_2(s))=\max_{t\in\mathbb{R}}(xh_1(t)+h_2(t)).$$

Exercise 3.56 (VTRMC 2013)**.** Let $I=3\sqrt{2}\int_0^x\frac{\sqrt{1+\cos t}}{17-8\cos t}dt$. If $0<x<\pi$ and $\tan I=\dfrac{2}{3}$, what is x?

Exercise 3.57 (Putnam 2013, A3)**.** Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and x, with $0 < x < 1$, satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove that there exists a real number y with $0 < y < 1$ such that

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

Exercise 3.58 (VTRMC 2014)**.** Evaluate $\int_0^2 \frac{(16-x^2)x}{16-x^2+\sqrt{(4-x)(4+x)(12+x^2)}}dx$.

Exercise 3.59 (Putnam 2014, A1)**.** Prove that every nonzero coefficient of the Taylor series of

$$(1-x+x^2)e^x$$

about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

Exercise 3.60 (Putnam 2014, B6)**.** Let $f : [0,1] \to \mathbb{R}$ be a function for which there exists a constant $K > 0$ such that $|f(x) - f(y)| \le K|x-y|$ for all $x, y \in [0,1]$. Suppose also that for each rational number $r \in [0,1]$, there exist integers a and b such that $f(r) = a + br$. Prove that there exist finitely many intervals $I_1, \ldots, I_n$ such that f is a linear function on each I_i and $[0,1] = \bigcup_{i=1}^n I_i$.

Exercise 3.61 (VTRMC 2015)**.** Evaluate $\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x} dx$ (where $0 \le \arctan(x) < \pi/2$ for $0 \le x < \infty$).

Exercise 3.62 (Putnam 2015, B1)**.** Let f be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

Exercise 3.63. Let $f : [0,1] \to \mathbb{R}$ be a continuous function that is differentiable over $(0,1)$. Suppose $f(0) = 0$ and $f'(x)$ is increasing. Is it always true that $\dfrac{f(x)}{x}$ is an increasing function over $(0,1)$?

Exercise 3.64 (VTRMC 2017)**.** Evaluate $\displaystyle\int_0^a \frac{dx}{1+\cos x+\sin x}$ for $-\pi/2 < a < \pi$. Use your answer to show that $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\cos x+\sin x} = \ln 2$.

Exercise 3.65 (Putnam 2017, A3)**.** Let a and b be real numbers with $a < b$, and let f and g be continuous functions from $[a,b]$ to $(0,\infty)$ such that $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ but $f \ne g$. For every positive integer n, define

$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n}\,dx.$$

Show that $I_1, I_2, I_3, \ldots$ is an increasing sequence with $\lim_{n\to\infty} I_n = \infty$.

Exercise 3.66 (VTRMC 2018)**.** It is known that $\int_1^2 \frac{\arctan(1+x)}{x} dx = q\pi\ln(2)$ for some rational number q. Determine q. Here $0 \le \arctan(x) < \pi/2$ for $0 \le x < \infty$.

Exercise 3.67 (VTRMC 2018)**.** A continuous function $f : [a,b] \to [a,b]$ is called piecewise monotone if $[a,b]$ can be subdivided into finitely many subintervals

$$I_1 = [c_0, c_1], I_2 = [c_1, c_2], \ldots, I_\ell = [c_{\ell-1}, c_\ell]$$

such that f restricted to each interval I_j is strictly monotone, either increasing or decreasing. Here we are assuming that $a = c_0 < c_1 < \cdots < c_{\ell-1} < c_\ell = b$. We are also assuming that each I_j is a maximal interval on which f is strictly monotone. Such a maximal interval is called a lap of the function f, and the number $\ell = \ell(f)$ of distinct laps is called the lap number of f. If $f : [a,b] \to [a,b]$ is a continuous piecewise-monotone function, show that the sequence $\left(\sqrt[n]{\ell(f^n)}\right)$ converges; here f^n means f composed with itself n-times, so $f^2(x) = f(f(x))$ etc.

Exercise 3.68 (Putnam 2018, A5)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function satisfying $f(0) = 0$, $f(1) = 1$, and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$.

Exercise 3.69 (VTRMC 2019)**.** Compute $\displaystyle\int_0^1 \frac{x^2}{x + \sqrt{1-x^2}}\, dx$ (the answer is a rational number).

Exercise 3.70 (VTRMC 2019)**.** Find the general solution of the differential equation

$$x^4 \frac{d^2y}{dx^2} + 2x^2 \frac{dy}{dx} + (1-2x)y = 0$$

valid for $0 < x < \infty$.

Exercise 3.71 (Putnam 2020, A6)**.** For a positive integer N, let f_N be the function defined by

$$f_N(x) = \sum_{n=0}^{N} \frac{N + 1/2 - n}{(N+1)(2n+1)} \sin((2n+1)x).$$

Determine the smallest constant M such that $f_N(x) \leq M$ for all N and all real x.

Exercise 3.72 (Putnam 2021, A2)**.** For every positive real number x, let

$$g(x) = \lim_{r \to 0} ((x+1)^{r+1} - x^{r+1})^{\frac{1}{r}}.$$

Find $\lim_{x\to\infty} \frac{g(x)}{x}$.

Exercise 3.73 (VTRMC 2022)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function whose second derivative is continuous. Suppose that f and f'' are bounded. Show that f' is also bounded.

Exercise 3.74. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function for which

$$\int_a^{f(b)} f(x)\, dx = \int_{f(a)}^{b} f(x)\, dx, \text{ for all } a, b \in \mathbb{R}.$$

Prove that $f(x) = x$ for all $x \in \mathbb{R}$.

Exercise 3.75 (Putnam 2023, A1)**.** For a positive integer n, let $f_n(x) = \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)$. Find the smallest n such that $|f_n''(0)| > 2023$.

Exercise 3.76 (Putnam 2023, A3)**.** Determine the smallest positive real number r such that there exist differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfying

(a) $f(0) > 0$,

(b) $g(0) = 0$,

(c) $|f'(x)| \leq |g(x)|$ for all x,

(d) $|g'(x)| \leq |f(x)|$ for all x, and

(e) $f(r) = 0$.

Exercise 3.77 (Putnam 2023, B4)**.** For a nonnegative integer n and a strictly increasing sequence of real numbers $t_0, t_1, \ldots, t_n$, let $f(t)$ be the corresponding real-valued function defined for $t \geq t_0$ by the following properties:

(a) $f(t)$ is continuous for $t \geq t_0$, and is twice differentiable for all $t > t_0$ other than $t_1, \ldots, t_n$;

(b) $f(t_0) = 1/2$;

(c) $\lim_{t \to t_k^+} f'(t) = 0$ for $0 \leq k \leq n$;

(d) For $0 \leq k \leq n-1$, we have $f''(t) = k+1$ when $t_k < t < t_{k+1}$, and $f''(t) = n+1$ when $t > t_n$.

Considering all choices of n and $t_0, t_1, \ldots, t_n$ such that $t_k \geq t_{k-1} + 1$ for $1 \leq k \leq n$, what is the least possible value of T for which $f(t_0 + T) = 2023$?

Chapter 4

Number Theory

4.1 Basics

4.1.1 Divisibility

Definition 4.1. For integers a and b we say a **divides** b whenever $b = ac$ for some integer c. In which case, we write $a \mid b$.

Theorem 4.1 (Properties of Divisibility)**.** *Let $a, b, c, x, y \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$.*

- *If $a \mid b$ and $a \mid c$, then $a \mid bx + cy$. Similar property holds for more integer.*
- *If $a \mid b$ and $b \mid c$, then $a \mid c$.*
- *If $a \mid b$ and $b \neq 0$, then $|a| \leq |b|$.*
- *If $a \mid b$, then $a^n \mid b^n$.*
- *If $a \mid b$ and $x \mid y$, then $ax \mid by$.*
- *If $ac \mid bc$ and $c \neq 0$, then $a \mid b$*
- *If $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.*
- *If $a \mid bc$ and $\gcd(a, b) = d$, then $\dfrac{a}{d} \mid c$.*

Theorem 4.2 (Division Algorithm)**.** *Given two integers a, b with $b \neq 0$, there are unique integers q and r satisfying $a = bq + r$ and $0 \leq r < |b|$.*

In the division algorithm above, a is called the dividend, b is called the divisor, r is called the remainder and q is called the quotient.

Definition 4.2. An integer $p > 1$ is called **prime** if its only positive divisors are 1 and p.

Theorem 4.3 (Unique Factorization Theorem)**.** *Given any integer $n > 1$, there are distinct primes $p_1, \dots, p_k$ and positive integers $a_1, \dots, a_k$ for which $n = p_1^{a_1} \cdots p_k^{a_k}$. Furthermore this factorization is unique.*

Theorem 4.4. *There are infinitely many primes.*

Theorem 4.5. *Let a, b be two non-zero integers. Then, $a \mid b$ iff the exponent of each prime p in the prime factorization of a does not exceed the exponent of p in the prime factorization of b.*

Theorem 4.6. *A positive integer a is a perfect k-th power iff the exponent of every prime in the prime factorization of a is a multiple of k.*

Theorem 4.7 (Sieve of Eratosthenes)**.** *An integer $p > 1$ is a prime iff p is not divisible by any integer k where $1 < k \leq \sqrt{p}$.*

Definition 4.3. For any positive integer n, the number of positive divisors of n and the sum of positive divisors of n are denoted by $\tau(n)$ and $\sigma(n)$, respectively.

Theorem 4.8. *Let n be a positive integer with prime factorization $n = p_1^{a_1} \cdots p_k^{a_k}$. Then,*

- $\tau(n) = (a_1 + 1) \cdots (a_k + 1)$.
- $\sigma(n) = (1 + p_1 + \cdots + p_1^{a_1}) \cdots (1 + p_k + \cdots + p_k^{a_k})$.

Definition 4.4. Let m, and n be integers, not both of them zero. An integer d is called the **greatest common divisor** of m and n whevener

- $d \mid n$ and $d \mid m$; and
- If $k \mid n$ and $k \mid m$, then $k \leq d$.

The integer d above is denoted by $\gcd(m, n)$. We say m and n are **relatively prime** whenever $\gcd(m, n) = 1$. We define $\gcd(0, 0) = 0$.

Definition 4.5. Let m and n be integers, not both zero. A positive integer r is called the **least common multiple** of m and n whenever

- $n \mid r$ and $m \mid r$; and
- If $n \mid k$ and $m \mid k$, for some positive integer k, then $r \leq k$.

r is denoted by $\operatorname{lcm}(m, n)$. We define $\operatorname{lcm}(0, 0) = 0$.

Theorem 4.9. *Given integers a, b and k, we have* $\gcd(a, b) = \gcd(a, b + ak)$.

Theorem 4.10 (Euclidean Algorithm)**.** *Let m and n be two positive integers. Using the Division Algorithm we can write*

$$\begin{aligned} m &= nq_1 + r_1 \qquad 0 \leq r_1 < n \\ n &= r_1 q_2 + r_2 \qquad 0 \leq r_2 < r_1 \\ &\vdots \\ r_k &= r_{k+1} q_{k+2} + 0 \end{aligned}$$

Then $\gcd(m, n) = r_{k+1}$

Theorem 4.11. *Let $m = p_1^{a_1} \cdots p_k^{a_k}$ and $n = p_1^{b_1} \cdots p_k^{b_k}$ be two prime factorizations of n and m, where $p_1, \ldots, p_k$ are all primes that appear in the prime factorization of either m or n and we allow the exponents a_i's and b_i's to be zero. Then* $\gcd(m,n) = p_1^{c_1} \cdots p_k^{c_k}$ *and* $\operatorname{lcm}(m,n) = p_1^{d_1} \cdots p_k^{d_k}$, *where* $c_i = \min(a_i, b_i)$ *and* $d_i = \max(a_i, b_i)$.

Theorem 4.12 (Bezout's Lemma)**.** *For every two integers m and n, there are integers a and b such that* $\gcd(m,n) = am + bn$.

4.1.2 Congruences

Definition 4.6. For integers a, b, and a positive integer n we write $a \equiv b \mod n$ whenever $n \mid a - b$.

Theorem 4.13 (Properties of Congruences)**.** *Let a, b, c, d be integers and n and k be positive integers,*

- *If* $a \equiv b \mod n$ *and* $c \equiv d \mod n$, *then* $a + c \equiv b + d \mod n$ *and* $ac \equiv bd \mod n$.
- *If* $a \equiv b \mod n$, *then* $a^k \equiv b^k \mod n$.
- *If* $ac \equiv bc \mod n$, *and n and c are relatively prime, then* $a \equiv b \mod n$.
- *If* $ac \equiv bc \mod n$, *and* $\gcd(n,c) = d$, *then* $a \equiv b \mod \frac{n}{d}$.

Definition 4.7. Let a be an integer and n be a positive integer. A **multiplicative inverse** of a modulo n is an integer b for which $ab \equiv 1 \mod n$.

A multiplicative inverse for a modulo n exists iff $\gcd(n,a) = 1$. This can be shown using the Bezout's Lemma.

Definition 4.8. A set $\{a_1, a_2, \ldots, a_n\}$ is called a **complete residue system** modulo n, (CRS for short) whenever every integer is congruent to precisely one a_i modulo n.

Often times, to prove a statement $\mod n$, it is enough to check the statement is true for all elements of a CRS modulo n.

Theorem 4.14 (Fermat's Theorem)**.** *Let p be a prime and a be an integer relatively prime to p. Then* $a^{p-1} \equiv 1 \mod p$.

Definition 4.9. For a positive integer n let $U(n)$ be the set of all integers $a, 1 \le a \le n$ for which a is relatively prime to n. For any positive integer n, $\phi(n)$ is the size of $U(n)$.

Theorem 4.15. *If m and n are relatively prime, then* $\phi(mn) = \phi(m)\phi(n)$.

Definition 4.10. Let n be a positive integer. A set $A = \{a_1, \ldots, a_{\phi(n)}\}$ is called a **reduced residue system** (RRS for short) modulo n whenever every integer that is relatively prime to n is congruent to precisely one element of A.

Theorem 4.16. *Let n be a positive integer. Then,*

- *n is even iff its units digit is even.*
- *n is divisible by 3 iff its sum of digits is divisible by 3.*
- *n is divisible by 9 iff its sum of digits is divisible by 9.*

- *n is divisible by 11 iff the alternating sum of its digits is divisible by 11.*

Definition 4.11. Let n, a be two relatively prime integers. The smallest positive integer m for which $a^m \equiv 1 \mod n$ is called the **order** of a modulo n and is denoted by $\text{ord}_n a$.

4.1.3 Diophantine Equations

A Diophantine Equation is an equation over naturals, integers or rationals. In other words, we are looking for all solutions of an equation that are in $\mathbb{N}, \mathbb{Z}$ or $\mathbb{Q}$.
The simplest Diophantine equations are the ones of form $xy = a$, where a is a given integer and we want to solve this equation over $\mathbb{Z}$. In that case, simply write down the prime factorization of a and list all possible value of x and y. Make sure you do not forget the negative integers.

Remark. Let n be a given integer. To solve $x^2 - y^2 = n$,

a. Factor the equation as $(x-y)(x+y) = n$.

b. If n is odd, write $x - y = r$ and $x + y = s$ where $rs = n$. Since both r and s are odd this system would always have integer solutions for x and y. (Show this.)

c. If $4 \mid n$, write the equation down as $\frac{x-y}{2} \cdot \frac{x+y}{2} = \frac{n}{4}$. Then factor $\frac{n}{4} = rs$ and set $\frac{x-y}{2} = r, \frac{x+y}{2} = s$. This system will always have integer solutions for x and y. (Why?)

d. If $n \equiv 2 \mod 4$, the equation has no integer solutions.

Many Diophantine equations can be factored. A very common example is Diophantine equations of form $ax + by + cxy = d$, where a, b, c, d are given integers. We would like to write

$$ax + by + cxy = (cx + \_)(y + \_) + \_$$

Therefore we must use $(cx + b)(y + \frac{a}{c}) = d + \frac{ab}{c}$. Clearing the denominator we get $(cx + b)(cy + a) = cd + ab$. This equation can be solved by finding all factors of $cd + ab$. Note that since $cx + b$ and $cy + a$ are a and b modulo c, this limits the choices of the factors of $cd + ab$ that we need to check.

Theorem 4.17. *Given integers a, b, and c, the Diophantine equation $ax + by = c$ has a solution for integers x, y iff* $\gcd(a, b) \mid c$.

To solve this equation first find a solution (x_0, y_0). This can be done by guessing to the Euclidean Algorithm. If $ax + by = c$, then $a(x - x_0) + b(y - y_0) = 0$. Therefore $b \mid x - x_0$ (why?) Thus all solutions can be found using $x = x_0 + kb$ and $y = y_0 - ka$. (why?)

4.1.4 Pythagorean Triples

Definition 4.12. A triple of positive integers (a, b, c) is called a **Pythagorean triple** whenever $a^2 + b^2 = c^2$. This Pythagorean Triple is called **primitive** whenever $\gcd(a, b, c) = 1$.

Remark. Any Pythagorean Triple (x,y,z) can be written as $(x,y,z) = (da,db,dc)$, where d is a positive integer and (a,b,c) is a primitive Pythagorean Triple.

Remark. Note that a Pythagorean triple (a,b,c) is primitive iff a,b,c are pairwise relatively prime. (Why?)

4.2 Important Theorems

Theorem 4.18 (Legendre's Formula)**.** *Let n be a positive integer and p be a prime. The power of p in the prime factorization of $n!$ is given by* $\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor$.

Theorem 4.19 (Postage Stamp Problem)**.** *Let a,b be positive relatively prime integers. Then for every $n \geq (a-1)(b-1)$, there are non-negative integers x,y for which $ax+by=n$.*

Theorem 4.20. *Any primitive Pythagorean triple is of the form $(2mn, m^2-n^2, m^2+n^2)$, where $m>n$ are two relatively prime positive integers, and precisely one of m or n is even.*

The proof of the above theorem can be found in this YouTube video. Note that the above theorem produces each primitive Pythagorean triple precisely once. Some examples are shown in the following table:

m	n	(a,b,c)
2	1	$(4,3,5)$
3	2	$(12,5,13)$
4	1	$(8,15,17)$
4	3	$(24,7,25)$
5	2	$(20,21,29)$
5	4	$(40,9,41)$

$$\vdots$$

Theorem 4.21 (Euler's Theorem)**.** *Let n be a positive integer and a be an integer relatively prime to n. Then $a^{\phi(n)} \equiv 1 \mod n$.*

Theorem 4.22 (Wilson's Theorem)**.** *For any prime p, we have $(p-1)! \equiv -1 \mod p$.*

Theorem 4.23 (Chinese Remainder Theorem)**.** *Let $a_1,\dots,a_k$ be pairwise relatively prime integers and $b_1,\dots,b_k$ be integers. Then there is an integer x for which $x \equiv b_i \mod a_i$ for all $i, 1 \leq i \leq k$. Furthermore if x,y are two solutions to this system, then $x \equiv y \mod a_1 \cdots a_k$. (In other words the solution is unique modulo $a_1 \cdots a_k$.)*

Remark. Note that if you can find one solution to the system given in the CRT, you know all solutions.

Theorem 4.24 (Primitive Roots)**.** *The group $U(n)$ is cyclic (in other words, there is an element of order $\phi(n)$ in $U(n)$) if and only if $n = 2, 4, p^k$, or $2p^k$, for an odd prime p.*

Theorem 4.25 (Orders)**.** *Suppose a,n are two relatively prime integers, and m be a positive integer. Then, $a^m \equiv 1 \mod n$ if and only if $\operatorname{ord}_n a \mid m$.*

Proof of the above theorem can be found in this YouTube video.

Definition 4.13. We say an integer a is a **quadratic residue** modulo an integer n, iff there is an integer x for which $x^2 \equiv a \mod n$. Otherwise, we say a is a **quadratic nonresidue** modulo n.

Definition 4.14. Given a prime p and an integer n we define the **Legendre symbol** $\left(\frac{n}{p}\right)$ as follows:

$$\left(\frac{n}{p}\right) = \begin{cases} 0 & \text{if } p \mid n \\ -1 & \text{if } n \text{ is a quadratic nonresidue mod } p \\ 1 & \text{if } n \text{ is a quadratic residue mod } p \text{ and } p \nmid n \end{cases}$$

Theorem 4.26 (Euler's Criterion)**.** *For every odd prime p and an integer a, we have $\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \mod p$. Consequently, if p does not divide integers a and b, then $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.*

Theorem 4.27 (Quadratic Reciprocity)**.** *Let p and q be two distinct odd primes. Then*

$$\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{q}{p}\right).$$

Furthermore,

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}, \textit{ and } \left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}.$$

For a discussion on how Quadratic Reciprocity is used to solve problems check out this YouTube Video.

Definition 4.15. Given two relatively prime integers a, b with $b > 1$ we define the **Jacobi symbol** $\left(\frac{a}{b}\right)$ as

$$\left(\frac{a}{b}\right) = \left(\frac{a}{p_1}\right) \cdots \left(\frac{a}{p_k}\right),$$

if $b = p_1 \cdots p_k$ is the prime factorization of b.

Note that

- $\left(\frac{a}{b}\right) = 1$ does not imply that a is a quadratic residue modulo b. For example $a = 2$ is not a quadratic residue mod $b = 15$, however $\left(\frac{2}{15}\right) = \left(\frac{2}{3}\right)\left(\frac{2}{5}\right) = (-1)(-1) = 1$.
- $\left(\frac{a}{b}\right) = -1$ implies that $\left(\frac{a}{p_j}\right) = -1$ for some prime p_j dividing b. Thus, a is not a quadratic residue modulo p_j, and hence a is not a quadratic residue modulo b.

Theorem 4.28. *Suppose a, b, c are integers with* $\gcd(a,c) = \gcd(b,c) = 1$.

- *If $c > 1$, then* $\left(\dfrac{ab}{c}\right) = \left(\dfrac{a}{c}\right)\left(\dfrac{b}{c}\right)$.
- *If $a \equiv b \mod c$, and $c > 1$, then* $\left(\dfrac{a}{c}\right) = \left(\dfrac{b}{c}\right)$.
- *If $a, b > 1$, then* $\left(\dfrac{c}{ab}\right) = \left(\dfrac{c}{a}\right)\left(\dfrac{c}{b}\right)$.

Theorem 4.29 (Extended Quadratic Reciprocity)**.** *Let $a, b > 1$ be two relatively prime odd integers. Then*

$$\left(\frac{a}{b}\right) = (-1)^{\frac{a-1}{2}\frac{b-1}{2}}\left(\frac{b}{a}\right).$$

Furthermore,

$$\left(\frac{2}{a}\right) = (-1)^{(a^2-1)/8}, \text{ and } \left(\frac{-1}{a}\right) = (-1)^{(a-1)/2}.$$

4.3 Classical Examples

Example 4.1. *Prove that for every prime p and every integer n with $0 < n < p$ the number $\binom{p}{n}$ is divisible by p.*

Solution. $\binom{p}{n} = \dfrac{p!}{n!(p-n)!}$. Since $0 < n < p$, and p is prime, the denominator is not divisible by p. On the other hand the numerator is divisible by p. Therefore, since p is prime, p divides the fraction above, as desired. $\square$

Example 4.2. *Show that for every two positive integers m and , n we have* $\gcd(m,n) \cdot \mathrm{lcm}(m,n) = mn$.

Solution. If α and β are the exponents of a prime p in the prime factorization of m and n, respectively, then the exponents of p in the prime factorizations of $\gcd(m,n)$ and $\mathrm{lcm}(m,n)$ are $\min(\alpha,\beta)$ and $\max(\alpha,\beta)$, respectively. The exponent of p in the prime factorization of $\gcd(m,n)\mathrm{lcm}(m,n)$ is then $\min(\alpha,\beta) + \max(\alpha,\beta)$ which is the same as $\alpha+\beta$. Since the exponent of p in the prime factorization of mn is $\alpha+\beta$, we have $\gcd(m,n)\cdot\mathrm{lcm}(m,n) = mn$. $\square$

Example 4.3. *Show that a positive integer is a perfect square iff $\tau(n)$ is odd.*

Solution. Suppose $n = p_1^{a_1} \cdots p_k^{a_k}$ is the prime factorization of n. We know $\tau(n) = (a_1+1)\cdots(a_k+1)$ is odd iff all a_i's are even, which is equivalent to n being a perfect square. $\square$

Example 4.4. *Let x be an odd integer. Prove that*

- $x^2 \equiv 1 \mod 8$, *and*
- $x^{2^n} \equiv 1 \mod 2^{n+2}$, *for every positive integer n.*

Solution. We will prove the second part by induction on n.

Basis step. If $n = 1$, and x is odd, then $x \equiv \pm 1 \mod 4$. Which means $x = 4k \pm 1$ for some integer k, and thus $x^2 = 16k^2 \pm 8k + 1 \equiv 1 \mod 8$.

Inductive step. Note that $x^{2^{n+1}} - 1 = (x^{2^n} - 1)(x^{2^n} + 1)$. The first terms is divisible by 2^n using the inductive hypothesis. The second term is even. Therefore, $x^{2^{n+1}} - 1 \equiv 0 \mod 2^{n+3}$, as desired. □

Example 4.5. *Prove that the product of every n consecutive integers is divisible by $n!$*

Solution. (Video Solution) If all of these integers are positive, their product would be of the form $a(a+1)\cdots(a+n-1)$. The ratio of this product and $n!$ is:

$$\frac{a(a+1)\cdots(a+n-1)}{n!} = \binom{a+n-1}{n} \in \mathbb{Z}$$

If all of these integers are negative, then their product is $(-1)^n$ times the product of n positive consecutive integers, which we know is divisible by $n!$. If neither of the above two cases happen, then one of the integers must be zero, and thus their product is zero, which is divisible by $n!$ □

Example 4.6. *The Fibonacci sequence is defined as $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Find the units digit of F_{2020}.*

Scratch: We know that after writing at most 101 pairs of consecutive terms there would be a repetition in $(F_n, F_{n+1}) \mod 10$, but that means if we get unlucky we would have to write 102 terms! That is not very fun. We realize that the Chinese Remainder Theorem allows us to break modulo 10 into modulo 2, and modulo 5. Therefore, we will do just that.

Solution. The answer is 5.

We will write $F_n \mod 2$, and $F_n \mod 5$.
$F_n \mod 2 : 0, 1, 1, 0, 1$. As soon as we see two terms are repeated we know everything after that is also going to be repeated. So, there is a cycle of length 3. $F_{2020} \equiv F_{2020 \mod 3} = F_1 = 1 \mod 2$.

$F_n \mod 5 : 0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1$. The cycle is of length 20. Therefore, $F_{2020} \equiv F_{2020 \mod 20} = F_0 = 0 \mod 5$.

Therefore, $F_{2020} \equiv 1 \mod 2$, and $F_{2020} \equiv 0 \mod 5$, which means $F_{2020} \equiv 5 \mod 10$. □

Example 4.7. *Suppose n is a positive integer and a is an integer more than 1. Prove that $n \mid \varphi(a^n - 1)$.*

Solution. (Video Solution) We claim that the order of a modulo $a^n - 1$ is n. Note that $a^n \equiv 1 \mod a^n - 1$. Furthermore, if k is a positive integer less than n, then $0 < a^k - 1 < a^n - 1$ which means $a^k - 1$ does not divide $a^n - 1$. Therefore, the order of a modulo $a^n - 1$ is n.

By Euler's Theorem, $a^{\varphi(a^n-1)} \equiv 1 \mod a^n - 1$. By Theorem 4.25 order of a mod $a^n - 1$ divides $\varphi(a^n - 1)$. By what we showed above this order is n. Hence $n \mid \varphi(a^n - 1)$, as desired. □

Example 4.8. *Let $a > 1$ be an integer. Prove that for every two positive integers m, n we have $a^m - 1 \mid a^n - 1$ if and only if $m \mid n$.*

Solution. (Video Solution) By an argument similar to the one used in the previous example, we can see that the order of a modulo $a^n - 1$ is n.

$a^m - 1 \mid a^n - 1$ if and only if $a^n \equiv 1 \mod (a^m - 1)$. By Theorem 4.25, this is equivalent to $\text{ord}_{a^m-1} a \mid n$. Since $\text{ord}_{a^m-1} a = m$, this is equivalent o $m \mid n$, as desired. □

Example 4.9. *Let p be a prime and k be a positive integer. Prove that*

$$1^k + 2^k + \cdots + (p-1)^k \overset{p}{\equiv} \begin{cases} 0 & \textit{if } (p-1) \nmid k \\ -1 & \textit{if } (p-1) \mid k \end{cases}$$

Solution.(Video Solution)

□

Using a similar strategy to the one described in the above example we can show the following:

Let F be a finite field with n elements, and let k be an integer. Then,

$$\sum_{x \in F \setminus \{0\}} x^k = \begin{cases} n-1 & \text{if } (n-1) \mid k \\ 0 & \text{if } (n-1) \nmid k \end{cases}$$

4.4 Further Examples

Example 4.10 (VTRMC 1979)**.** *Show, for all positive integers $n = 1, 2, \ldots$, that 14 divides $3^{4n+2} + 5^{2n+1}$.*

Scratch: One way of showing divisibility is by factoring. So, if we can factor the expression with one factor of 14, we will be able to solve the problem. The expression looks like $x^n + y^n$, except the exponents are not the same, but we do notice that the exponents can be made the same. Given that, the following is a possible solution:

Solution. $3^{4n+2} + 5^{2n+1} = 9^{2n+1} + 5^{2n+1}$. We know for every odd positive integer $x^n + y^n = (x+y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$. This $9^{2n+1} + 5^{2n+1} = (9+5)m = 14m$ for some integer m, which completes the proof. □

Another way of solving the problem is to take $3^{4n+2} + 5^{2n+1}$ modulo 14. We know $5 \equiv -9 \mod 14$. Thus, $3^{4n+2} + 5^{2n+1} \equiv 3^{4n+2} + (-9)^{2n+1} \equiv 0 \mod 14$, as desired.

Example 4.11 (Putnam 1999, B6). *Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s,n) = 1$ or $\gcd(s,n) = s$. Show that there exist $s,t \in S$, not necessarily distinct, such that $\gcd(s,t)$ is prime.*

Solution.(Video Solution) Note that the product of all elements of S is a positive integer that is not relatively prime to any element of S. Let a be the smallest positive integer that is not relatively prime to any of the elements of S. By assumption, there is some $s \in S$ for which $\gcd(a,s) = s$, i.e. $s \mid a$. Let p be a prime dividing s. Since a/p is less than a, by minimality of a, there is some $t \in S$ for which $\gcd(a/p,t) = 1$. Since by the choice of a, we know $\gcd(a,t) > 1$, and $\gcd(a,t) = \gcd(\frac{a}{p}p,t) = \gcd(p,t)$, we must have $\gcd(a,t) = p$. Note that since $s \mid a$, $gcd(s,t) = 1$ or p. On the other hand we know p divides both s and t, thus, $\gcd(s,t) = p$ is prime, as desired. $\square$

Example 4.12 (Putnam 2019, A1). *Determine all possible values of the expression*

$$A^3 + B^3 + C^3 - 3ABC$$

where A, B, and C are nonnegative integers.

Scratch: Here are a few ideas that come to mind:

- Try and see what small values are possible.
- We know we can factor $A^3 + B^3 + C^3 - 3ABC = (A+B+C)\dfrac{((A-B)^2 + (B-C)^2 + (C-A)^2)}{2}$.
- We note that the second expression is quadratic and we'd better make sure it is small in order to cover more integers. For that, we make sure A, B, and C are close to one another.
- Setting $A = B = C+1$ and $A = B = C-1$. This gives us a lot of possibilities. This takes care of all cases that are either 1 or -1 modulo 3.
- Next we try $A = B+1 = C+2$, and see that we can generate all multiples of 9.

This yields the following solution:

Solution. We will prove that the answer is

$$\boxed{\text{All nonnegative integers } n \text{ for which } n \equiv \pm 1 \mod 3 \text{ or } n \equiv 0 \mod 9.}$$

First note that $A^3 + B^3 + C^3 - 3ABC = (A+B+C)\dfrac{((A-B)^2 + (B-C)^2 + (C-A)^2)}{2}$.

Let k be a nonnegative integer.

Setting $A = B = k$, and $C = k+1$, we obtain $A^3 + B^3 + C^3 - 3ABC = 3k+1$.

Setting $A = B = k+1$, and $C = k$, we obtain $A^3 + B^3 + C^3 - 3ABC = 3k+2$.

Setting $A = k, B = k+1$, and $C = k+2$, we obtain $A^3 + B^3 + C^3 - 3ABC = 9k+9$.

Setting $A = B = C = 0$, we obtain $A^3 + B^3 + C^3 - 3ABC = 0$.

This shows all nonnegative integers n with $n \equiv \pm 1 \mod 3$ or $n \equiv 0 \mod 9$ can be obtained.

It is left to prove if $n = A^3 + B^3 + C^3 - 3ABC$ for integers A, B, C and n is a multiple of 3, then $9 \mid n$. By Fermat's Theorem, we see $A^3 + B^3 + C^3 \equiv A + B + C \equiv 0 \mod 3$. Therefore, $C \equiv -A - B \mod 3$. This implies $C = 3m - A - B$ for some integer m.

$$\begin{aligned} C^3 &= (3m - A - B)^3 \\ &= (3m)^3 - 3(3m)^2(A+B) + 3(3m)(A+B)^2 - (A+B)^3 \\ &\equiv -(A+B)^3 \mod 9 \\ &\equiv -A^3 - 3AB(A+B) - B^3 \mod 9 \end{aligned}$$

Therefore, $A^3 + B^3 + C^3 - 3ABC \equiv -3AB(A+B) - 3ABC \equiv -3AB(-C) - 3ABC \equiv 0 \mod 9$. This completes the proof. □

Example 4.13 (Putnam 2019, A5)**.** *Let p be an odd prime number, and let $\mathbb{F}_p$ denote the field of integers modulo p. Let $\mathbb{F}_p[x]$ be the ring of polynomials over $\mathbb{F}_p$, and let $q(x) \in \mathbb{F}_p[x]$ be given by*

$$q(x) = \sum_{k=1}^{p-1} a_k x^k,$$

where

$$a_k = k^{(p-1)/2} \mod p.$$

Find the greatest nonnegative integer n such that $(x-1)^n$ divides $q(x)$ in $\mathbb{F}_p[x]$.

Scratch: Here are some ideas that come to mind:

- The question is essentially asking us to find the multiplicity of 1. How do we find the multiplicity of a root of a polynomial? We must evaluate the polynomial and its derivatives at $x = 1$. When it stops being zero we know the multiplicity.
- Perhaps we could find an explicit formula for $q(x)$ and then use that? This seems to be a long shot, but is worth keeping in mind.
- We have seen $k^{(p-1)/2} \mod p$ when we want to see whether or not k is a quadratic residue. So, perhaps that might somehow be relevant?

The first idea seems to be plausible. So, let's evaluate $q(1) = \sum_{k=1}^{p-1} k^{(p-1)/2}$, $q'(1) = \sum_{k=1}^{p-1} k^{(p+1)/2}$, ... I have previously evaluated sums like that so at this point I know I can solve the problem, but if you have not seen this, note that if we use a primitive root, the sum turns into a geometric sum which can be evaluated. Putting these together we will obtain the following solution:

Solution. The answer is $\boxed{\frac{p-1}{2}}$

First we will prove the following claim:

Claim: For every positive integer $\ell < p-1$, $\sum_{k=1}^{p-1} k^\ell = 0$ in $\mathbb{F}_p$ and $\sum_{k=1}^{p-1} k^{p-1} \neq 0$ in $\mathbb{F}_p$.

Let g be a primitive root modulo p. We have $\sum_{k=1}^{p-1} k^\ell = \sum_{n=1}^{p-1} g^{n\ell} = \frac{g^{p\ell}-g^\ell}{g^\ell - 1}$. Note that since $0 < \ell < p-1$, the denominator is nonzero. The numerator is 0 modulo p, by Fermat's Little Theorem. This proves the claim when $\ell < p$. When $\ell = p-1$, each term is 1 modulo p, and thus the sum is $p-1 = -1 \neq 0$ in $\mathbb{F}_p$, as desired. $\square$

Let $n < (p-1)/2$. We will prove that $q^{(n)}(1) = 0$ in $\mathbb{F}_p$. Note that

$$q^{(n)}(1) = \sum_{k=1}^{p-1} k^{(p-1)/2} k(k-1)\cdots(k-n+1)$$

This expression is a linear combination of sums of form $\sum_{k=1}^{p-1} k^{(p-1)/2+m}$, where $m \leq n < (p-1)/2$. Since $(p-1)/2 + m < p-1$, by the claim proved above all of these sums are zero modulo p. Therefore, $q^{(n)}(1) = 0$.

When $n = (p-1)/2$, similar to what we did above all terms of $q^{((p-1)/2)}(1)$ are zero except $\sum_{k=1}^{p-1} k^{p-1}$ which is nonzero. Thus the multiplicity of 1 is $(p-1)/2$. $\square$

Example 4.14 (Putnam 1992, A3)**.** *For a given positive integer m, find all triples (n,x,y) of positive integers, with n relatively prime to m, which satisfy*

$$(x^2+y^2)^m = (xy)^n.$$

Scratch: Here are my first thought: The exponents are relatively prime. This means both sides must be perfect mn-th power. I don't really know what we should do next, but I do know that what I notices is worth getting started with. So, lets write $x^2+y^2 = z^n$ and $xy = z^m$. I now see that if we knew x and y were relatively prime, then they both must be m-th power. So, let's write $x = dx_1$ and $y = dy_1$, where d is their greatest common factor. This gives us $d^2(x_1^2+y_1^2) = z^n$ and $d^2x_1y_1 = z^m$. We know $x^2+y^2 \geq 2xy > xy$, which means $n > m$ or x_1y_1 must divide $x_1^2+y_1^2$. This yields the following solution:

Solution. We will show that

$$\boxed{\text{There are no solutions when } m \text{ is odd and when } m \text{ is even } x = y = 2^{m/2}, n = m+1 \text{ is the only solutions.}}$$

First note that when m is even, $x = y = 2^{m/2}$, and $n = m+1$, we have $(x^2+y^2)^m = (2^m+2^m)^m = (2^{m+1})^m = 2^{m^2+m}$. On the other hand, $(xy)^n = (2^{m/2}2^{m/2})^{m+1} = 2^{m^2+m}$, thus $(x^2+y^2)^m = (xy)^n$, as desired.

Now, we will show these are the only solutions.

Suppose (n,x,y) is a solution. Note that since m and n are relatively prime and the two sides are perfect m-th and n-th powers, they both must be perfect mn-th power. Thus, there is a positive integer z for which $x^2+y^2 = z^n$ and $xy = z^m$. By AM-GM we have $x^2+y^2 \geq 2xy$ or $z^n \geq 2z^m > z^m$, which implies $n > m$. Therefore, xy divides x^2+y^2. Let $d = \gcd(x,y)$ and write $x = dx_1, y = dy_1$. We have $d^2x_1y_1$ divides $d^2(x_1^2+y_1^2)$ or $x_1y_1 \mid (x_1^2+y_1^2)$. This implies

$x_1 \mid x_1^2 + y_1^2$. Since $x_1 \mid x_1^2$, we have $x_1 \mid y_1^2$, but we know x_1 and y_1 are relatively prime. Therefore, $x_1 = 1$. Similarly, $y_1 = 1$. Thus, $x = y$. This implies $2x^2 = z^n$ and $x^2 = z^m$, hence $2z^m = z^n$ or $z^{n-m} = 2$, which is only possible when $z = 2$ and $n = m+1$. Therefore, $x^2 = 2^m$, which means m is even and $x = 2^{m/2}$, as desired. $\square$

Example 4.15 (USTST, 2020)**.** *Find all pairs of positive integers (a,b) satisfying all of the following conditions:*

(a) a divides $b^4 + 1$.

(b) b divides $a^4 + 1$.

(c) $\lfloor \sqrt{a} \rfloor = \lfloor \sqrt{b} \rfloor$.

Scratch: Initial thoughts:

- Perhaps we could look at some small examples.
- Usually for problems such as these we may be able to combine the two divisibility conditions into one. Can we do that here?
- The last condition tells us a and b are pretty close, so maybe that matters. For example if a and b were the same then $a \mid a^4 + 1$ implies $a = 1$.

First of all note that $a \mid b^4 + 1$ implies a and b do not share any common factors more than 1. Next, for simplicity assume $a < b$.

$a = 1$ yields $b \mid 2$, i.e. $(a,b) = (1,1),(1,2)$ are two possible solutions.

$a = 2$ yields $b \mid 5$, however since $\lfloor \sqrt{b} \rfloor = 2$, this is impossible.

$a = 3$ yields $3 \mid b^4 + 1$, but that is impossible, because b^2 is either 0 or 1 modulo 3. Thus, $b^4 + 1$ is either 1 or 2 modulo 3.

$a = 4$ yields $4 \mid b^4 + 1$, which is similarly impossible.

$a = 5$ yields $5 \mid b^4 + 1$, which is similarly impossible.

$a = 6$ is impossible for the same reason that $a = 3$ is not possible.

$a = 7$ yields $7 \mid b^4 + 1$, which is also not possible.

This means a cannot be a multiple of 3, 4, 5 or 7. This means the next smallest possible values of a are 11 and 13. Neither of which works for similar reasons.

It may be tempting to think $a \mid b^4+1$ is never possible, but clearly this is not the case, since first of all $a=2$ was possible and second, if that were the case the third condition would be unnecessary. So, we should somehow use the given assumption that a and b are somehow "close" to each other.

$a \mid b^4+1$ and $b \mid a^4+1$ imply that both a and b divide a^4+b^4+1, but that may not be very helpful. Since we know a and b are close we know $b-a$ is "small". So, perhaps we could use this fact. Note that all terms of the expansion of $(b-a)^4$ are divisible by ab except a^4+b^4, which means $(b-a)^4+1$ is divisible by ab. It looks like we are getting closer to a solution. Let's not approximate ab and $(b-a)^4+1$ to see if this latter observation could help.

Assume $\lfloor\sqrt{a}\rfloor = \lfloor\sqrt{b}\rfloor = n$. This means $n^2 \le a,b < n^2+2n+1$. So, the maximum value of $(b-a)^4+1$ is $(2n-0)^4+1=16n^4+1$, while ab is roughly n^4. This means their ratio should not exceed 16. So, we are left with finitely many possibilities and we can hopefully get those cases done. Now, we are ready to write down a full solution:
Solution. The only pairs (a,b) are $(1,1),(1,2)$ and $(2,1)$.

First, note that clearly these pairs all work. Next, without loss of generality assume $a \le b$.

Since $a \mid b^4+1$, no prime dividing a can divide b and thus $\gcd(a,b)=1$. Note that $(b-a)^4+1 \equiv (0-a)^4+1 \equiv 0 \mod b$. Similarly $(b-a)^4+1$ is divisible by a. Therefore, $(b-a)^4+1$ is divisible by ab. Assume $(b-a)^4+1=mab$ for some positive integer m.

Let $\lfloor\sqrt{a}\rfloor = \lfloor\sqrt{b}\rfloor = n$. We know

$$n \le \sqrt{a},\sqrt{b} < n+1 \Rightarrow n^2 \le a,b \le n^2+2n.$$

This means $b-a \le 2n$. Since b and a are relatively prime, they cannot be 0 and $2n$ and thus $(b-a)^4+1 \le (2n-1)^4+1 \le 16n^4$. On the other hand $ab \ge n^4$ and thus $m \le 16$.

Note that for every integer x the integer x^4+1 is not divisible by $3,4,5,7,11$ or 13. Therefore, m cannot be any of the integers $3,4,5,6,7,8,9,10,11,12,13,14,15,16$. Thus, $m=1,2$.

When $m=2$ we obtain $(b-a)^4+1=2ab$ must be even, and thus $b-a$ must be odd. Therefore, a or b is even which means $2ab$ is divisible by 4. This means $(b-a)^4+1$ is divisible by 4, which we showed is impossible.

This means $(b-a)^4+1=ab$. Setting $b-a=c$ we will obtain $c^4+1=a^2+ac$. Writing this down as a quadratic equation $a^2+ca-c^4-1=0$ and using the fact that the discriminant must be a perfect square we obtain: $c^2+4c^4+c=k^2$ for some positive integer k. Note that $k^2>4c^4$, which means it must be at least $(2c^2+1)^2=4c^4+4c^2+1$. This implies

$$c^2+4c^4+4 \ge 4c^4+4c^2+1 \Rightarrow 3 \ge 3c^2 \Rightarrow c=1,0.$$

This shows $a=1$. The solution is complete. $\square$

Example 4.16 (IMC 2022, Problem 3)**.** *Let p be a prime number. A flea is staying at point 0 of the real line. At each minute, the flea has three possibilities: to stay at its position, or to move by 1 to the left or to the right. After $p-1$ minutes, it wants to be at 0 again. Denote by $f(p)$ the number of its strategies to do this (for example, $f(3)=3$: it may either stay at 0 for the entire time, or go to the left and then to the right, or go to the right and then to the left). Find $f(p)$ modulo p.*

Solution. (Video Solution) We claim the answer is

$$f(p) \pmod{p} = \begin{cases} 0 & \text{if } p=3 \\ 1 & \text{if } p \equiv 1 \pmod{3} \\ p-1 & \text{if } p \equiv -1 \pmod{3} \end{cases}$$

We see that $f(2)=1$ and $f(3)=3$ satisfy the above. From now on assume $p>3$.

Since the flea returns to the original position, the number of moves to the left and right must be the same. Suppose there are k moves to the right, k moves to the left and $p-1-2k$ minutes that the flea does not move. This can be done in $\dfrac{(p-1)!}{k!k!(p-1-2k)!}$ ways. Therefore,

$$f(p) = \sum_{k=0}^{(p-1)/2} \frac{(p-1)!}{k!k!(p-1-2k)!}.$$

We will now evaluate $\dfrac{(p-1)!}{k!k!(p-1-2k)!}$ in $\mathbb{Z}_p$, the field of integers modulo p. Note the following:

$$\frac{(p-1)!}{k!k!(p-1-2k)!} = \frac{(p-2k)(p-2k+1)\cdots(p-1)}{k!k!} = \frac{(-2k)(-2k+1)\cdots(-1)}{k!k!} = \frac{(-1)^{2k}(2k)!}{(k!)^2}$$

Separating $(2k)!$ into even and odd terms, we obtain the following:

$$\frac{(-1)^{2k}(2k)!}{(k!)^2} = \frac{2\cdot 4\cdots(2k)\cdot 1\cdot 3\cdots(2k-1)}{(k!)^2} = \frac{2^k k!\cdot 1\cdot 3\cdots(2k-1)}{(k!)^2} = \frac{2^k\cdot 1\cdot 3\cdots(2k-1)}{k!}$$

Replacing each odd integer ℓ in the numerator by $-(p-\ell)$ we obtain the following:0

$$\frac{2^k(-1)^k\cdot(p-1)\cdot(p-3)\cdots(p-2k+1)}{k!} = \frac{2^k(-1)^k 2^k\cdot(\frac{p-1}{2})\cdot(\frac{p-3}{2})\cdots(\frac{p-2k+1}{2})}{k!} = (-4)^k\binom{\frac{p-1}{2}}{k}.$$

Therefore,

$$f(p) \equiv \sum_{k=0}^{(p-1)/2} (-4)^k \binom{\frac{p-1}{2}}{k} = (1-4)^{\frac{p-1}{2}} = (-3)^{\frac{p-1}{2}} = (-1)^{\frac{p-1}{2}} 3^{\frac{p-1}{2}}.$$

By the Euler's Criterion, $3^{(p-1)/2} \equiv \left(\frac{3}{p}\right) \pmod{p}$, the legendre symbol that determines if 3 is a quadratic residue or a quadratic non-residue modulo p. By the Quadratic Reciprocity, we have

$$\left(\frac{3}{p}\right) = (-1)^{(p-1)/2}\left(\frac{p}{3}\right) \Rightarrow (-1)^{(p-1)/2}\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) \Rightarrow f(p) \equiv \left(\frac{p}{3}\right) \pmod{p}.$$

We know $\left(\frac{p}{3}\right)=1$ if $p\equiv 1 \mod 3$ and $\left(\frac{p}{3}\right)=-1$ if $p\equiv -1 \mod 3$. This completes the proof. $\square$

Example 4.17 (Putnam 2018, B3). *Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides 2^n-1, and $n-2$ divides 2^n-2.*

Solution. (Video Solution) We will use the following well-known fact from number theory. For a proof of this fact see Example 4.8.

$$\text{For positive intgeers } m,n \text{ we have } (2^m-1) \mid (2^n-1) \text{ if and only if } m \mid n. \qquad (*)$$

Note that $n \mid 2^n$ if and only if n is a power of 2, since $2^n > n$. So, let's assume $n = 2^m$ for some integer $m \geq 0$.

$n-1$ dvides 2^n-1 if and only if $(2^m-1) \mid (2^n-1)$ if and only if $m \mid n$, using $(*)$. Since n is a power of 2, we must have $m = 2^k$ for some integer $k \geq 0$. In which case, since $n = 2^m > m$, we would have $m \mid n$.

$n-2 = 2^{2^k}-2$ divides 2^n-2 if and only if $(2^{2^k-1}-1) \mid (2^{n-1}-1)$ if and only if $(2^k-1) \mid (n-1)$, by $(*)$. Since $n-1 = 2^{2^k}-1$, applying $(*)$ again we conclude, this is equivalent to $k \mid 2^k$. Therefore, similar to above k must be a power of 2. Therefore, the given condistions are equivalent to $n = 2^{2^{2^\ell}}$ for some integer $\ell \geq 0$. Since $n < 10^{100}$ we must have

$$n = 2^{2^{2^\ell}} < 10^{100}$$

$\ell = 0,1,2$ yield $n = 4, 16, 2^{16}$ which are all clearly less than 10^{100}. Next, we have $\ell = 3$, which yields $n = 2^{2^8} = 2^{256} = 8^{256/3} < 10^{100}$. When we substitute $\ell = 4$ we obtain $n = 2^{2^{16}} = 16^{2^{14}} > 10^{100}$. So the only values of n are $n = 4, 16, 2^{16}$ and 2^{256}. □

Example 4.18. *Let p be a prime, and m be a positive integer. Define the set S by*

$$S = \{n \in \mathbb{N} \mid p \nmid n \text{ and } 1 \leq n \leq p^m\}.$$

Evaluate

$$\prod_{n \in S} n \pmod{p^m}$$

Solution. (Video Solution) Note that for every $n \in S$, since $p \nmid n$ we have $\gcd(n, p^m) = 1$. Therefore, each element in S has a multiplicative inverse in S. We can pair up each element of S with its inverse and replace their product by 1, as long as that element is not the same as its inverse. In other words, the product of elements of S is the same as the product of all $n \in S$ for which $n^2 \equiv 1 \pmod{p^m}$. Let us now find out all $n \in S$ with $n^2 \equiv 1 \pmod{p^m}$. This is equivalence to $p^m \mid n^2-1$ if and only if $p^m \mid (n-1)(n+1)$. We have $\gcd(n+1, n-1) = \gcd(n+1, 2) = 1$ or 2. If p is odd, then p^m must either divide $n-1$ or $n+1$, or $n \equiv \pm 1 \pmod{p^m}$. Therefore, the product is -1 when p is odd.

For $p = 2$ we have $2^m \mid (n-1)(n+1)$, and thus $(n-1)(n+1)$ is even and hence $\gcd(n-1, n+1) = 2$. If $m \geq 2$, then

$$2^m \mid (n-1)(n+1) \Leftrightarrow 2^{m-2} \mid \frac{n-1}{2} \cdot \frac{n+1}{2} \Rightarrow 2^{m-2} \mid \frac{n-1}{2} \text{ or } 2^{m-2} \mid \frac{n+1}{2}$$

Therefore, $n-1 = 2^{m-1}k$ or $n+1 = 2^{m-1}k$ for an integer k. Since $1 \leq n \leq 2^m$ we have $n \equiv \pm 1, 2^{m-1} \pm 1 \pmod{2^m}$. Therefore, $\prod_{n \in S} n \equiv 1(-1)(2^{m-1}+1)(2^{m-1}-1) = -2^{2m-2}+1 \equiv 1 \pmod{2^m}$, since $2m-2 \geq m$.

For $p=2, m=1$ the given product is 1. Therefore,

$$\prod_{n\in S} n \pmod{p^m} = \begin{cases} p^m-1 & \text{if } p \text{ is odd or } (p=2 \text{ and } m\geq 2) \\ 1 & \text{if } p=2 \text{ and } m=1 \end{cases}$$

□

Example 4.19. *Prove that there is no right triangle with integer sides, where both legs are perfect squares.*

Solution.(Video Solution) We will do so by contradiction. Assume among all non-trivial positive integer solutions, (a,b,c) is one with the smallest c. We note that $\gcd(a,b,c)=1$. Otherwise, if $p>1$ is a common factor of a,b,c, then $p^4 \mid a^4+b^4=c^2$ and thus $p^2 \mid c$. Therefore, $(a/p)^4+(b/p)^4=(c/p^2)^2$, which contradicts the minimality of c.

Since (a^2,b^2,c) is a primitive Pythagorean triple, there are positive, relatively prime positive integers m,n for which precisely one of m, or n is even, and that

$$c=m^2+n^2,\ a^2=2mn,\ b^2=m^2-n^2$$

Note that since $n^2+b^2=m^2$, and $\gcd(m,n)=1$, the triple (n,b,m) is also a primitive Pythagorean triple. This means m is odd and n is even. Therefore, there are positive relatively prime integers u,v for which

$$m=u^2+v^2,\ n=2uv,\ b=u^2+v^2.$$

Since $a^2=(2n)m$ is a perfect square, m,n are relatively prime, m is odd and n is even, $\gcd(m,2n)=1$. Thus m and $2n$ are both perfect squares. From the fact that $2n=4uv$ is a perfect square and $\gcd(u,v)=1$ we conclude that both u and v are perfect squares. Since m is a perfect square, we conclude $m=x^2, u=y^2, v=z^2$ for some positive integers x,y,z. The equality $m=u^2+v^2$ implies $x^2=y^4+z^4$. We will now show $x<c$, and that yields a contradiction.

$$x\leq x^2=m<m^2+n^2=c \Rightarrow x<c.$$

This contradicts the minimality of c. □

The method used in the above example is called the method of Infinite Descent. This is a special kind of proof by contradiction that is used to prove certain Diophantine equations do not have non-trivial solutions. We start with choosing a minimal solution and then find a smaller one, hence yielding a contradiction.

Example 4.20 (IMO 2023, Problem 1)**.** *Determine all composite integers $n>1$ that satisfy the following property:*

If $1=d_1<d_2<\cdots<d_k=n$ are all the positive divisors of n, then d_i divides $d_{i+1}+d_{i+2}$ for every $1\leqslant i\leqslant k-2$.

Solution.(Video Solution)

□

Example 4.21 (Shortlisted IMO, 2019)**.** *Find all triples (a,b,c) of positive integers such that $a^3+b^3+c^3=(abc)^2$.*

Scratch: Generally when solving Diophantine equations, we would use congruences or inequalities to restrict the possibilities of variables. In other words, either algebraic properties or number theoretical properties–and usually both– are used. The first thing that I tried was to try to make a,b,c relatively prime. That would allow us to use number theoretical properties of these numbers. Setting $d = \gcd(a,b)$ we can write $a = dx, b = dy$. We know $d^3 \mid a^3 + b^3$ and $d^4 \mid a^2b^2c^2$. This means $d^3 \mid c^3$. Writing $c = dz$ we obtain $x^3 + y^3 + z^3 = d^3z^2y^2z^2$ and we also know $\gcd(x,y) = 1$. If x,z had a common factor p then p^3 would divide x^3. So, we may assume x,y,z are pairwise relatively prime. I also note that if I have a solution to $x^3 + y^3 + z^3 = d^3(xyz)^2$, then (dx,dy,dz) would be a solution to $a^3 + b^3 + c^3 = (abc)^2$. So, we are not making the problem more difficult by focusing on the new equation $x^3 + y^3 + z^3 = d^3(xyz)^2$ under the additional condition that x,y,z are pairwise relatively prime. In order to be able to use the fact that x,y,z are pairwise relatively prime, we probably would need to use the fact that

$$m \mid kn, \text{ and } \gcd(m,n) = 1 \Rightarrow m \mid k$$

The equation $x^3 + y^3 + z^3 = d^3(xyz)^2$ implies $x^2 \mid y^3 + z^3$. But the fact that x,y,z are pairwise relatvely prime does not immediately give us anything. At this point I decided to try another approach while keeping what we did at the back of our mind.

I noticed that the right hand side $(abc)^2$ is a degree 6 polynomial while the left hand side $a^3 + b^3 + c^3$ is a cubic. In order to take advantage of this we assume $a \geq b \geq c$. That yields the following solution:

Solution.(Video Solution) Suppose $a \geq b \geq c$. We note that

$$a^2 \mid a^3 - (abc)^2 \Rightarrow a^2 \mid b^3 + c^3 \Rightarrow a^2 \leq b^3 + c^3 \leq 2b^3 \Rightarrow \sqrt[3]{\frac{a^2}{2}} \leq b.$$

This yields

$$3a^3 \geq a^3 + b^3 + c^3 = a^2b^2c^2 \geq a^2\sqrt[3]{\frac{a^4}{4}}c^2 = a^3\sqrt[3]{\frac{a}{4}}c^2$$

Therefore, $3 \geq \sqrt[3]{\frac{a}{4}}c^2$. If $c \geq 2$, then $3 \geq \sqrt[3]{\frac{2}{4}}4 \Rightarrow 27 \geq 32$, a contradiction. Therefore, $c = 1$. We can now re-write the given equation as $a^3 + b^3 + 1 = a^2b^2$. We also know $a \geq b$. Similar to above $a^2 \mid b^3 + 1$. If $a^2 = b^3 + 1$, then we would get

$$a^3 + a^2 = a^2b^2 \Rightarrow a + 1 = b^2 \Rightarrow (b^2 - 1)^2 = b^3 + 1$$

Dividing both sides by $b + 1$ we obtain

$$(b^2 - 1)(b - 1) = b^2 - b + 1 \Rightarrow b^3 - b^2 - b + 1 = b^2 - b + 1 \Rightarrow b^3 - 2b^2 = 0 \Rightarrow b = 2.$$

So, we obtain $(1,2,3)$ as a solution. Now, assume $b^3 + 1 \geq 2a^2$. Note that $b = 1$ implies $a^3 + 2 = a^2$ which is not possible. Therefore,

$$2b^3 > b^3 + 1 \geq 2a^2 \Rightarrow b^3 > a^2 \Rightarrow b^2 > a\sqrt[3]{a}.$$

$$2a^3 \geq a^3 + b^3 + 1 = a^2b^2 > a^3\sqrt[3]{a} \Rightarrow 8 > a.$$

Now, there are a few values of a that we need to test. This is not very difficult to do given that $2 \le b < a \le 7$. We can check $b = 2,3,4,5,6$ and we conclude that $(1,2,3)$ and all of its permutations is the only solution to this Diophantine equation. □

Example 4.22. *For every integer $n \geq 1$ consider the $n \times n$ table with entry $\left\lfloor \frac{ij}{n+1} \right\rfloor$ at the intersection of row i and column j, for every $i = 1, \ldots, n$ and $j = 1, \ldots, n$. Determine all integers $n \geq 1$ for which the sum of the n^2 entries in the table is equal to $\frac{1}{4}n^2(n-1)$.*

Solution.(Video Solution) The equality holds if and only if $n+1$ is prime.

Let $S = \sum_{i,j=1}^{n} \left\lfloor \frac{ij}{n+1} \right\rfloor$. Since $1 \le j \le n$ if and only if $1 \le n+1-j \le n$, we can re-write the sum as follows:

$$S = \sum_{i,j=1}^{n} \left\lfloor \frac{i(n+1-j)}{n+1} \right\rfloor = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(i + \left\lfloor \frac{-ij}{n+1} \right\rfloor\right) = \sum_{i-1}^{n} ni + \sum_{i,j=1}^{n} \left\lfloor \frac{-ij}{n+1} \right\rfloor = \frac{n^2(n+1)}{2} + \sum_{i,j=1}^{n} \left\lfloor \frac{-ij}{n+1} \right\rfloor .$$

Adding this to the original sum we obtain the following:

$$2S = \frac{n^2(n+1)}{2} + \sum_{i,j=1}^{n}\left(\left\lfloor \frac{ij}{n+1} \right\rfloor + \left\lfloor \frac{-ij}{n+1} \right\rfloor\right) \Rightarrow S = \frac{n^2(n+1)}{4} + \frac{1}{2}\sum_{i,j=1}^{n}\left(\left\lfloor \frac{ij}{n+1} \right\rfloor + \left\lfloor \frac{-ij}{n+1} \right\rfloor\right)$$

Therefore, $S = \frac{n^2(n-1)}{4}$, if and only if

$$\frac{1}{2}\sum_{i,j=1}^{n}\left(\left\lfloor \frac{ij}{n+1} \right\rfloor + \left\lfloor \frac{-ij}{n+1} \right\rfloor\right) = \frac{n^2(n-1)}{4} - \frac{n^2(n+1)}{4} = -\frac{n^2}{2}.$$

This is equivalent to

$$\sum_{i,j=1}^{n}\left(\left\lfloor \frac{ij}{n+1} \right\rfloor + \left\lfloor \frac{-ij}{n+1} \right\rfloor\right) == -n^2 \qquad (*)$$

Note that for every real number x, we have

$$\lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} 0 & \text{if } x \in \mathbb{Z} \\ -1 & \text{if } x \notin \mathbb{Z} \end{cases}$$

Since there are n^2 terms in the sum $(*)$, each of which is either 0 or -1, the sum is equal to $-n^2$ if and only if

$$\left\lfloor \frac{ij}{n+1} \right\rfloor + \left\lfloor \frac{-ij}{n+1} \right\rfloor = -1, \text{ for all } 1 \le i,j \le n.$$

This is equivalent to $\frac{ij}{n+1} \notin \mathbb{Z}$. Which is equivalent to $n+1$ being a prime. □

Example 4.23 (IMO 2021, Problem 1)**.** *Let $n \geq 100$ be an integer. The numbers $n, n+1, \ldots, 2n$ are written on $n+1$ cards, one number per card. The cards are shuffled and divided into two piles. Prove that one of the piles contains two cards such that the sum of their numbers is a perfect square.*

Scratch: After trying this for 100, We will find three distinct numbers between n and $2n$ inclusive for which the sum of each pair of them is a perfect square. Let's

Solution.(Video Solution) □

Example 4.24 (IMO 2022, Shortlisted Problem, N2). *Find all integers $n > 2$ for which*

$$n! \mid \prod_{\substack{p<q\le n, \\ p,\ q \text{ primes}}} (p+q).$$

Solution.(Video Solution)

□

Example 4.25 (IMC 2023, Problem 4). *Let p be a prime number and let k be a positive integer. Suppose that the numbers $a_i = i^k + i$ for $i = 0, 1, \ldots, p-1$ form a complete residue system modulo p. What is the set of possible remainders of a_2 upon division by p?*

Scratch: First, I tried adding a_i's to obtain

$$\sum_{i=1}^{p-1}(i^k+i) = \sum_{i=1}^{p=1} i \Rightarrow \sum_{i=1}^{p-1} i^k = 0.$$

Solution.(Video Solution)

□

Example 4.26 (IMO Shortlisted Problem, N8). *Prove that $5^n - 3^n$ does not divide $2^n + 65$ for any positive integer n,*

Solution. (Video Solution)

□

Example 4.27 (IMO 2022, Shortlisted Problem, N3). *Let a and d be relatively prime integers more than 1. Define a sequence x_k, recursively, by $x_1 = 1$ and*

$$x_{k+1} = \begin{cases} x_k + d & \text{if } a \nmid x_k \\ x_k/a & \text{if } a \mid x_k \end{cases}$$

Find the greatest positive integer n for which there exists an index k such that x_k is divisible by a^n.

Solution.(Video Solution)

□

Example 4.28 (Putnam 1989, A1). *How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?*

Solution.(Video Solution)

□

Example 4.29. *Find all positive integers n for which $\varphi(n)$ divides n.*

Solution.(Video Solution)

□

4.5 General Strategies

- One method that we can use to show a Diophantine equation does not have an integer solution is Infinite Descent. Another one is to show the equation fails in a certain mod.
- As is the case in many instances, starting with small examples help us get intuition.
- When dealing with Diophantine equations or finding integers satisfying certain divisibility conditions, taking advantage of the fact that divisors of a positive integer do not exceed that integer is often helpful. This often helps us control the growth of one side of the Diophantine equation.

4.6 Exercises

Exercise 4.1. Let $f(x) = ax^2 + bx + c$ be a quadratic function with integer coefficients, for which for every integer n, there is an integer c_n for which n divides $f(c_n)$. Prove that both roots of $f(x)$ are rational.

Exercise 4.2. Suppose a, b, c are positive integers for which $\dfrac{a\sqrt{3}+b}{b\sqrt{3}+c}$ is an integer. Prove that $a+b+c$ divides $a^2+b^2+c^2$.

Exercise 4.3 (VTRMC 1981)**.** The number $2^{48}-1$ is exactly divisible by what two numbers between 60 and 70?

Exercise 4.4 (VTRMC 1983)**.** A positive integer N (in base 10) is called special if the operation C of replacing each digit d of N by its nine's-complement $9-d$, followed by the operation R of reversing the order of the digits, results in the original number. (For example, 3456 is a special number because $R[(C3456)] = 3456$.) Find the sum of all special positive integers less than one million which do not end in zero or nine.

Exercise 4.5 (VTRMC 1984)**.** Find the base 10 units digit of the sum $\sum\limits_{k=1}^{99} k!$.

Exercise 4.6 (VTRMC 1984)**.** Consider any three consecutive positive integers. Prove that the cube of the largest cannot be the sum of the cubes of the other two.

Exercise 4.7 (VTRMC 1986)**.** Find all pairs N, M of positive integers, $N < M$, such that

$$\sum_{j=N}^{M} \frac{1}{j(j+1)} = \frac{1}{10}$$

Exercise 4.8 (VTRMC 1987)**.** Let $a_1, a_2, \ldots, a_n$ be an arbitrary rearrangement of $1, 2, \ldots, n$. Prove that if n is odd, then $(a_1-1)(a_2-2)\ldots(a_n-n)$ is even.

Exercise 4.9 (VTRMC 1988)**.** A man goes into a bank to cash a check. The teller mistakenly reverses the amounts and gives the man cents for dollars and dollars for cents. (Example: if the check was for \$5.10, the man was given \$10.05.). After spending five cents, the man finds that he still has twice as much as the original check amount. What was the original check amount? Find all possible solutions.

Exercise 4.10 (VTRMC 1988)**.** Let a be a positive integer. Find all positive integers n such that $b = a^n$ satisfies the condition that a^2+b^2 is divisible by $ab+1$.

Exercise 4.11 (VTRMC 1989)**.** Let a,b,c,d be distinct integers such that the equation

$$(x-a)(x-b)(x-c)(x-d)-9=0$$

has an integer root r. Show that $4r=a+b+c+d$.

Exercise 4.12 (Putnam 1991, B4)**.** Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j}\equiv 2^p+1 \pmod{p^2}.$$

Exercise 4.13 (Putnam 1991, B5)**.** Let p be an odd prime and let $\mathbb{Z}_p$ denote (the field of) integers modulo p. How many elements are in the set

$$\{x^2 : x\in\mathbb{Z}_p\}\cap\{y^2+1 : y\in\mathbb{Z}_p\}?$$

Exercise 4.14 (Putnam 1993, B1)**.** Find the smallest positive integer n such that for every integer m with $0<m<1993$, there exists an integer k for which

$$\frac{m}{1993}<\frac{k}{n}<\frac{m+1}{1994}.$$

Exercise 4.15 (Putnam 1994, B1)**.** Find all positive integers n that are within 250 of exactly 15 perfect squares.

Exercise 4.16 (Putnam 1994, B6)**.** For any integer n, set

$$n_a=101a-100\cdot 2^a.$$

Show that for $0\le a,b,c,d\le 99$, $n_a+n_b\equiv n_c+n_d \pmod{10100}$ implies $\{a,b\}=\{c,d\}$.

Exercise 4.17 (VTRMC 1995)**.** If n is a positive integer larger than 1 , let $n=\prod p_i^{k_i}$ be the unique prime factorization of n, where the p_i 's are distinct primes, $2,3,5,7,11,\ldots$, and define $f(n)$ by $f(n)=\sum k_ip_i$ and $g(n)$ by $g(n)=\lim_{m\to\infty}f^m(n)$, where f^m is meant the m-fold application of f. Then n is said to have property H if $n/2<g(n)<n$.

(i) Evaluate $g(100)$ and $g\left(10^{10}\right)$.

(ii) Find all positive odd integers larger than 1 that have property H.

Exercise 4.18 (Putnam 1995, A3)**.** The number $d_1d_2\ldots d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1e_2\ldots e_9$ is such that each of the nine 9-digit numbers formed by replacing just one of the digits d_i is $d_1d_2\ldots d_9$ by the corresponding digit e_i ($1\le i\le 9$) is divisible by 7. The number $f_1f_2\ldots f_9$ is related to $e_1e_2\ldots e_9$ is the same way: that is, each of the nine numbers formed by replacing one of the e_i by the corresponding f_i is divisible by 7. Show that, for each i, d_i-f_i is divisible by 7. [For example, if $d_1d_2\ldots d_9=199501996$, then e_6 may be 2 or 9, since 199502996 and 199509996 are multiples of 7.]

Exercise 4.19 (Putnam 1995, B6)**.** For a positive real number α, define

$$S(\alpha)=\{\lfloor n\alpha\rfloor : n=1,2,3,\ldots\}.$$

Prove that $\{1,2,3,\ldots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha),S(\beta)$ and $S(\gamma)$. [As usual, $\lfloor x\rfloor$ is the greatest integer $\le x$.]

Exercise 4.20 (Putnam 1996, A5)**.** If p is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.

Exercise 4.21 (Putnam 1997, B3)**.** For each positive integer n, write the sum $\sum_{m=1}^{n} 1/m$ in the form p_n/q_n, where p_n and q_n are relatively prime positive integers. Determine all n such that 5 does not divide q_n.

Exercise 4.22 (Putnam 1997, B5)**.** Prove that for $n \geq 2$,

$$\overbrace{2^{2^{\cdots^{2}}}}^{n \text{ terms}} \equiv \overbrace{2^{2^{\cdots^{2}}}}^{n-1 \text{ terms}} \pmod{n}.$$

Exercise 4.23 (Putnam 1998, A4)**.** Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

Exercise 4.24 (Putnam 1998, B5)**.** Let N be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = \underbrace{1111\cdots111}_{1998 \text{ times}}.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$.

Exercise 4.25 (Putnam 1998, B6)**.** Prove that, for any integers a, b, c, there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.

Exercise 4.26 (VTRMC 1999)**.** A set S of distinct positive integers has property ND if no element x of S divides the sum of the integers in any subset of $S \backslash \{x\}$. Here $S \backslash \{x\}$ means the set that remains after x is removed from S.

(i) Find the smallest positive integer n such that $\{3, 4, n\}$ has property ND.

(ii) If n is the number found in (i), prove that no set S with property ND has $\{3, 4, n\}$ as a proper subset.

Exercise 4.27 (Putnam 2000, A2)**.** Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

Exercise 4.28 (Putnam 2000, A6)**.** Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

Exercise 4.29 (Putnam 2000, B2)**.** Prove that the expression

$$\frac{gcd(m,n)}{n} \binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

Exercise 4.30 (VTRMC 2001)**.** Let a_n be the n-th positive integer k such that the greatest integer not exceeding $\sqrt{k}$ divides k, so the first few terms of $\{a_n\}$ are $\{1,2,3,4,6,8,9,12,...\}$. Find a_{10000} and give reasons to substantiate your answer.

Exercise 4.31 (Putnam 2001, A5)**.** Prove that there are unique positive integers a, n such that $a^{n+1}-(a+1)^n = 2001$.

Exercise 4.32 (Putnam 2003, B3)**.** Show that for each positive integer n,

$$n! = \prod_{i=1}^{n} \operatorname{lcm}\{1,2,\ldots,\lfloor n/i \rfloor\}.$$

(Here lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.)

Exercise 4.33 (VTRMC 2005)**.** Find the largest positive integer n with the property that $n+6(p^3+1)$ is prime whenever p is a prime number such that $2 \leq p < n$. Justify your answer.

Exercise 4.34 (Putnam 2005, B2)**.** Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n-4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

Exercise 4.35 (VTRMC 2006)**.** Find, and give a proof of your answer, all positive integers n such that neither n nor n^2 contain a 1 when written in base 3.

Exercise 4.36 (VTRMC 2006)**.** Recall that the Fibonacci numbers $F(n)$ are defined by $F(0) = 0, F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Determine the last digit of $F(2006)$ (e.g. the last digit of 2006 is 6).

Exercise 4.37 (VTRMC 2008)**.** Find all pairs of positive integers a,b such that $ab-1$ divides $a^4 - 3a^2 + 1$.

Exercise 4.38 (Putnam 2008, A3)**.** Start with a finite sequence $a_1, a_2, \ldots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\operatorname{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means greatest common divisor and lcm means least common multiple.)

Exercise 4.39 (Putnam 2008, B1)**.** What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

Exercise 4.40 (Putnam 2008, B4)**.** Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \ldots, h(p^2-1)$ are distinct modulo p^2. Show that $h(0), h(1), \ldots, h(p^3-1)$ are distinct modulo p^3.

Exercise 4.41 (VTRMC 2009)**.** Given that

$$40! = abcdef283247897734345611269596115894272pqrstuvwx$$

find p,q,r,s,t,u,v,w,x, and then find a,b,c,d,e,f.

Exercise 4.42 (VTRMC 2009)**.** Let n be a nonzero integer. Prove that $n^4 - 7n^2 + 1$ can never be a perfect square (i.e. of the form m^2 for some integer m).

Exercise 4.43 (VTRMC 2010)**.** For n a positive integer, define $f_1(n) = n$ and then for i a positive integer, define $f_{i+1}(n) = f_i(n)^{f_i(n)}$. Determine $f_{100}(75)$ mod 17 (i.e. determine the remainder after dividing $f_{100}(75)$ by 17, an integer between 0 and 16). Justify your answer.

Exercise 4.44 (Putnam 2010, A4)**.** Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

Exercise 4.45 (Putnam 2011, B6)**.** Let p be an odd prime. Show that for at least $\dfrac{p+1}{2}$ values of n in $\{0, 1, \dots, p-1\}$, $\sum_{k=0}^{p-1} k!n^k$ is not divisible by p.

Exercise 4.46 (VTRMC 2011)**.** Let m, n be positive integers and let $[a]$ denote the residue class mod mn of the integer a (thus $\{[r] \mid r \text{ is an integer }\}$ has exactly mn elements). Suppose the set $\{[ar] \mid r \text{ is an integer }\}$ has exactly m elements. Prove that there is a positive integer q such that q is prime to mn and $[nq] = [a]$.

Exercise 4.47 (VTRMC 2012)**.** Define $f(n)$ for n a positive integer by $f(1) = 3$ and $f(n+1) = 3^{f(n)}$. What are the last two digits of $f(2012)$?

Exercise 4.48 (VTRMC 2012)**.** Define a sequence (a_n) for n a positive integer inductively by $a_1 = 1$ and $a_n = \dfrac{n}{\prod\limits_{\substack{1 \le d < n \\ d \mid n}} a_d}$. Thus $a_2 = 2, a_3 = 3, a_4 = 2$, etc. Find a_{999000}.

Exercise 4.49 (VTRMC 2013)**.** A positive integer n is called *special* if it can be represented in the form $n = \dfrac{x^2+y^2}{u^2+v^2}$, for some positive integers x, y, u, v. Prove that

(a) 25 is special;

(b) 2013 is not special;

(c) 2014 is not special.

Exercise 4.50. Determine all positive integers n for which there is an infinite subset A of positive integers such that for all n distinct $a_1, \dots, a_n \in A$ the numbers $a_1 + \cdots + a_n$ and $a_1 \cdots a_n$ are relatively prime.

Exercise 4.51 (VTRMC 2014)**.** Find the least positive integer n such that 2^{2014} divides $19^n - 1$.

Exercise 4.52 (VTRMC 2014)**.** Let $n \ge 1$ and $r \ge 2$ be positive integers. Prove that there is no integer m such that $n(n+1)(n+2) = m^r$.

Exercise 4.53 (VTRMC 2015)**.** Find all integers n for which $n^4 + 6n^3 + 11n^2 + 3n + 31$ is a perfect square.

Exercise 4.54 (Putnam 2015, A5)**.** Let q be an odd positive integer, and let N_q denote the number of integers a such that $0 < a < q/4$ and $\gcd(a, q) = 1$. Show that N_q is odd if and only if q is of the form p^k with k a positive integer and p a prime congruent to 5 or 7 modulo 8.

Exercise 4.55 (Putnam 2015, B2)**.** Given a list of the positive integers $1, 2, 3, 4, \dots$, take the first three numbers $1, 2, 3$ and their sum 6 and cross all four numbers off the list. Repeat with the three smallest remaining numbers $4, 5, 7$ and their sum 16. Continue in this way, crossing off the three smallest remaining numbers and their sum, and consider the sequence of sums produced: $6, 16, 27, 36, \dots$. Prove or disprove that there is some number in the sequence whose base 10 representation ends with 2015.

Exercise 4.56 (VTRMC 2016)**.** For a positive integer a, let $P(a)$ denote the largest prime divisor of a^2+1. Prove that there exist infinitely many triples (a,b,c) of distinct positive integers such that $P(a) = P(b) = P(c)$.

Exercise 4.57 (VTRMC 2016)**.** Suppose that m,n,r are positive integers such that

$$1+m+n\sqrt{3} = (2+\sqrt{3})^{2r-1}.$$

Prove that m is a perfect square.

Exercise 4.58 (Putnam 2016, B2)**.** Define a positive integer n to be *squarish* if either n is itself a perfect square or the distance from n to the nearest perfect square is a perfect square. For example, 2016 is squarish, because the nearest perfect square to 2016 is $45^2 = 2025$ and $2025-2016 = 9$ is a perfect square. (Of the positive integers between 1 and 10, only 6 and 7 are not squarish.)
For a positive integer N, let $S(N)$ be the number of squarish integers between 1 and N, inclusive. Find positive constants α and β such that

$$\lim_{N\to\infty} \frac{S(N)}{N^\alpha} = \beta,$$

or show that no such constants exist.

Exercise 4.59 (VTRMC 2017)**.** Find all pairs (m,n) of nonnegative integers for which $m^2+2\cdot 3^n = m(2^{n+1}-1)$.

Exercise 4.60 (Putnam 2017, A4)**.** A class with $2N$ students took a quiz, on which the possible scores were $0,1,\ldots,10$. Each of these scores occurred at least once, and the average score was exactly 7.4. Show that the class can be divided into two groups of N students in such a way that the average score for each group was exactly 7.4.

Exercise 4.61 (Putnam 2017, B2)**.** Suppose that a positive integer N can be expressed as the sum of k consecutive positive integers

$$N = a+(a+1)+(a+2)+\cdots+(a+k-1)$$

for $k = 2017$ but for no other values of $k > 1$. Considering all positive integers N with this property, what is the smallest positive integer a that occurs in any of these expressions?

Exercise 4.62 (Putnam 2018, A1)**.** Find all ordered pairs (a,b) of positive integers for which

$$\frac{1}{a}+\frac{1}{b} = \frac{3}{2018}.$$

Exercise 4.63 (VTRMC 2018)**.** Let m,n be integers such that $n \geq m \geq 1$. Prove that $\frac{\gcd(m,n)}{n}\begin{pmatrix} n \\ m \end{pmatrix}$ is an integer. Here gcd denotes greatest common divisor and $\begin{pmatrix} n \\ m \end{pmatrix} = \frac{n!}{m!(n-m)!}$ denotes the binomial coefficient.

Exercise 4.64 (VTRMC 2019)**.** For each positive integer n, define $f(n)$ to be the sum of the digits of 2771^n (so $f(1) = 17$). Find the minimum value of $f(n)$ (where $n \geq 1$). Justify your answer.

Exercise 4.65 (Putnam 2020, A1)**.** How many positive integers N satisfy all of the following three conditions?

(i) N is divisible by 2020.

(ii) N has at most 2020 decimal digits.

(iii) The decimal digits of N are a string of consecutive ones followed by a string of consecutive zeros.

Exercise 4.66 (Putnam 2020, B1)**.** For a positive integer N, let f_N be the function defined by

$$f_N(x) = \sum_{n=0}^{N} \frac{N+1/2-n}{(N+1)(2n+1)} \sin((2n+1)x).$$

Determine the smallest constant M such that $f_N(x) \leq M$ for all N and all real x.

For a positive integer n, define $d(n)$ to be the sum of the digits of n when written in binary (for example, $d(13) = 1+1+0+1 = 3$). Let

$$S = \sum_{k=1}^{2020} (-1)^{d(k)} k^3.$$

Determine S modulo 2020.

Exercise 4.67 (Putnam 2021, B4)**.** Let $F_0, F_1, \ldots$ be the sequence of Fibonacci numbers, with $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For $m > 2$, let R_m be the remainder when the product $\prod_{k=1}^{F_m-1} k^k$ is divided by F_m. Prove that R_m is also a Fibonacci number.

Exercise 4.68 (Putnam 2021, A5)**.** Let A be the set of all integers n such that $1 \leq n \leq 2021$ and $\gcd(n, 2021) = 1$. For every nonnegative integer j, let

$$S(j) = \sum_{n \in A} n^j.$$

Determine all values of j such that $S(j)$ is a multiple of 2021.

Exercise 4.69 (VTRMC 2022)**.** Find all positive integers a,b,c,d, and n satisfying $n^a + n^b + n^c = n^d$ and prove that these are the only such solutions.

Exercise 4.70 (VTRMC 2022)**.** Give all possible representations of 2022 as a sum of at least two consecutive positive integers and prove that these are the only representations.

Exercise 4.71. Prove that if a,b,c are integers for which $a^2 + b^4 = c^4$, then $ab = 0$.

Exercise 4.72. Prove that the only integer solution to the equation $a^3 + 3b^3 + 9c^3 = 0$ is the trivial solution $a = b = c = 0$.

Exercise 4.73 (Putnam 2023, B2)**.** For each integer n, let $k(n)$ be the number of ones in the binary representation of $2023 \cdot n$. What is the minimum value of $k(n)$?

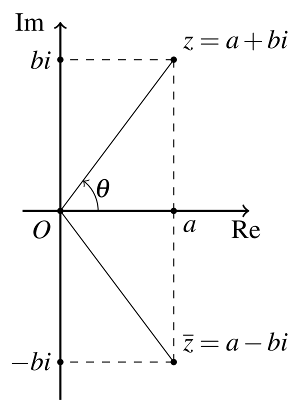
Chapter 5

Complex Numbers

5.1 Basics

A video summary of what follows can be found in this YouTube Video.

Any complex number z can be written as $z = a + bi$, where $a, b \in \mathbb{R}$. This is called the standard form of z. The number a is called the real part of z and is denoted by $a = \text{Re } z$ and b is called the imaginary part of z and is denoted by $b = \text{Im } z$. Two complex numbers are equal iff their real parts and imaginary parts are equal.
For any complex number $z = a + bi$ (with $a, b \in \mathbb{R}$), its conjugate is defined as $\bar{z} = a - bi$ and its absolute value is defined as $|z| = \sqrt{a^2 + b^2}$. Note that $|z|^2 = z\bar{z}$.



Major Operations: $(a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i$, $\quad (a+bi)(c+di) = (ac - bd) + (ad + bc)i$, and $\dfrac{z}{w} = \dfrac{z\overline{w}}{|w|^2}$
Each complex number can be represented on the complex plane. The horizontal axis is considered the real axis and the vertical axis is the imaginary axis.

Given a complex number $z = a + bi$, we know $|z| = \sqrt{a^2 + b^2}$, which is the distance between z and the origin. Let θ be the angle between the segment Oz and the positive real axis. We obtain $\cos\theta = \frac{a}{|z|}$ and $\sin\theta = \frac{b}{|z|}$. Thus

$z = |z|(\cos\theta + i\sin\theta)$. One can easily see that

$$(\cos\theta + i\sin\theta)(\cos\alpha + i\sin\alpha) = \cos(\theta+\alpha) + i\sin(\theta+\alpha)$$

This identity motivates the notation $\cos\theta + i\sin\theta = e^{i\theta}$ – which can also be justified using Taylor series for e^x, $\sin x$ and $\cos x$.

$$\begin{cases} e^{i\theta} = 1 + \dfrac{i\theta}{1!} + \dfrac{(i\theta)^2}{2!} + \dfrac{(i\theta)^3}{3!} + \cdots \\ \cos\theta = 1 - \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} - \cdots \\ \sin\theta = \dfrac{\theta}{1!} - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \cdots \end{cases}$$

As a result we obtain the following identities:

$$\begin{cases} \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \operatorname{Im}\left(e^{i\theta}\right) \\ \\ \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \operatorname{Re}\left(e^{i\theta}\right) \end{cases}$$

Remark. Note that for real numbers α, θ,

$$\boxed{e^{i\theta} = e^{i\alpha} \text{ iff } \theta - \alpha = 2k\pi \text{ for some } k \in \mathbb{Z}.}$$

Taking Roots: Let $c \in \mathbb{C}$ and $n \in \mathbb{Z}^+$ be given. Suppose we want to find all $n-$th roots of c. In other words, we want to find all $z \in \mathbb{C}$ for which $z^n = c$.

To find $n-$th roots of c

- Write $c = |c|\, e^{i\alpha}$, where $0 \le \alpha < 2\pi$.
- Write $z = |z|e^{i\theta}$, where $0 \le \theta < 2\pi$. The equality $z^n = c$ implies $|z|^n = |c|$. Therefore, $|z| = \sqrt[n]{|c|}$.
- Thus, $z = \sqrt[n]{|c|}\, e^{i\theta}$. This implies $e^{in\theta} = e^{i\alpha}$, which means $n\theta = \alpha + 2k\pi$ for some integer k.
- Therefore all $n-$th roots of c are $\sqrt[n]{|c|}\, e^{(2k\pi+\alpha)i/n}$, where $k = 0, 1, \ldots, n-1$. Note that these roots are distinct, if $c \neq 0$.

$$\boxed{\text{The } n\text{-th roots of unity are given by } e^{2\pi i k/n} \text{ with } k = 0, 1, \ldots, n-1.}$$

Adding a complex number u to a complex number z results in translating z in the direction of Ou.

The result of rotating z about the origin with an angle θ is $ze^{i\theta}$. The reason is that under rotation the distance to the origin remains unchanged and the argument is increased by θ.

Multiplying a complex number z by a real number λ keeps z in the same line through the origin, however it changes the distance of z to the origin if $\lambda \neq \pm 1$. If λ is negative, it reflects z about the origin along with re-scaling it.

$$\boxed{\text{The result of rotating } z \text{ about } u \text{ with an angle } \theta \text{ is the complex number } (z-u)e^{i\theta} + u.}$$

5.2 Important Theorems

Theorem 5.1 (Conjugate Roots Theorem)**.** *Let $p(x)$ be a polynomial with real coefficients.*

- *If z is a complex root of $p(x)$, then $\overline{z}$ is also a root of $p(x)$.*
- *If all coefficients of $p(x)$ are rational and $a+\sqrt{b}$ is an irrational root of $p(x)$, with $a,b \in \mathbb{Q}$ and $b>0$, then so is $a-\sqrt{b}$.*

Proof. Prove the first part of the above theorem by following these steps:

- Use properties of complex conjugate to show $\overline{p(z)} = p(\overline{z})$.
- Deduce $\overline{z}$ is also a root for $p(x)$.

□

Theorem 5.2 (Fundamental Theorem of Algebra)**.** *Every polynomial with complex coefficients can be completely factored into linear polynomials. In other words if $p(z) = a_n z^n + \cdots + a_0$ is a polynomial with complex coefficients, then there are complex numbers $r_1, \ldots, r_n$ for which $p(z) = a_n(z-r_1)\cdots(z-r_n)$.*

5.3 Classical Examples

Example 5.1. *Find a formula in closed form for* $\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\cdots\sin\left(\frac{(n-1)\pi}{n}\right)$.

Scratch: Typically the way I would like to find sums and products of trigonometric functions is by using complex numbers. We are familiar with the following identities:

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \mathrm{Im}\,(e^{i\theta})$$

The second identity does not seem useful for this particular problem, since the product of imaginary parts of complex numbers is not easy to simplify. So, let's make use of the first one. For simplicity let $\theta = \frac{\pi}{n}$, and denote by P the given product:

$$P = \prod_{k=1}^{n-1} \sin(k\theta)$$

Note that P is a positive real number since all angles are in the first or second quadrants. We can simplify this product by using the fact that $P = |P|$ and thus simplify the terms i and $e^{-ik\theta}$. The rest can be evaluated using n-th roots of unity. This can now be turned into a solution as follows:

Solution. (Video Solution) The answer is $\frac{n}{2^{n-1}}$.

Let P be the given product, and for simplicity let $\theta = \frac{\pi}{n}$. Note also that $\sin(k\theta)$ is positive for $k = 1, \ldots, n-1$, and hence P is a positive real number. Using the identity $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ we obtain the following:

$$P = \prod_{k=1}^{n-1}\left(\frac{e^{ik\theta}-e^{-ik\theta}}{2i}\right) = \frac{1}{(2i)^{n-1}}\prod_{k=1}^{n-1} e^{-ik\theta}(e^{2ik\theta}-1).$$

Therefore,

$$P = |P| = \frac{1}{2^{n-1}}\left|\prod_{k=1}^{n-1}(1-e^{2ik\theta})\right|.$$

Note that $z = e^{2ik\theta}$ for $k = 1,\ldots,n-1$ are distinct roots of the polynomial $z^n - 1 = 0$. Also, note that none of these roots are 1. Therefore,

$$z^n - 1 = (z-1)\prod_{k=1}^{n-1}(z-e^{2ik\theta}) \Rightarrow \frac{z^n-1}{z-1} = \prod_{k=1}^{n-1}(z-e^{2ik\theta}) \Rightarrow z^{n-1}+\cdots+z+1 = \prod_{k=1}^{n-1}(z-e^{2ik\theta}).$$

Sunstiuting $z = 1$ we conclude:

$$\prod_{k=1}^{n-1}(1-e^{2ik\theta}) = n.$$

Therefore, the given product is

$$P = \frac{1}{2^{n-1}}\left|\prod_{k=1}^{n-1}(1-e^{2ik\theta})\right| = \frac{n}{2^{n-1}}.$$

The final answer is $\boxed{\dfrac{n}{2^{n-1}}}$. □

Example 5.2. *Evaluate the sum* $\sum\limits_{k=1}^{n} \sin k$*, where the angles are measured in radians.*

Solution.(Video Solution) We will use the fact that $\sin\theta$ is the imaginary part of $e^{i\theta}$.

$\sum\limits_{k=1}^{n} \sin k = \text{Im}\,(\sum\limits_{k=1}^{n} e^{ki})$. However the sum $\sum\limits_{k=1}^{n} e^{ki}$ is a geometric sum, which is equal to

$$\frac{e^i - e^{(n+1)i}}{1-e^i} = \frac{e^{i(n/2+1)}}{e^{i/2}}\left(\frac{e^{-ni/2}-e^{ni/2}}{e^{-i/2}-e^{i/2}}\right) = e^{i(n/2+1/2)}\left(\frac{-2i\ \sin(n/2)}{-2i\ \sin(1/2)}\right).$$

The answer is $\dfrac{\sin(n/2+1/2)\sin(n/2)}{\sin(1/2)}$. □

5.4 Further Examples

Example 5.3 (VTRMC 1980)**.** *Let $z = x+iy$ be a complex number with x and y rational and with $|z| = 1$.*

(a) Find two such complex numbers.

(b) Show that $|z^{2n}-1| = 2|\sin n\theta|$, where $z = e^{i\theta}$.

(c) Show that $|z^{2n}-1|$ is rational for every n.

Scratch: For the first part we need $x^2+y^2 = 1$. Letting $x = a/c$ and $y = b/c$, we need $a^2+b^2 = c^2$, or we need Pythagorean triples. For the second part note that $|z| = 1$ implies z is on the unit circle or $z = e^{i\theta}$ for some angle θ. We know $z^{2n} = e^{i2\theta}$. Note that $e^{2in\theta}-1 = e^{in\theta}(e^{in\theta}-e^{-in\theta}) = e^{in\theta}2i\sin(n\theta)$. The last part follows from the fact that $\sin(n\theta)$ can be written as a polynomial of $\sin\theta$ and $\cos\theta$ using De'Movire's Formula.

Solution. (a) $z = \frac{3}{5} + \frac{4}{5}i$ and $z = \frac{4}{5} + \frac{3}{5}i$ are two such examples.
(b) $|z^{2n} - 1| = |e^{2in\theta} - 1| = |e^{in\theta}(e^{in\theta} - e^{-in\theta})| = |e^{in\theta} 2i \sin(n\theta)| = 2|\sin(n\theta)|$, since $|i| = |e^{in\theta}| = 1$.
(c) Note that $\cos(n\theta) + i\sin(n\theta) = e^{in\theta} = (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = \sum_{j=0}^{n} \binom{n}{j} (\cos\theta)^{n-j}(i\sin\theta)^j$. Therefore by looking at the imaginary part we obtain

$$\sin(n\theta) = \binom{n}{1}(\cos\theta)^{n-1}\sin\theta - \binom{n}{3}(\cos\theta)^{n-3}(\sin\theta)^3 + \binom{n}{5}(\cos\theta)^{n-5}(\sin\theta)^5 - \cdots.$$

Since both $\cos\theta$ and $\sin\theta$ are rational, $\sin(n\theta)$ is rational. Therefore, by part (b), $|z^{2n} - 1|$ is rational. □

Example 5.4. *Let $z_1, \ldots, z_n$ be complex numbers for which $\sum_{j=1}^{n} |z_j|^2 = 1$. Prove that there are $\varepsilon_j \in \{\pm 1\}$ for which $|\sum_{j=1}^{n} \varepsilon_j z_j| \leq 1$.*

Scratch: Note that we can always assume $\varepsilon_1 = 1$, otherwise we could negate all of the ε_j's. Our first step would be to try some small cases. $n = 1$ is obvious. For $n = 2$, we have $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\text{Re }(z_1\bar{z}_2) = 1 \pm 2\text{Re }(z_1\bar{z}_2)$, which does not exceed 1 for one of the choices of $\pm$. In fact we just showed $|z_1 + \varepsilon_2 z_2|^2 \leq |z_1|^2 + |z_2|^2$.
For $n = 3$, we have more possibilities of ε_j's and thus the problem gets more complicated. We would like to rely on the previous case of $n = 2$, however we realize that we only know $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$, which is difficult to use. We do notice that the initial inequality could be rescaled to get $|\sum_{j=1}^{n} \varepsilon_j z_j|^2 \leq \sum_{j=1}^{n} |z_j|^2$ if we remove the assumption that $\sum_{j=1}^{n} |z_j|^2 = 1$. In other words, we have $|z_1 + \varepsilon_2 z_2 + \varepsilon_3 z_3|^2 \leq |z_1 + \varepsilon_2 z_2|^2 + |z_3|^2$ by the case $n = 2$ and repeating it again we can solve the problem. So, in order to solve the problem we will prove a stronger version of the problem by induction.

Solution.(Video Solution) We will prove the following stronger version of the problem by induction on n.

"For complex numbers $z_1, \ldots, z_n$, there are $\varepsilon_j \in \{\pm 1\}$ for which $|\sum_{j=1}^{n} \varepsilon_j z_j|^2 \leq \sum_{j=1}^{n} |z_j|^2$."

For $n = 1$, $\varepsilon_1 = 1$ works.
For $n = 2$, if $\varepsilon = \pm 1$ we get $|z_1 + \varepsilon z_2|^2 = |z_1|^2 + |z_2|^2 + 2\varepsilon \text{Re }(z_1\bar{z}_2)$. If $\text{Re }(z_1\bar{z}_2) \leq 0$, then we let $\varepsilon = 1$ and otherwise we let $\varepsilon = -1$. Thus, $2\varepsilon\text{Re }(z_1\bar{z}_2) \leq 0$. Thus, $|z_1 + \varepsilon z_2|^2 \leq |z_1|^2 + |z_2|^2$. This proves the statement for $n = 2$.
Suppose the statement is true for a natural number n and let $z_1, \ldots, z_{n+1}$ be complex number. We know, by inductive hypothesis that for some $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$ we have $|\sum_{j=1}^{n} \varepsilon_j z_j|^2 \leq \sum_{j=1}^{n} |z_j|^2$. Applying the basis step proved above to $\sum_{j=1}^{n} \varepsilon_j z_j$ and z_{n+1} we know there is $\varepsilon_{n+1} \in \{\pm 1\}$ such that $|\sum_{j=1}^{n+1} \varepsilon_j z_j|^2 \leq |\sum_{j=1}^{n} \varepsilon_j z_j|^2 + |z_{n+1}|^2$. Using the inductive hypothesis $|\sum_{j=1}^{n} \varepsilon_j z_j|^2 \leq \sum_{j=1}^{n} |z_j|^2$, which implies $|\sum_{j=1}^{n+1} \varepsilon_j z_j|^2 \leq \sum_{j=1}^{n+1} |z_j|^2$. This proves the statement. The result follows by applying what we proved to the given complex numbers. □

Example 5.5 (Putnam 2019, A3). *Given real numbers $b_0, b_1, \ldots, b_{2019}$ with $b_{2019} \neq 0$, let $z_1, z_2, \ldots, z_{2019}$ be the roots in the complex plane of the polynomial*

$$P(z) = \sum_{k=0}^{2019} b_k z^k.$$

Let $\mu = \dfrac{|z_1|+\cdots+|z_{2019}|}{2019}$ *be the average of the distances from* $z_1, z_2, \ldots, z_{2019}$ *to the origin. Determine the largest constant* M *such that* $\mu \geq M$ *for all choices of* $b_0, b_1, \ldots, b_{2019}$ *that satisfy*

$$1 \leq b_0 < b_1 < b_2 < \cdots < b_{2019} \leq 2019.$$

Here are my initial thoughts.

- Is there anything special about 2019? Maybe, but I'd like to see if I can solve this problem for small cases.
- I need to show two things. First, $\mu \geq M$, and second there is an example of a polynomial that give us M, or perhaps a sequence of examples that give us values as close to M as possible.

When the degree is 1, we must have $1 \leq b_0 < b_1 \leq 1$, which is impossible.

When the degree is 2, we must have $1 \leq b_0 < b_1 < b_2 \leq 2$, and we need to find the minimum of $|z_1|+|z_2|$. Since this is a quadratic we can go ahead and use quadratic formula and find z_1, z_2, but this idea cannot be generalized. I can however relate z_1, z_2 with the coefficients:

$$z_1+z_2 = -\frac{b_1}{b_2}, z_1 z_2 = \frac{b_0}{b_2}.$$

We can relate $|z_1|+|z_2|$ and their product using the AM-GM inequality.

$$|z_1|+|z_2| \geq 2\sqrt{|z_1||z_2|} = 2\sqrt{b_0/b_2} \geq 2/\sqrt{2} = \sqrt{2}.$$

The equality only holds when $|z_1| = |z_2| = 1/\sqrt{2}$, $b_0 = 1$, and $b_2 = 2$. This means $|\sqrt{2}z_1| = |\sqrt{2}z_2| = 1$. I can choose $\sqrt{2}z_1$ and $\sqrt{2}z_2$ to be roots of unity. Since I want them to satisfy a quadratic I will naturally choose 3rd roots of unity. So, the equation for $\sqrt{2}z_1$ and $\sqrt{2}z_2$ would be $x^2+x+1=0$. Replacing x by $\sqrt{2}z$ we obtain the equation $2z^2+\sqrt{2}z+1=0$. This gives us $b_0 = 1, b_1 = \sqrt{2}, b_2 = 2$, which matches the given inequalities.

When $n = 3$, we need to minimize $|z_1|+|z_2|+|z_3|$. It is hard to relate this with the coefficients, but I know the product of the roots can easily be related to the coefficients. In other words, I know $|z_1 z_2 z_3| = b_0/b_3 \geq 1/3$. How can I relate this and the sum of $|z_i|$'s? The AM-GM inequality comes in handy. $|z_1|+|z_2|+|z_3| \geq 3\sqrt[3]{|z_1 z_2 z_3|} \geq 3/\sqrt[3]{3}$. Similar to the quadratic case we see that we need $|z_1| = |z_2| = |z_3| = \dfrac{1}{\sqrt[3]{3}}$ or we can choose $\sqrt[3]{3}z_i$'s to be 4th roots of unity. This yields the equation $x^3+x^2+x+1=0$ for $\sqrt[3]{3}z_i$'s, and hence $3t^3+\sqrt[3]{9}t^2+\sqrt[3]{3}t+1=0$ is an equation whose roots satisfy the required conditions. At this point I can see how this solution can be generalized. So let's write this down.

Solution.(Video Solution) The answer is $\boxed{\dfrac{1}{\sqrt[2019]{2019}}}$.

First, for simplicity let $n = 2019$.

Consider the polynomial $p(x) = \sum\limits_{k=0}^{n} a_k x^k$, where $a_k = \sqrt[n]{n^k}$ for all k. We see that

$$1 \leq a_k = (\sqrt[n]{n})^k < (\sqrt[n]{n})^{k+1} = a_{k+1} \leq n,$$

and $p(x) = \sum\limits_{k=0}^{n} (\sqrt[n]{n}x)^k = \dfrac{(\sqrt[n]{n}x)^{n+1}-1}{x-1}$, and thus each z_i satisfies $(\sqrt[n]{n}z_i)^{n+1} = 1$ which means $|\sqrt[n]{n}z_i| = 1$. Therefore, $\dfrac{|z_1|+\cdots+|z_n|}{n} = 1/\sqrt[n]{n}$.

Next, note that for any such polynomial we have $(|z_1|+\cdots+|z_n|)/n \geq \sqrt[n]{|z_1\cdots z_n|} = \sqrt[n]{b_0/b_n} \geq \sqrt[n]{1/n}$. This completes the proof. □

Example 5.6 (AIME I 2015, Problem 13)**.** *Evaluate the product*

$$\prod_{k=1}^{45} \sin(2k-1)^\circ$$

Scratch: Let $P = \prod_{k=1}^{45} \sin(2k-1)^\circ$. We start by writing $\sin(2k-1)^\circ$ in terms of complex numbers:

$$P = \prod_{k=1}^{45} \frac{e^{i(2k-1)} - e^{-i(2k-1)}}{2i} = \prod_{k=1}^{45} \frac{e^{-i(2k-1)}}{2i}(e^{i(4k-2)} - 1).$$

We note that $z = e^{i(4k-2)}$ satisfies $z^{90} = e^{180i(2k-1)} = (-1)^{2k-1} = -1$, however these are only 45 roots, but $z^{90}+1=0$ has 90 complex roots. To resolve this issue we note that if $k = 46, 47, \ldots, 90$ we have

$$\sin(91) = \sin(180-91) = \sin(89), \sin(93) = \sin(87), \ldots$$

In other words, if we allow k to range between 1 and 90 we obtain all roots of $z^{90}+1=0$. This yields the following solution:

Solution. (Video Solution) The answer is $\frac{1}{\sqrt{2}^{89}}$.

Let P denote the given product, and note that P is a positive real number. So, $P = |P|$. Note that since $\sin(180-(2k-1)) = \sin(2k-1)$ we can write:

$$P^2 = \prod_{k=1}^{45} \sin^2(2k-1) = \prod_{k=1}^{45} \sin(2k-1)\sin(180-2k+1) = \prod_{k=1}^{90} \sin(2k-1)$$

Using the identity $\sin(2k-1) = \frac{e^{i(2k-1)} - e^{-i(2k-1)}}{2i}$, we obtain the following:

$$P^2 = \prod_{k=1}^{90} \left(\frac{e^{i(2k-1)} - e^{-i(2k-1)}}{2i} \right) = \frac{1}{(2i)^{90}} \prod_{k=1}^{90} e^{-i(2k-1)}(e^{i(4k-2)} - 1).$$

Since P is a positive real number we have

$$P^2 = |P^2| = \frac{1}{2^{90}} \prod_{k=1}^{90} |1 - e^{i(4k-2)}|$$

Since $0 < 4k-2 < 360$, the complex numbers $e^{i(4k-2)}$ in the above product are all distinct. These 90 numbers all satisfy $(e^{i(4k-2)})^{90} = e^{i(360k-180)} = -1$. Therefore, the polynomial $z^{90}+1$ can be factored as

$$z^{90} + 1 = \prod_{k=1}^{90} (z - e^{i(4k-2)}).$$

Substituting $z = 1$ we conclude that:

$$2 = \prod_{k=1}^{90} (1 - e^{i(4k-2)})$$

Using this we obtain $P^2 = \dfrac{2}{2^{90}}$ and hence $\boxed{P = \dfrac{1}{\sqrt{2^{89}}}}$. □

Example 5.7. *Suppose z_1, z_2, z_3, z_4 are four complex numbers satisfying the following:*

$$|z_1| = |z_2| = |z_3| = |z_4| = 1, \text{ and } z_1 + z_2 + z_3 + z_4 = 0.$$

Prove that these four complex numbers are either vertices of a rectangle on the complex plane, or they are two pairs of identical complex numbers.

Solution. (Video Solution) Note that z_1, z_2, z_3, z_4 lie on the unit circle. If they were all the same, then we would have $4z_1 = 0$, which contradicts $|z_1| = 1$. Thus, they cannot all be the same. Suppose $z_1 \neq z_2$. We know $z_1 + z_2 = -z_3 - z_4$. This is equivalent to $\dfrac{z_1 + z_2}{2} = \dfrac{-z_3 - z_4}{2}$, which is equivalent to the fact that the midpoint of segment connecting z_1 and z_2 coincides with the midpoint of the segment connecting $-z_3$ and $-z_4$. Let w be this common midpoint. If w is not the origin, then the segment connecting the origin and w must be perpendicular to both chords $z_1 z_2$ and $(-z_3)(-z_4)$. Therefore, the two chords must be the same. In other words, possibly after relabeling z_3 and z_4, we have $z_1 = -z_3$ and $2_z = -z_4$, hence $z_1 + z_3 = z_2 + z_4 = 0$. If $w = 0$, then $z_1 + z_2 = z_3 + z_4 = 0$. Thus, either way after possibly relabeling the complex numbers, we may assume $z_1 + z_2 = z_3 + z_4 = 0$. Thus, the segments $z_1 z_2$ and $z_3 z_4$ are diameters of the unit circle. If these diameters were the same, then $z_1, z_2, z_3 . z_4$ would be two pairs of identical complex numbers. Otherwise, they would form a rectangle, as desired. □

Example 5.8 (Putnam 1989, A3)**.** *Prove that if a complex number z satisfies:*

$$11z^{10} + 10iz^9 + 10iz - 11 = 0.$$

Then $|z| = 1$.

Solution. (Video Solution) □

Example 5.9. *Consider n equally spaced points on a unit circle. Prove that the product of the $n-1$ distances from one of them to each of the rest is equal to n.*

Solution. (Video Solution)

□

5.5 General Strategies

- Be aware of over-using the standard form. Treat each complex number as ONE variable z as much as possible, instead of treating it as $a + bi$.

- Use the power of geometry when dealing with complex numbers.

- The identities $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \text{Re}\ (e^{i\theta})$, $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \text{Im}\ (e^{i\theta})$, and $e^{i\theta} = \cos\theta + i\sin\theta$ often help in evaluation of sums or products involving trigonometric functions.

5.6 Exercises

Exercise 5.1 (VTRMC 1989). (i) Prove that $f_0(x) = 1 + x + x^2 + x^3 + x^4$ has no real zero.

(ii) Prove that, for every integer $n \geq 0, f_n(x) = 1 + 2^{-n}x + 3^{-n}x^2 + 4^{-n}x^3 + 5^{-n}x^4$ has no real zero. (Hint: consider $(d/dx)(xf_n(x))$.)

Exercise 5.2 (Putnam 1990, B2). Prove that for every two complex numbers x, z with $|x| < 1$, $|z| > 1$, we have:

$$1 + \sum_{j=1}^{\infty} (1 + x^j) P_j = 0,$$

where P_j is

$$\frac{(1-z)(1-zx)(1-zx^2)\cdots(1-zx^{j-1})}{(z-x)(z-x^2)(z-x^3)\cdots(z-x^j)}.$$

Exercise 5.3 (Putnam 2004, B4). Let n be a positive integer, $n \geq 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k, 0)$ in the xy-plane, for $k = 1, 2, \ldots, n$. Let R_k be the map that rotates the plane counterclockwise by the angle θ about the point P_k. Let R denote the map obtained by applying, in order, R_1, then $R_2, \ldots$, then R_n. For an arbitrary point (x, y), find, and simplify, the coordinates of $R(x, y)$.

Exercise 5.4 (Putnam 2005, A3). Let $p(z)$ be a polynomial of degree n all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

Exercise 5.5 (VTRMC 2012). Find five nonzero complex numbers a, b, c, d, e such that

$$\begin{cases} a+b+c+d+e = -1 \\ a^2+b^2+c^2+d^2+e^2 = 15 \\ \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}+\dfrac{1}{e} = -1 \\ \dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}+\dfrac{1}{d^2}+\dfrac{1}{e^2} = 15 \\ abcde = -1 \end{cases}$$

Exercise 5.6. Find the product of the length of all diagonals of a regular n-gon whose side length is 1.

Exercise 5.7. Let A_n be the average length of all diagonals of a regular n-gon inscribed in the unit circle. Evaluate $\lim\limits_{n\to\infty} A_n$.

Exercise 5.8. Prove that for every angle θ, we have

$$\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n} = \frac{4-2\cos\theta}{5-4\cos\theta}.$$

Exercise 5.9. Find all rational numbers $r \in (0, 1)$ for which $\sin(r\pi)$ is rational.

Exercise 5.10. Let n be a positive integer. Find the number of pairs $(p(x), q(x))$ of polynomials with real coefficients for which $\deg p > \deg q$ and $(p(x))^2 + (q(x))^2 = x^{2n} + 1$.

Exercise 5.11. Show that there does not exist any equilateral triangles in the plane whose vertices are all lattice points.

Exercise 5.12. Let $w, z_1, z_2, \ldots, z_n$ be complex numbers for which $w^2 = z_1^2 + z_2^2 + \cdots + z_n^2$. Prove that

$$|\text{Re } w| \leq |\text{Re } z_1| + |\text{Re } z_2| + \cdots + |\text{Re } z_n|,$$

where Re z is the real part of a complex number z.

Exercise 5.13 (Putnam 2015, A3)**.** Compute

$$\log_2\left(\prod_{a=1}^{2015}\prod_{b=1}^{2015}(1 + e^{2\pi iab/2015})\right)$$

Here i is the imaginary unit (that is, $i^2 = -1$).

Exercise 5.14 (Putnam 2018, B2)**.** Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} \colon |z| \leq 1\}$.

Exercise 5.15 (Putnam 2020, B5)**.** For $j \in \{1, 2, 3, 4\}$, let z_j be a complex number with $|z_j| = 1$ and $z_j \neq 1$. Prove that

$$3 - z_1 - z_2 - z_3 - z_4 + z_1 z_2 z_3 z_4 \neq 0.$$

Exercise 5.16. Let n be a positive integer. Prove that $4^{n-2} \prod_{k=1}^{n} \left(3\cos^2\left(\frac{k\pi}{n}\right) + 1\right)$ is the square of an integer.

Chapter 6

Geometry

6.1 Basics

The cross product of two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ is given by:

$$\mathbf{u} \times \mathbf{v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}.$$

The dot product of two vectors $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ is given by $\mathbf{u} \cdot \mathbf{v} = \sum_{j=1}^{n} u_j v_j$.

Definition 6.1. Suppose $\mathbb{F}$ is the field of real or complex numbers. Let V be a vector space over $\mathbb{F}$. A function $\langle , \rangle$ that assigns a scalar $\langle \mathbf{u}, \mathbf{v} \rangle$ to every (ordered) pair of vectors $\mathbf{u}, \mathbf{v} \in V$ is called an **inner product** if it satisfies the following for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and all $c \in \mathbb{F}$:

1. $\langle \mathbf{u} + c\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + c\langle \mathbf{v}, \mathbf{w} \rangle$. (Linearity)

2. $\langle \mathbf{u}, \mathbf{u} \rangle > 0$ if $\mathbf{u}$ is a nonzero vector in V. (Positivity)

3. $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$. (Conjugate Symmetry)

Definition 6.2. Suppose $\mathbb{F}$ is the field of real or complex numbers. Let V be a vector space over $\mathbb{F}$. A function $|| \cdot ||$ that assigns a real number $||\mathbf{u}||$ to every vector $\mathbf{u} \in V$ is called a **norm** if it satisfies the following for all $\mathbf{u}, \mathbf{v} \in V$, and all $c \in \mathbb{F}$:

1. $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$. (Triangle Inequality)

2. $||\mathbf{u}|| > 0$ if $\mathbf{u}$ is a nonzero vector in V. (Positivity)

3. $||c\mathbf{u}|| = |c| \; ||\mathbf{u}||$. (Homogeneity)

6.2 Important Theorems

Theorem 6.1 (Properties of the Cross Product)**.** *Suppose* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. *Then,*

(a) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

(b) $\mathbf{u} \times \mathbf{v}$ *is perpendicular to both* $\mathbf{u}$ *and* $\mathbf{v}$.

(c) The volume of the parallelepiped formed by $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *is given by* $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$.

(d) $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| \, ||\mathbf{v}|| \sin\theta$, *where* θ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$.

(e) $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| \, ||\mathbf{v}|| \cos\theta$.

Here are some useful facts from Euclidean geometry:

Theorem 6.2. *Let ABC be a triangle:*

(a) The three altitudes of ABC intersect at a point called the **orthocenter**.

(b) The three angle bisectors of ABC intersect at a point called the **incenter**.

(c) The three medians of ABC intersect at a point called the **centroid**.

(d) The perpendicular bisectors of three sides of ABC intersect at a point called the **circumcenter**.

Theorem 6.3 (Law of Sines)**.** *In every triangle ABC we have*

$$\frac{|AB|}{\sin C} = \frac{|AC|}{\sin B} = \frac{|BC|}{\sin A} = 2R,$$

where R is the circumradius of ABC.

Theorem 6.4 (Law of Cosines)**.** *In every triangle ABC we have:*

$$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB| \, |AC| \cos A.$$

Theorem 6.5. *The area of a triangle ABC can be evaluated using the following formulas:*

$$[ABC] = \frac{1}{2}|AB| \, |AC| \sin A = \sqrt{s(s-a)(s-b)(s-c)}.$$

Here, a, b, c *are the side lengths and* $s = \dfrac{a+b+c}{2}$ *is the semiperimeter of ABC.*

Theorem 6.6. *Let M be the midpoint of side BC of triangle ABC, and G be the centroid of ABC. Then,* $\dfrac{|AG|}{|GM|} = 2$.

Theorem 6.7 (Angle-Bisector Theorem)**.** *Let AD be the bisector of angle A in triangle ABC, where D lies on BC. Then,* $\dfrac{|BD|}{|CD|} = \dfrac{|AB|}{|AC|}$.

Theorem 6.8 (Norm Obtained from an Inner Product)**.** *In an inner product vector space the function* $||\cdot||$ *defined by* $||\mathbf{u}|| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$ *is a norm.*

Theorem 6.9 (Cauchy-Schwarz Inequality)**.** *In any inner product vector space we have*

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \le ||\mathbf{u}|| \, ||\mathbf{v}||.$$

The equality holds if and only if $\mathbf{u} = c\,\mathbf{v}$ *for some scalar* $c \in \mathbb{F}$ *or* $\mathbf{u} = \mathbf{0}$.

6.3 Classical Examples

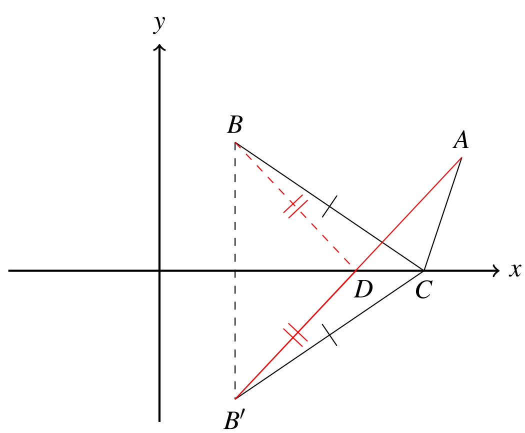
Example 6.1. *The following are examples of inner products:*

1. $\langle f,g\rangle = \int_0^1 f(t)\overline{g(t)}\,dt$ *for every* $f,g \in C[0,1]$, *the vector space of continuous functions from* $[0,1]$ *to* $\mathbb{C}$.

2. $\langle (z_1,\dots,z_n),(w_1,\dots,w_n)\rangle = \sum_{j=1}^{n} z_j\overline{w_j}$ *is an inner product over* $\mathbb{R}^n$ *and* $\mathbb{C}^n$.

Example 6.2 (VTRMC 1992)**.** *Assume that* $x_1 > y_1 > 0$ *and* $y_2 > x_2 > 0$. *Find a formula for the shortest length* ℓ *of a planar path that goes from* (x_1,y_1) *to* (x_2,y_2) *and that touches both the x-axis and the y-axis. Justify your answer.*

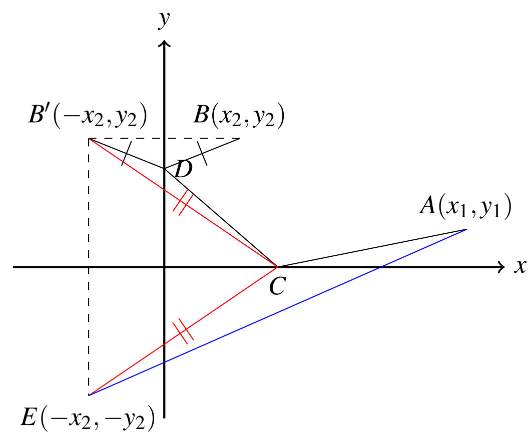
Scratch: First we would see if we can simplify the problem. We can simplify the condition by requiring the path to only touch one of the two axes. Let's say we require the path to touch the x-axis. What that means is that we are looking for a point C on the x-axis to minimize $AC+CB$, where $A=(x_1,y_1)$ and $B=(x_2,y_2)$. If somehow we could require C to be between A and B that would be ideal, but given that both A and B are in the first quadrant that is impossible. However we can fix this by reflecting B about the x-axis! This means $AC+CB$ is at least the distance between A and $B'=(x_2,-y_2)$. So, this solves the problem in this simpler case.



For the more general case we could apply reflection twice. This yields the following solution:

Solution. The answer is $\boxed{\sqrt{(x_1+x_2)^2+(y_1+y_2)^2}.}$

Suppose C and D are two points on x and y-axes, respectively. Reflect $B(x_2,y_2)$ about the y-axis to obtain the point $B'(-x_2,y_2)$, and then reflect B' about the x-axis to obtain the point $E(-x_2,-y_2)$. We have $BD=B'D$, and thus $BD+DC \geq B'C$. Equality holds when D is the intersection of $B'C$ and the y-axis. We have $B'C+CA = EC+CA \geq EA$ and equality holds when C is the intersection of EA and the x-axis. Therefore the minimum of $AC+CD+DB$ is EA, which is $\sqrt{(x_1+x_2)^2+(y_1+y_2)^2}$.



□

6.4 Further Examples

Example 6.3 (Putnam 2019, A2)**.** *In triangle ABC, let G be the centroid, and let I be the center of the inscribed circle. Let α and β be the angles at the vertices A and B, respectively. Suppose that the segment IG is parallel to AB and that* $\beta = 2\tan^{-1}(1/3)$*. Find α.*

Scratch: IG being parallel to AB immediately tells us that the inradius is the same as the distance between G and AB. Since this is a necessary and sufficient condition I feel comfortable using this condition without worrying about not having used all of the given assumptions. Now, from geometry I recall how to find r, the inradius. I also know that the distance from G to AB is one-third the altitude from C. So I feel comfortable that I should be able to solve the problem. Here is how we write the solution:

Solution. We will prove that $\boxed{\alpha = \frac{\pi}{2}}$.

Since IG is parallel to AB, the distance from I to AB is the same as the distance from G to AB. We know $r = \dfrac{[ABC]}{s}$, where $[ABC]$ and s are the area and semiperimeter of ABC, respectively. The distance from G to AB is one-third the distance from C to AB by a property of the centroid. Therefore, what we have is $h_c = 3r$. Multiplying both sides by c we obtain $ch_c = 2[ABC] = 3cr = 3c\dfrac{[ABC]}{s}$, we obtain $2s = 3c$ or $a+b = 2c$. Using Law of sines and the fact that $\sin(\alpha+\beta) = \sin(\pi-\alpha-\beta)$ we obtain $\sin\alpha + \sin\beta = 2\sin(\alpha+\beta)$. This implies $2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2}) = 4\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha+\beta}{2})$. Therefore, $\cos(\frac{\alpha-\beta}{2}) = 2\cos(\frac{\alpha+\beta}{2})$. This implies

$$\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right) + \sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right) = 2\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right) - 2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right) \Rightarrow \tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right) = \frac{1}{3}.$$

Since by assumption $\tan(\frac{\beta}{2}) = \frac{1}{3}$, we conclude that $\tan(\frac{\alpha}{2}) = 1$. Since $\alpha < \pi$, we conclude $\alpha = \frac{\pi}{2}$. □

Example 6.4 (Putnam 2000, B6)**.** *Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form*

$(\pm1,\pm1,\ldots,\pm1)$ *in n-dimensional space with* $n \geq 3$. *Show that there are three distinct points in B which are the vertices of an equilateral triangle.*

We will try this for $n = 3$. I notice that these eight points are vertices of a cube. We can find eight equilateral triangles and all sides of these equilateral triangles are $2\sqrt{2}$. From each one of these triangles at most two vertices can be selected, and thus at most $2\times 8 = 16$ vertices may be selected. However, each vertex belongs to precisely three equilateral triangles and thus we cannot select more than $16/3$ vertices without having an equilateral triangle. This is precisely what we were trying to prove.

Now the question is: how would I generalize this? We notice that each segment of length $2\sqrt{2}$ is the hypotenuse of a triangle with side length 2. This can certainly be generalized by looking at the neighbors of a vertex.

Solution. For every point $(a_1,\ldots,a_n)$ with $a_i = \pm1$ we let $N(a_1,\ldots,a_n)$ be the set of all points $(b_1,\ldots,b_n)$ for which $b_i = a_i$ for every i, except for one j for which $b_j = -a_j$.

Note that all elements of $N(a_1,\ldots,a_n)$ are of distance $2\sqrt{2}$ of one another, since each distance is

$$\sqrt{(1-(-1))^2+(-1-1)^2} = 2\sqrt{2}.$$

Note that $N(a_1,\ldots,a_n)$ contains precisely n points. Also, each point $(b_1,\ldots,b_n)\in N(a_1,\ldots,a_n)$ if and only if $(a_1,\ldots,a_n)\in N(b_1,\ldots,b_n)$. Therefore, each $(b_1,\ldots,b_n)$ belongs to precisely n sets of the form $N(a_1,\ldots,a_n)$.

Suppose on the contrary B contains no equilateral triangle. Therefore, B may not have more than 2 points from each set $N(a_1,\ldots,a_n)$. This means B can have at most 2×2^n points, however each point $(b_1,\ldots,b_n)$ in $N(a_1,\ldots,a_n)$ is in precisely n sets of the form $N(c_1,\ldots,c_n)$. Thus, each point is counted n times. Therefore, B may not have more than $2^{n+1}/n$ points, which is a contradiction. □

Example 6.5 (Putnam 2022, B2)**.** *Let* $\times$ *represent the cross product in* $\mathbb{R}^3$. *For what positive integers n does there exist a set* $S \subset \mathbb{R}^3$ *with exactly n elements such that*

$$S = \{\mathbf{v}\times\mathbf{w} : \mathbf{v},\mathbf{w}\in S\}?$$

Solution.(Video Solution) We show $n = 1,7$ are the only such possible integers.

First, note that if $\mathbf{v}\in S$, then $\mathbf{v}\times\mathbf{v}\in S$, which implies $\mathbf{0}\in S$. Also, note that $\mathbf{v} = \mathbf{u}\times\mathbf{w}$ for some $\mathbf{u},\mathbf{w}\in S$ and thus $-\mathbf{v} = \mathbf{w}\times\mathbf{u}\in S$. To summarize, we proved the following:

(i) $\mathbf{0}\in S$.

(ii) If $\mathbf{v}\in S$, then $-\mathbf{v}\in S$.

For $n = 1$, $S = \{\mathbf{0}\}$ satisfies the conditions of the theorem. From now on assume $n > 1$. Note that S must contain vectors that are not collinear. Otherwise $S = \{\mathbf{v}\times\mathbf{w} : \mathbf{v},\mathbf{w} \in S\} = \{\mathbf{0}\}$. We claim all nonzero vectors in S are unit vectors. Assume $\mathbf{0} \neq \mathbf{w} \in S$ and select some vector $\mathbf{x}$ not collinear with $\mathbf{w}$. We construct a sequence of vectors $\mathbf{w}_n$ in S with length is given by $||\mathbf{w}_n|| = ||\mathbf{w}||^{n-1}||\mathbf{w}\times\mathbf{x}||$. Set $\mathbf{w}_1 = \mathbf{w}\times\mathbf{x} \in S$. Inductively define $\mathbf{w}_n = \mathbf{w}_{n-1}\times\mathbf{w} \in S$. The length of $\mathbf{w}_n$ is $||\mathbf{w}_{n-1}||\cdot||\mathbf{w}|| = ||\mathbf{w}||^n||\mathbf{w}\times\mathbf{x}||$. If $\mathbf{w} \neq 1$, either the length of $\mathbf{w}_n$ tends to zero or it tends to infinity, which means S is not finite. Thus, we proved the following:
(iii) If $\mathbf{0} \neq \mathbf{v} \in S$, then $||\mathbf{v}|| = 1$.
Assume $\mathbf{v},\mathbf{w}$ are two non-collinear vectors in S. By assumption $\mathbf{v}\times\mathbf{w} \in S$. By (iii), $||\mathbf{v}\times\mathbf{w}|| = 1$. Since $\mathbf{v}$ and $\mathbf{w}$ are unit vectors, using the formula for $||\mathbf{v}\times\mathbf{w}||$ we deduce that $\mathbf{v}$ and $\mathbf{w}$ must be orthogonal.

We now pair up all nonzero vectors of S. We obtain $(n-1)/2$ pairs of parallel vectors. If we choose one vector from each pair, we obtain an orthonormal set of vectors in $\mathbb{R}^3$. Thus, $(n-1)/2 = 1,2,3$. Given two non-collinear vectors $\mathbf{u},\mathbf{v} \in S$, the vector $\mathbf{u}\times\mathbf{v}$ is in S and is not parallel to either $\mathbf{u}$ or $\mathbf{v}$. Thus, the only possible case is $(n-1)/2 = 3$, or $n = 7$. The set $S = \{\mathbf{0}, \pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3\}$ shows $n = 7$ is possible □

Example 6.6. *(Putnam 1985, A2) Let ABC be an acute-angled triangle with area 1. A rectangle $R = R_1R_2R_3R_4$ is inscribed in ABC so that R_1 and R_2 lie on side BC, the vertex R_3 lies on side AC and R_4 lies on side AB. Similarly, a rectangle S is inscribed in the triangle AR_3R_4, with two vertices on side R_3R_4 and one on each of the other two sides AR_3 and AR_4. What is the maximum total area of R and S over all possible choices of triangles ABC and rectangles R and S?*

Solution. (Video Solution)

□

Example 6.7. *Show that the plane cannot be covered by the interiors of finitely many parabolas.*

Solution. (Video Solution) □

Example 6.8. *Prove that there are infinitely many points on the unit circle such that the distance between any two of them is a rational number.*

Solution. (Video Solution)

□

6.5 Exercises

Exercise 6.1 (VTRMC 1983)**.** Let a triangle have vertices at $O(0,0)$, $A(a,0)$, and $B(b,c)$ in the (x,y)-plane.

(a) Find the coordinates of a point $P(x,y)$ in the exterior of $\triangle OAB$ satisfying area $(OAP) = \text{area}(OBP) = \text{area}(ABP)$.

(b) Find a point $Q(x,y)$ in the interior of $\triangle OAQ$ satisfying area $(OAQ) = \text{area}(OBQ) = \text{area}(ABQ)$

Exercise 6.2 (VTRMC 1985)**.** Consider an infinite sequence $\{c_k\}_{k=0}^{\infty}$ of circles. The largest, C_0, is centered at $(1,1)$ and is tangent to both the x and y-axes. Each smaller circle C_n is centered on the line through $(1,1)$ and $(2,0)$ and is tangent to the next larger circle C_{n-1} and to the x-axis. Denote the diameter of C_n by d_n for $n = 0,1,2,\ldots$ Find

(a) d_1

(b) $\sum_{n=0}^{\infty} d_n$

Exercise 6.3 (VTRMC 1987)**.** A path zig-zags from $(1,0)$ to $(0,0)$ along line segments $\overline{P_nP_{n+1}}$, where P_0 is $(1,0)$ and P_n is $(2^{-n},(-2)^{-n})$, for $n > 0$. Find the length of the path.

Exercise 6.4 (VTRMC 1987)**.** A triangle with sides of lengths a,b, and c is partitioned into two smaller triangles by the line which is perpendicular to the side of length c and passes through the vertex opposite that side. Find integers $a < b < c$ such that each of the two smaller triangles is similar to the original triangle and has sides of integer lengths.

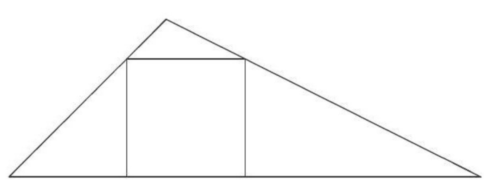
Exercise 6.5 (VTRMC 1987)**.** On Halloween, a black cat and a witch encounter each other near a large mirror positioned along the y-axis. The witch is invisible except by reflection in the mirror. At $t = 0$, the cat is at $(10,10)$ and the witch is at $(10,0)$. For $t \geq 0$, the witch moves toward the cat at a speed numerically equal to their distance of separation and the cat moves toward the apparent position of the witch, as seen by reflection, at a speed numerically equal to their reflected distance of separation. Denote by $(u(t),v(t))$ the position of the cat and by $(x(t),y(t))$ the position of the witch.

(a) Set up the equations of motion of the cat and the witch for $t \geq 0$.

(b) Solve for $x(t)$ and $u(t)$ and find the time when the cat strikes the mirror. (Recall that the mirror is a perpendicular bisector of the line joining an object with its apparent position as seen by reflection.)

Exercise 6.6 (VTRMC 1988)**.** A circle C of radius r is circumscribed by a parallelogram S. Let θ denote one of the interior angles of S, with $0 < \theta \leq \pi/2$. Calculate the area of S as a function of r and θ.

Exercise 6.7 (VTRMC 1989)**.** A square of side a is inscribed in a triangle of base b and height h as shown. Prove that the area of the square cannot exceed one-half the area of the triangle.



Exercise 6.8 (Putnam 1990, A3)**.** Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area greater than or equal to 5/2.

Exercise 6.9 (Putnam 1990, B6)**.** Let S be a nonempty closed bounded convex set in the plane. Let K be a line and t a positive number. Let L_1 and L_2 be support lines for S parallel to K_1, and let $\overline{L}$ be the line parallel to K and midway between L_1 and L_2. Let $B_S(K,t)$ be the band of points whose distance from $\overline{L}$ is at most $(t/2)w$, where w is the distance between L_1 and L_2. What is the smallest t such that

$$S \cap \bigcap_K B_S(K,t) \neq \emptyset$$

for all S? (K runs over all lines in the plane.)

Note: A support line ℓ of a convex set S is a line ℓ that passes through at least one boundary point of S, but no interior point of S.

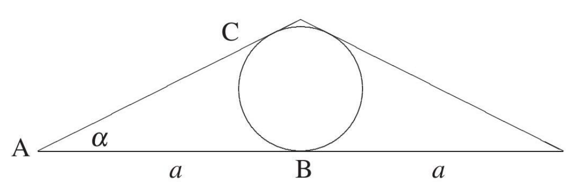
Exercise 6.10 (Putnam 1990, A4)**.** Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?

Exercise 6.11 (Putnam 1991, A1)**.** A 2×3 rectangle has vertices as $(0,0),(2,0),(0,3)$, and $(2,3)$. It rotates 90° clockwise about the point $(2,0)$. It then rotates 90° clockwise about the point $(5,0)$, then 90° clockwise about the point $(7,0)$, and finally, 90° clockwise about the point $(10,0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is (1,1).

Exercise 6.12 (Putnam 1991, A4)**.** Does there exist an infinite sequence of closed discs $D_1, D_2, D_3, \ldots$ in the plane, with centers $c_1, c_2, c_3, \ldots$, respectively, such that

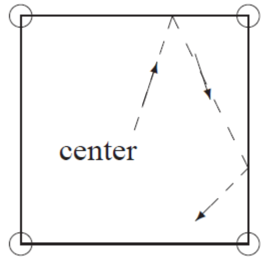
1. the c_i have no limit point in the finite plane,
2. the sum of the areas of the D_i is finite, and
3. every line in the plane intersects at least one of the D_i?

Exercise 6.13 (VTRMC 1991)**.** An isosceles triangle with an inscribed circle is labeled as shown in the figure. Find an expression, in terms of the angle α and the length a, for the area of the curvilinear triangle bounded by sides AB and AC and the arc BC.



Exercise 6.14 (VTRMC 1993)**.** Prove that a triangle in the plane whose vertices have integer coordinates cannot be equilateral.

Exercise 6.15 (VTRMC 1993)**.** On a small square billiard table with sides of length 2 ft., a ball is played from the center and after rebounding off the sides several times, goes into a cup at one of the corners. Prove that the total distance travelled by the ball is not an integer number of feet.



Exercise 6.16 (Putnam 1993, B5)**.** Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

Exercise 6.17 (Putnam 1994, B2)**.** For which real numbers c is there a straight line that intersects the curve

$$x^4+9x^3+cx^2+9x+4$$

in four distinct points?

Exercise 6.18 (VTRMC 1995)**.** A straight rod of length 4 inches has ends which are allowed to slide along the perimeter of a square whose sides each have length 12 inches. A paint brush is attached to the rod so that it can slide between the two ends of the rod. Determine the total possible area of the square which can be painted by the brush.

Exercise 6.19 (Putnam 1995, B2)**.** An ellipse, whose semi-axes have lengths a and b, rolls without slipping on the curve $y = c\sin\left(\frac{x}{a}\right)$. How are a,b,c related, given that the ellipse completes one revolution when it traverses one period of the curve?

Exercise 6.20 (Putnam 1996, A1)**.** Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed in the rectangle (without interior overlap). You may assume that the sides of the squares are parallel to the sides of the rectangle.

Exercise 6.21 (Putnam 1996, A2)**.** Let C_1 and C_2 be circles whose centers are 10 units apart, and whose radii are 1 and 3. Find, with proof, the locus of all points M for which there exists points X on C_1 and Y on C_2 such that M is the midpoint of the line segment XY.

Exercise 6.22 (Putnam 1996, B6)**.** Let $(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive real numbers x and y such that

$$(a_1,b_1)x^{a_1}y^{b_1}+(a_2,b_2)x^{a_2}y^{b_2}+\cdots+(a_n,b_n)x^{a_n}y^{b_n}=(0,0).$$

Exercise 6.23 (VTRMC 1997)**.** A disk of radius 1 *cm*. has a small hole at a point half way between the center and the circumference. The disk is lying inside a circle of radius 2 *cm*. A pen is put through the hole in the disk, and then the disk is moved once round the inside of the circle, keeping the disk in contact with the circle without slipping, so the pen draws a curve. What is the area enclosed by the curve?

Exercise 6.24 (Putnam 1997, A1)**.** A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC?

Exercise 6.25 (VTRMC 1998)**.** The radius of the base of a right circular cone is 1 . The vertex of the cone is V, and P is a point on the circumference of the base. The length of PV is 6 and the midpoint of PV is M. A piece of string is attached to M and wound tightly twice round the cone finishing at P. What is the length of the string?

Exercise 6.26 (VTRMC 1998)**.** Find the volume of the region which is common to the interiors of the three circular cylinders $y^2+z^2=1, z^2+x^2=1$ and $x^2+y^2=1$.

Exercise 6.27 (VTRMC 1998)**.** Let ABC be a triangle and let P be a point on AB. Suppose $\angle BAC = 70^\circ$, $\angle APC = 60^\circ, AC = \sqrt{3}$ and $PB = 1$. Prove that ABC is an isosceles triangle.

Exercise 6.28 (Putnam 1998, A1)**.** A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

Exercise 6.29 (Putnam 1998, A6)**.** Let A,B,C denote distinct points with integer coordinates in $\mathbb{R}^2$. Prove that if

$$(|AB|+|BC|)^2 < 8\cdot[ABC]+1$$

then A,B,C are three vertices of a square. Here $|XY|$ is the length of segment XY and $[ABC]$ is the area of triangle ABC.

Exercise 6.30 (Putnam 1998, B2)**.** Given a point (a,b) with $0<b<a$, determine the minimum perimeter of a triangle with one vertex at (a,b), one on the x-axis, and one on the line $y=x$. You may assume that a triangle of minimum perimeter exists.

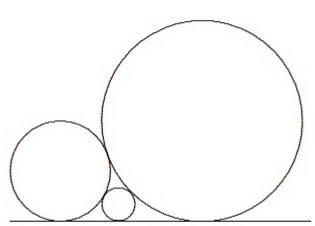
Exercise 6.31 (VTRMC 1999)**.** A rectangular box has sides of length 3,4 , 5. Find the volume of the region consisting of all points that are within distance 1 of at least one of the sides.

Exercise 6.32 (Putnam 1999, B1)**.** Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta\to 0}|EF|$.

Exercise 6.33 (VTRMC 2000)**.** Two diametrically opposite points P,Q lie on an infinitely long cylinder which has radius $2/\pi$. A piece of string with length 8 has its ends joined to P, is wrapped once round the outside of the cylinder, and then has its midpoint joined to Q (so there is length 4 of the string on each side of the cylinder). A paint brush is attached to the string so that it can slide along the full length the string. Find the area of the outside surface of the cylinder which can be painted by the brush.

Exercise 6.34 (Putnam 2000, A5)**.** Three distinct points with integer coordinates lie in the plane on a circle of radius $r>0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

Exercise 6.35 (VTRMC 2001)**.** Two circles with radii 1 and 2 are placed so that they are tangent to each other and a straight line. A third circle is nestled between them so that it is tangent to the first two circles and the line. Find the radius of the third circle.



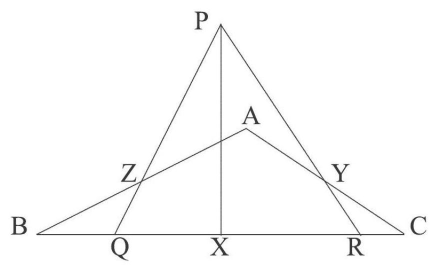
Exercise 6.36 (Putnam 2001, A4)**.** Triangle ABC has an area 1. Points E, F, G lie, respectively, on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of the triangle RST.

Exercise 6.37 (Putnam 2001, A6)**.** Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

Exercise 6.38 (VTRMC 2002)**.** Let a, b be positive constants. Find the volume (in the first octant) which lies above the region in the xy-plane bounded by $x = 0, x = \pi/2, y = 0$, $y\sqrt{b^2\cos^2 x + a^2 \sin^2 x} = 1$, and below the plane $z = y$.

Exercise 6.39 (VTRMC 2003)**.** Let T be a solid tetrahedron whose edges all have length 1 . Determine the volume of the region consisting of points which are at distance at most 1 from some point in T (your answer should involve $\sqrt{2}, \sqrt{3}, \pi$).

Exercise 6.40 (VTRMC 2003)**.** In the diagram below, X is the midpoint of BC, Y is the midpoint of AC, and Z is the midpoint of AB. Also $\angle ABC + \angle PQC = \angle ACB + \angle PRB = 90°$. Prove that $\angle PXC = 90°$.



Exercise 6.41 (Putnam 2003, B5)**.** Let A, B, and C be equidistant points on the circumference of a circle of unit radius centered at O, and let P be any point in the circle's interior. Let a, b, c be the distance from P to A, B, C, respectively. Show that there is a triangle with side lengths a, b, c, and that the area of this triangle depends only on the distance from P to O.

Exercise 6.42 (Putnam 2004, A2)**.** For $i = 1, 2$ let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \leq A_2$?

Exercise 6.43 (Putnam 2004, B3)**.** Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region

$$R = \{(x, y); 0 \leq x \leq a, 0 \leq y \leq f(x)\}$$

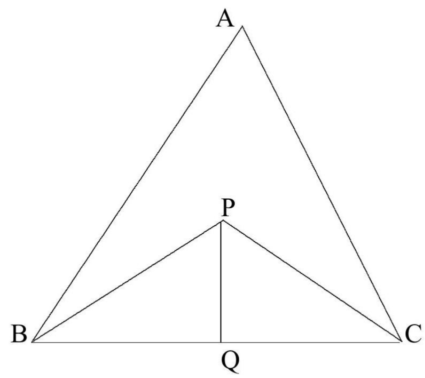
has perimeter k units and area k square units for some real number k.

Exercise 6.44 (VTRMC 2005)**.** A cubical box with sides of length 7 has vertices at

$$(0,0,0), (7,0,0), (0,7,0), (7,7,0), (0,0,7), (7,0,7), (0,7,7), (7,7,7).$$

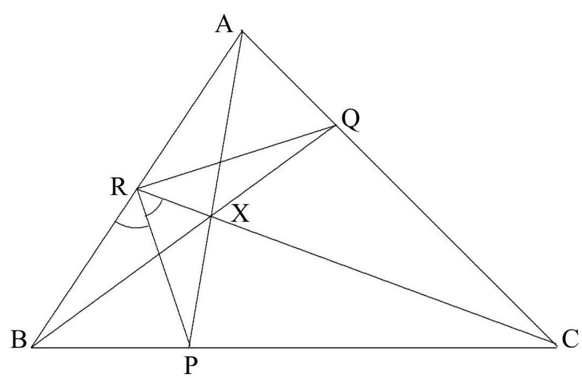
The inside of the box is lined with mirrors and from the point $(0,1,2)$, a beam of light is directed to the point $(1,3,4)$. The light then reflects repeatedly off the mirrors on the inside of the box. Determine how far the beam of light travels before it first returns to its starting point at $(0,1,2)$.

Exercise 6.45 (VTRMC 2006)**.** In the diagram below BP bisects $\angle ABC, CP$ bisects $\angle BCA$, and PQ is perpendicular to BC. If $BQ \cdot QC = 2PQ^2$, prove that $AB + AC = 3BC$.



Exercise 6.46 (Putnam 2006, B3)**.** Let S be a finite set of points in the plane. A linear partition of S is an unordered pair $\{A, B\}$ of subsets of S such that $A \cup B = S$, $A \cap B = \emptyset$, and A and B lie on opposite sides of some straight line disjoint from S (A or B may be empty). Let L_S be the number of linear partitions of S. For each positive integer n, find the maximum of L_S over all sets S of n points.

Exercise 6.47 (VTRMC 2007)**.** In the diagram below, P, Q, R are points on BC, CA, AB respectively such that the lines AP, BQ, CR are concurrent at X. Also PR bisects $\angle BRC$, i.e. $\angle BRP = \angle PRC$. Prove that $\angle PRQ = 90°$.



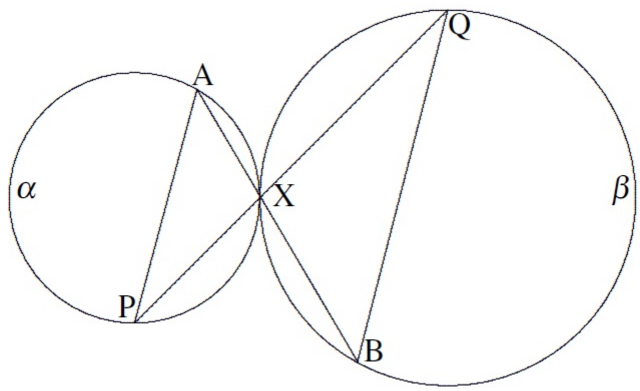
Exercise 6.48 (Putnam 2007, A2)**.** Find the least possible area of a convex set in the plane that intersects both branches of the hyperbola $xy = 1$ and both branches of the hyperbola $xy = -1$. (A set S in the plane is called *convex* if for any two points in S the line segment connecting them is contained in S.)

Exercise 6.49 (Putnam 2007, A6)**.** A *triangulation* $\mathscr{T}$ of a polygon P is a finite collection of triangles whose union is P, and such that the intersection of any two triangles is either empty, or a shared vertex, or a shared side. Moreover, each side is a side of exactly one triangle in $\mathscr{T}$. Say that $\mathscr{T}$ is *admissible* if every internal vertex is shared by 6 or more triangles. For example, [figure omitted.] Prove that there is an integer M_n, depending only on n, such that any admissible triangulation of a polygon P with n sides has at most M_n triangles.

Exercise 6.50 (VTRMC 2008)**.** Let ABC be a triangle, let M be the midpoint of BC, and let X be a point on AM. Let BX meet AC at N, and let CX meet AB at P. If $\angle MAC = \angle BCP$, prove that $\angle BNC = \angle CPA$.

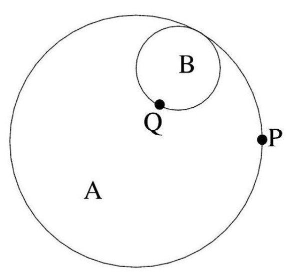
Exercise 6.51 (Putnam 2008, B3)**.** What is the largest possible radius of a circle contained in a 4-dimensional hypercube of side length 1?

Exercise 6.52 (VTRMC 2009)**.** Two circles α, β touch externally at the point X. Let A, P be two distinct points on α different from X, and let AX and PX meet β again in the points B and Q respectively. Prove that AP is parallel to QB.



Exercise 6.53 (VTRMC 2010)**.** Let $\triangle ABC$ be a triangle with sides a, b, c and corresponding angles A, B, C (so $a = BC$ and $A = \angle BAC$ etc.). Suppose that $4A + 3C = 540°$. Prove that $(a-b)^2(a+b) = bc^2$.

Exercise 6.54 (VTRMC 2010)**.** Let A, B be two circles in the plane with B inside A. Assume that A has radius $3, B$ has radius $1, P$ is a point on A, Q is a point on B, and A and B touch so that P and Q are the same point. Suppose that A is kept fixed and B is rolled once round the inside of A so that Q traces out a curve starting and finishing at P. What is the area enclosed by this curve?



Exercise 6.55 (Putnam 2010, B2)**.** Given that A, B, and C are noncollinear points in the plane with integer coordinates such that the distances AB, AC, and BC are integers, what is the smallest possible value of AB?

Exercise 6.56 (Putnam 2011, A1)**.** Define a *growing spiral* in the plane to be a sequence of points with integer coordinates $P_0 = (0,0), P_1, \ldots, P_n$ such that $n \geq 2$ and:

- the directed line segments $P_0P_1, P_1P_2, \ldots, P_{n-1}P_n$ are in the successive coordinate directions east (for P_0P_1), north, west, south, east, etc.;
- the lengths of these line segments are positive and strictly increasing.

[Picture omitted.] How many of the points (x, y) with integer coordinates $0 \leq x \leq 2011, 0 \leq y \leq 2011$ *cannot* be the last point, P_n of any growing spiral?

Exercise 6.57 (Putnam 2012, B2)**.** Let P be a given (non-degenerate) polyhedron. Prove that there is a constant $c(P) > 0$ with the following property: If a collection of n balls whose volumes sum to V contains the entire surface of P, then $n > c(P)/V^2$.

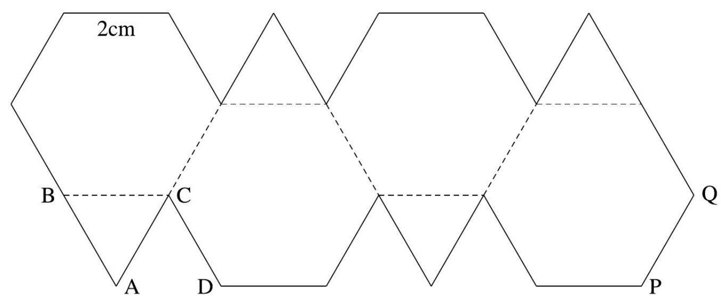
Exercise 6.58 (VTRMC 2013)**.** Let ABC be a right-angled triangle with $\angle ABC = 90°$, and let D be on AB such that $AD = 2DB$. What is the maximum possible value of $\angle ACD$?

Exercise 6.59 (Putnam 2013, A5)**.** For $m \geq 3$, a list of $\binom{m}{3}$ real numbers a_{ijk} $(1 \leq i < j < k \leq m)$ is said to be *area definite* for $\mathbb{R}^n$ if the inequality

$$\sum_{1\leq i<j<k\leq m} a_{ijk} \cdot \text{Area}(\Delta A_i A_j A_k) \geq 0$$

holds for every choice of m points $A_1, \ldots, A_m$ in $\mathbb{R}^n$. For example, the list of four numbers $a_{123} = a_{124} = a_{134} = 1$, $a_{234} = -1$ is area definite for $\mathbb{R}^2$. Prove that if a list of $\binom{m}{3}$ numbers is area definite for $\mathbb{R}^2$, then it is area definite for $\mathbb{R}^3$.

Exercise 6.60 (VTRMC 2015)**.** The planar diagram below, with equilateral triangles and regular hexagons, sides length 2 cm., is folded along the dashed edges of the polygons, to create a closed surface in three dimensional Euclidean spaces. Edges on the periphery of the planar diagram are identified (or glued) with precisely one other edge on the periphery in a natural way. Thus for example, BA will be joined to QP and AC will be joined to DC. Find the volume of the three-dimensional region enclosed by the resulting surface.



Exercise 6.61 (Putnam 2015, A1)**.** Let A and B be points on the same branch of the hyperbola $xy = 1$. Suppose that P is a point lying between A and B on this hyperbola, such that the area of the triangle APB is as large as possible. Show that the region bounded by the hyperbola and the chord AP has the same area as the region bounded by the hyperbola and the chord PB.

Exercise 6.62 (Putnam 2016, B3)**.** Suppose that S is a finite set of points in the plane such that the area of triangle $\triangle ABC$ is at most 1 whenever A, B, and C are in S. Show that there exists a triangle of area 4 that (together with its interior) covers the set S.

Exercise 6.63 (VTRMC 2017)**.** Let ABC be a triangle and let P be a point in its interior. Suppose $\angle BAP = 10°$, $\angle ABP = 20°$, $\angle PCA = 30°$ and $\angle PAC = 40°$. Find $\angle PBC$.

Exercise 6.64 (VTRMC 2017)**.** Let P be an interior point of a triangle of area T. Through the point P, draw lines parallel to the three sides, partitioning the triangle into three triangles and three parallelograms. Let a, b and c be the areas of the three triangles. Prove that $\sqrt{T} = \sqrt{a} + \sqrt{b} + \sqrt{c}$.

Exercise 6.65 (Putnam 2017, A6)**.** The 30 edges of a regular icosahedron are distinguished by labeling them $1, 2, \dots, 30$. How many different ways are there to paint each edge red, white, or blue such that each of the 20 triangular faces of the icosahedron has two edges of the same color and a third edge of a different color? [Note: the top matter on each exam paper included the Mathematical Association of America, which is itself an icosahedron.]

Exercise 6.66 (Putnam 2017, B1)**.** Let L_1 and L_2 be distinct lines in the plane. Prove that L_1 and L_2 intersect if and only if, for every real number $\lambda \neq 0$ and every point P not on L_1 or L_2, there exist points A_1 on L_1 and A_2 on L_2 such that $\overrightarrow{PA_2} = \lambda \overrightarrow{PA_1}$.

Exercise 6.67 (Putnam 2017, B5)**.** A line in the plane of a triangle T is called an *equalizer* if it divides T into two regions having equal area and equal perimeter. Find positive integers $a > b > c$, with a as small as possible, such that there exists a triangle with side lengths a, b, c that has exactly two distinct equalizers.

Exercise 6.68 (Putnam 2018, A6)**.** Suppose that A, B, C, and D are distinct points, no three of which lie on a line, in the Euclidean plane. Show that if the squares of the lengths of the line segments AB, AC, AD, BC, BD, and CD are rational numbers, then the quotient

$$\frac{\text{area}(\triangle ABC)}{\text{area}(\triangle ABD)}$$

is a rational number.

Exercise 6.69 (VTRMC 2019)**.** Let X be the point on the side AB of the triangle ABC such that $BX/XA = 9$. Let M be the midpoint of BX and let Y be the point on BC such that $\angle BMY = 90^\circ$. Suppose AC has length 20 and that the area of the triangle XYC is $9/100$ of the area of the triangle ABC. Find the length of BC.

Exercise 6.70 (Putnam 2021, A3)**.** Determine all positive integers N for which the sphere

$$x^2 + y^2 + z^2 = N$$

has an inscribed regular tetrahedron whose vertices have integer coordinates.

Exercise 6.71 (VTRMC 2022)**.** Let A and B be the two foci of an ellipse and let P be a point on this ellipse. Prove that the focal radii of P (that is, the segments AP and BP) form equal angles with the tangent to the ellipse at P.

Exercise 6.72 (Putnam 2023, A4)**.** Let $v_1, \dots, v_{12}$ be unit vectors in $\mathbb{R}^3$ from the origin to the vertices of a regular icosahedron. Show that for every vector $v \in \mathbb{R}^3$ and every $\varepsilon > 0$, there exist integers $a_1, \dots, a_{12}$ such that

$$\|a_1 v_1 + \cdots + a_{12} v_{12} - v\| < \varepsilon.$$

Chapter 7

Inequalities

7.1 Basics

Some basic properties of inequalities are listed below:

- If $a > b$ and $c > 0$, then $ac > bc$.
- If $a > b$ and $c < 0$, then $ac < bc$.
- if $a > b$ and $ab > 0$, then $\frac{1}{a} < \frac{1}{b}$
- If $a > b$ and $c > d$, then $a + c > b + d$.
- If $a > b > 0$ and $c > d > 0$, then $ac > bd$.
- If $a > b$ and n is odd, then $a^n > b^n$.
- If $a > b > 0$, then $a^n > b^n$.
- The Trivial Inequality: $a^2 \geq 0$.
- $|x| < a$ if and only if $-a < x < a$.
- Triangle Inequality: $|x + y| \leq |x| + |y|$.

7.2 Important Theorems

Theorem 7.1 (AM-GM Inequality)**.** *Let $x_1, \ldots, x_n$ be nonnegative real numbers. Then*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}.$$

Furthermore, equality holds iff $x_1 = x_2 = \cdots = x_n$.

A video proof of the AM-GM Inequality can be found here.

Theorem 7.2 (Weighted AM-GM Inequality). *Suppose $w_1,\dots,w_n > 0$ satisfy $w_1+\cdots+w_n = 1$, and $x_1,\dots,x_n$ are positive real numbers. Then.*

$$x_1^{w_1}\cdots x_n^{w_n} \le w_1x_1+\cdots+w_nx_n.$$

Note that when $w_1 = \cdots = w_n = 1\frac{1}{n}$, the Weighted AM-GM turns into the regular AM-GM Inequality.

Theorem 7.3 (Cauchy-Schwarz Inequality). *Let V be an inner product vector space. For every $\mathbf{v},\mathbf{w} \in V$, we have*

- $|\langle \mathbf{v},\mathbf{w}\rangle| \le ||\mathbf{v}||\cdot||\mathbf{w}||$.
- $||\mathbf{v}+\mathbf{w}|| \le ||\mathbf{v}||+||\mathbf{w}||$.

Using the standard inner product on $\mathbb{R}^n$ we obtain the following:

Theorem 7.4 (Cauchy-Schwarz Inequality). *Let $a_1,\dots,a_n,b_1,\dots,b_n$ be real numbers. Then*

$$(a_1b_1+a_2b_2+\cdots+a_nb_n)^2 \le (a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2).$$

Furthermore, the equality holds iff the vectors $(a_1,a_2,\dots,a_n)$ and $(b_1,b_2,\dots,b_n)$ are scalar multiples of each other. In other words, the equality holds iff there is a constant $c \in \mathbb{R}$, for which $a_k = cb_k$ for all k, or $b_k = 0$ for all k.

Remark. When a_k and b_k are complex numbers, the Cauchy-Schwarz Inequality is true as follows: $|\sum_{k=1}^{n} a_k\overline{b}_k|^2 \le (\sum_{k=1}^{n}|a_k|^2)(\sum_{k=1}^{n}|b_k|^2)$. The equality holds iff the vectors $(a_1,\dots,a_n)$ and $(b_1,\dots,b_n)$ are scalar multiples of each other. In other words, there is $c \in \mathbb{C}$ for which $a_j = cb_j$ for all j, or $b_j = 0$ for all j.

Theorem 7.5 (HM-GM-AM-SRM Inequality). *Let $x_1,\dots,x_n$ be positive real numbers. Then*

$$\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}} \le \sqrt[n]{x_1x_2\cdots x_n} \le \frac{x_1+x_2+\cdots+x_n}{n} \le \sqrt{\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}}.$$

Furthermore, each equality holds iff $x_1 = x_2 = \cdots = x_n$.

Definition 7.1.

Theorem 7.6 (Jensen's Inequality). *Let I be an open interval and $f: I \to \mathbb{R}$ be a function that is concave up. Then, for every $x_1,\dots,x_n \in I$, and every $w_1,\dots,w_n \in (0,1)$ satisfying $w_1+\cdots+w_n = 1$, we have*

$$f(w_1x_1+\cdots+w_nx_n) \le w_1f(x_1)+\cdots+w_nf(x_n).$$

A video proof of the Jensen's inequality can be found in this YouTube Video.

Theorem 7.7. *Suppose $f(x) \ge g(x)$ for all $x \in [a,b]$. Then, $\int_a^b f(x) \ge \int_a^b g(x)$.*

Theorem 7.8 (Power Mean Inequality). *Suppose $x_1,\dots,x_n$ are positive real numbers and $w_1,\dots,w_n \in (0,1)$ satisfy $\sum_{j=1}^{n} w_j = 1$. Then, for every two nonzero $\alpha \ge \beta$ we have*

$$(w_1x_1^\alpha+\cdots+w_nx_n^\alpha)^{1/\alpha} \ge \left(w_1x_1^\beta+\cdots+w_nx_n^\beta\right)^{1/\beta}$$

Furthermore, if $\alpha > 0 > \beta$, then

$$(w_1x_1^\alpha+\cdots+w_nx_n^\alpha)^{1/\alpha} \ge x_1^{w_1}\cdots x_n^{w_n} \ge \left(w_1x_1^\beta+\cdots+w_nx_n^\beta\right)^{1/\beta}.$$

7.3 Classical Examples

Example 7.1 (Bernoulli's inequality)**.** *Prove that for every real number $x \geq -1$ and every positive integer n we have $(1+x)^n \geq 1+nx$. When does the equality hold?*

Solution. We will prove the inequality by induction on n.

Basis step. For $n = 1$ both sides are $1+x$, and thus the equality holds.

Inductive step. Suppose $(1+x)^n \geq 1+nx$ for some positive integer n and all $x \geq -1$. Since $1+x \geq 0$ we can multiply both sides of the inequality by $1+x$ to obtain

$$(1+x)^{n+1} \geq (1+nx)(1+x) = 1+nx^2+(n+1)x \geq 1+(n+1)x.$$

This inequality is valid since $x^2 \geq 0$. This completes the proof.

We see that the equality holds for $n = 1$. In the inductive step, if the equality holds then we must have $x^2 = 0$, in which case we have $x = 0$. Furthermore, we notice that for $x = 0$ both sides are 1. Thus, the equality holds if and only if $x = 0$ or $n = 1$. $\square$

7.4 Further Examples

Example 7.2 (VTRMC 1979)**.** *Show that the right circular cylinder of volume V which has the least surface area is the one whose diameter is equal to its altitude. (The top and bottom are part of the surface.)*

Solution. Let r be the radius of the base and h be the height. The volume $V = \pi r^2 h$ is a constant. The surface area is

$$2\pi rh + 2\pi r^2 = 2\frac{V}{r} + 2\frac{V}{h} = \frac{V}{r} + \frac{V}{r} + \frac{2V}{h} \geq 3\sqrt[3]{\frac{V^3}{r^2 h}} = \pi\sqrt[3]{V^2}.$$

Here we used the AM-GM inequality. The minimum occurs when $\dfrac{V}{r} = \dfrac{2V}{h}$, or $2r = h$, as desired. $\square$

Example 7.3 (Putnam 1950, B1)**.** *Let $P_1, \dots, P_n$ be, not necessarily distinct, points on the number line. For what point P is the sum $\sum_{i=1}^{n} |PP_i|$ minimized?*

Scratch: We will try a few small cases.

For $n = 1$, the minimum is clearly obtained when $P = P_1$.

For $n = 2$, we see that $|PP_1| + |PP_2| \geq |P_1P_2|$ and the equality holds when P is between P_1 and P_2.

For $n = 3$, we see that it becomes important where the points are, so let's assume the points are in order from the smallest to the largest. We see that $|PP_1| + |PP_3| \geq |P_1P_3|$. We can make $|PP_2| = 0$, by taking $P = P_2$. So the minimum is obtained when $P = P_2$.

We can see that each time we can ensure the sum $|PP_1| + |PP_n|$ is minimized by making sure P is between P_1 and P_n. Then the same argument can be repeated. So clearly the answer is different when n is odd and when n is even. However, we only need one such P, which we can obtain by taking the median point of $P_{\lfloor (n+1)/2 \rfloor}$.

Solution. Suppose $P_1, \ldots, P_n$ are represented by real numbers $x_1 \leq \cdots \leq x_n$ and P is represented by the real number x. We claim the minimum of $\sum_{i=1}^{n} |PP_i| = \sum_{i=1}^{n} |x - x_i|$ is obtained at $x = x_{\lfloor (n+1)/2 \rfloor}$. We will prove this by induction on n.

For $n = 1$, $|x - x_1| \geq 0$, and equality holds when $x = x_1$, as desired.

For $n = 2$, by the triangle inequality we have $|x - x_1| + |x - x_2| \geq |x_1 - x_2|$, and the equality occurs when $x = x_1$, as desired.

By inductive hypothesis know $\sum_{k=2}^{n-1} |x - x_k|$ is minimized when $x = x_{\lfloor (n+1)/2 \rfloor}$. Since $x_1 \leq x_{\lfloor (n+1)/2 \rfloor} \leq x_n$, we have $|x_{\lfloor (n+1)/2 \rfloor} - x_1| + |x_{\lfloor (n+1)/2 \rfloor} - x_n| = x_n - x_1$ and that by the triangle inequality for every x, we have $|x - x_1| + |x - x_n| \geq |x_1 - x_n| = x_n - x_1$. This completes the proof. $\square$

Example 7.4 (IMC 2019, Problem 3)**.** *Let $f : (-1, 1) \to \mathbb{R}$ be a twice differentiable function such that*

$$2f'(x) + xf''(x) \geq 1 \quad \textit{for } x \in (-1, 1).$$

Prove that

$$\int_{-1}^{1} xf(x)\,\mathrm{d}x \geq \frac{1}{3}.$$

Scratch: We can integrate inequalities. If we write the left side as a derivative of a function then integrating would be easier. For that we need an integrating factor. So, we want to make sure $2\mu f'(x) + \mu x f''(x)$ is derivative of $\mu x f'(x)$, which means we need $\mu' x + \mu = 2\mu$, and thus $\mu = x$ is an integrating factor. Multiplying the inequality by x and integrating from 0 to x we obtain $x^2 f'(x) \geq \frac{x^2}{2}$, or $2f'(x) \geq 1$. Integrating again we get $2f(x) - 2f(0) \geq x$. Multiplying by x we obtain $2xf(x) \geq 2f(0)x + x^2$ (we have to be careful about when x is negative). Integrating again gives us the inequality.

Solution. Note that $\dfrac{d}{dx}(x^2 f'(x)) = 2xf'(x) + x^2 f''(x) = x(2f'(x) + xf''(x)) \geq x$ if $x > 0$. Integrating from 0 to x yields $x^2 f'(x) \geq x^2/2$. similarly when $x < 0$, we have $\dfrac{d}{dx}(x^2 f'(x)) \leq x$, and integrating from x to 0 we get the same inequality. Therefore, $x^2 f'(x) \geq x^2/2$ for all $x \in [-1, 1]$. This means $f'(x) \geq 1/2$ for all $x \in [-1, 1]$. Integrating from 0 to x we get $f(x) - f(0) \geq x/2$, when $x > 0$ and $f(x) - f(0) \leq x/2$, when $x < 0$. Therefore, for all $x \in [-1, 1]$, we obtain $xf(x) \geq xf(0) + x^2/2$. Integrating this from -1 to 1 we obtain

$$\int_{-1}^{1} xf(x)\,\mathrm{d}x + f(0)\int_{-1}^{1} x\,\mathrm{d}x \geq \int_{-1}^{1} \frac{x^2}{2}\,\mathrm{d}x = 1/3 \Rightarrow \int_{-1}^{1} xf(x)\,\mathrm{d}x \geq 1/3.$$

$\square$

Example 7.5 (IMO 2020, Problem 2)**.** *The real numbers a,b,c,d are such that $a \geq b \geq c \geq d > 0$ and $a+b+c+d = 1$. Prove that*

$$(a+2b+3c+4d)a^a b^b c^c d^d < 1.$$

Scratch: First note that using the given condition, the inequality can be written as

$$(1+b+2c+3d)a^a b^b c^c d^d < 1.$$

We will try the analogous problem for two and three variables to get some insight. For two variables the problem becomes the following:

$$0 < b \leq a, \text{ and } a+b = 1 \Rightarrow (a+2b)a^a b^b < 1.$$

$a+2b = 1+b$. So we need to prove $(1+b)a^a b^b < 1$. At this point to simplify $a^a b^b$ I use the fact that $b \leq a$ to get $a^a b^b \leq a^a a^b = a^{a+b} = a$. This means

$$(1+b)a^a b^b \leq (1+b)a = a+ba < a+b = 1.$$

This is precisely what we wanted to show. Now, let's work on the case for three variables a,b,c. We need to prove the following:

$$0 < c \leq b \leq a, \text{ and } a+b+c = 1 \Rightarrow (a+2b+3c)a^a b^b c^c < 1.$$

Similar to above using $a+b+c = 1$ and the inequality $a^a b^b c^c \leq a^a a^b a^c = a^{a+b+c} = a$ we obtain

$$(a+2b+3c)a^a b^b c^c \leq (1+b+2c)a = a+ba+2ca$$

It is not clear if this quantity is less than 1. In fact after setting $b = c$ and a bit of exploring we realize for $a = \frac{5}{6}$, $b = c = \frac{1}{12}$ we have $a+ba+2ca = \frac{25}{24}$, which is larger than 1. So, we need to make the iequality $a^a b^b c^c < a$ stronger, and frankly that is not surprising because otherwise problem would have probably turned out to be too easy for the IMO. I do notice that the terms ba and $2ca$ in the sum $a+ba+2ca$ which are quadratic are relatively small. So perhaps we can work on making the term a smaller. Applying the weighted AM-GM we have

$$a^a b^b c^c \leq a \cdot a + b \cdot b + c\dot{c} = a^2 + b^2 + c^2.$$

This means, we get

$$(1+b+2c)a^a b^b c^b \leq a^a b^b c^c + ba + 2ca \leq a^2 + b^2 + c^2 + ba + 2ca < (a+b+c)^2 = 1.$$

Now, we will employ a similar technique solve the given problem as follows:

Solution. (Video Solution) By the weighted AM-GM we have

$$a^a b^b c^c d^d \leq a \cdot a + b \cdot b + c \cdot c + d \cdot d = a^2 + b^2 + c^2 + d^2.$$

On the other hand since $a \geq b,c,d$ we also have

$$a^a b^b c^c d^d \leq a^a a^b a^c a^d = a^{a+b+c+d} = a.$$

Therefore, we get the following

$$\begin{aligned}(a+2b+3c+4d)a^a b^b c^c d^c &= (1+b+2c+3d)a^a b^b c^c d^d \\ &\leq a^2+b^2+c^2+d^2+ba+2ca+3da \\ &\leq a^2+b^2+c^2+d^2+2ba+2ca+2da \\ &< (a+b+c+d)^2 = 1\end{aligned}$$

$\square$

Example 7.6 (IMO 2023, Problem 4)**.** *Let $x_1, x_2, \ldots, x_{2023}$ be pairwise different positive real numbers such that*

$$a_n = \sqrt{(x_1+x_2+\cdots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \ldots, 2023$. Prove that $a_{2023} \geq 3034$.

Scratch: Is there anything special about 2023? or can we prove a similar inequality for every a_n? Let's try small values of n. $a_1 = 1 \geq 1$.

$$a_2^2 = 2+\frac{x_1}{x_2}+\frac{x_2}{x_1} \geq 2+2 = 4 \Rightarrow a_2^2 \geq 4 \Rightarrow a_2 \geq 2.$$

Here we used the AM-GM Inequality. The equality does not occur since $x_1 \neq x_2$. Thus $a_2 \geq 3$. If we try something similar for a_3 we obtain

$$a_3^2 = 3+\sum_{i\neq j}\frac{x_i}{x_j} \geq 3+6\sqrt[6]{\prod_{i\neq j}\frac{x_i}{x_j}} = 9.$$

Similarly the equality cannot happen since x_1, x_2, x_3 are distinct. Therefore, $a_3 \geq 4$. Let's try a_4.

$$a_4^2 = 4+\sum_{i\neq j}\frac{x_i}{x_j} > 4+12 \Rightarrow a_4 \geq 5.$$

We notice that 2023 is about two-thirds of 3034, so it looks like the progress is not fast enough. In fact if we try the same thing for a_n^2 we get

$$a_n^2 = n+\sum_{i\neq j}\frac{x_i}{x_j} > n+n(n-1) = n^2 \Rightarrow a_n \geq n+1.$$

This is too weak to give us anything meaningful. Perhaps we can use induction to obtain something better:

$$a_{n+1}^2 = a_n^2+1+\sum_{i=1}^{n}\left(\frac{x_{n+1}}{x_i}+\frac{x_i}{x_{n+1}}\right) > a_n^2+1+2n.$$

Again this is not meaningful, since $a_n^2+1+2n < (a_n+1)^2$. So, we can only get $a_{n+1} \geq a_n+1$. Next, I realized there is a way to relate the sum $\sum_{i=1}^{n}\left(\frac{x_{n+1}}{x_i}+\frac{x_i}{x_{n+1}}\right)$ with a_n by writing it down as follows:

$$x_{n+1}\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)+\frac{1}{x_{n+1}}(x_1+\cdots+x_n) \geq 2\sqrt{x_{n+1}\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)\frac{1}{x_{n+1}}(x_1+\cdots+x_n)} = 2a_n$$

If we can show the inequality does not hold, then we would have

$$a_{n+1}^2 > a_n^2 + 1 + 2a_n = (a_n+1)^2 \Rightarrow a_{n+1} \geq a_n + 2.$$

This is great, except it is too good to be true!! I notice if we prove the above, it means we will have proven something stronger than what they asked us to prove. Note that $3034 \approx \frac{3}{2} \times 2023$. So, roughly speaking, we only need to prove a_n increases by $3/2$ as n increases by 1. So, instead we will relate a_{n+2} and a_n in a similar manner to obtain the following solution:

Solution.(Video Solution) We prove that for every $n \geq 1$ we have $a_{n+2} \geq a_n + 3$.

$$\begin{aligned}
a_{n+2}^2 &= a_n^2 + \frac{x_{n+1}}{x_{n+1}} + \frac{x_{n+2}}{x_{n+2}} + \frac{x_{n+1}}{x_{n+2}} + \frac{x_{n+2}}{x_{n+1}} + \\
&\quad + x_{n+1}\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right) + \frac{1}{x_{n+1}}(x_1 + \cdots + x_n) + x_{n+2}\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right) + \frac{1}{x_{n+2}}(x_1 + \cdots + x_n) \\
&\geq a_n^2 + 4 + 4\sqrt[4]{x_{n+1}\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)\frac{1}{x_{n+1}}(x_1 + \cdots + x_n)x_{n+2}\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)\frac{1}{x_{n+2}}(x_1 + \cdots + x_n)} \\
&= a_n^2 + 4 + 4a_n = (a_n+2)^2 \Rightarrow a_{n+2} \geq a_n + 2
\end{aligned}$$

Note that equality does not occur since $x_{n+1} \neq x_{n+2}$. Therefore, $a_{n+2} \geq a_n + 3$. Thus, we will have

$$a_{2023} \geq a_{2021} + 3 \geq a_{2019} + 3 \times 2 \geq a_{2017} + 3 \times 3 \geq a_{2015} + 3 \times 4 \geq \cdots \geq a_1 + 3 \times 1011 = 3034.$$

Therefore, $a_{2023} \geq 3034$. □

Example 7.7 (IMO 2021, Shortlisted Problem, A3)**.** *Given a positive integer n, find the smallest value of*

$$\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_n}{n} \right\rfloor$$

over all permutations $a_1, a_2, \ldots, a_n$ *of* $1, 2, \ldots, n$

Scratch: Let's denote the answer to the problem by x_n, and let's evaluate x_n for small values of n.

We have $x_1 = 1$, $x_2 = 2$ and $x_3 = 2$ which can easily be checked.

By the time I reached $n = 4$ I realized I can find the minimum for n, by checking which of the a_j's is n. Here is an example for $n = 6$:

$$a_4 = 6 \Rightarrow \underbrace{\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \left\lfloor \frac{a_3}{3} \right\rfloor}_{\geq x_3} + \left\lfloor \frac{6}{4} \right\rfloor + \left\lfloor \frac{a_5}{5} \right\rfloor + \left\lfloor \frac{a_6}{6} \right\rfloor \geq x_3 + 1.$$

This is true since a_1, a_2, a_3 may be made smaller if necessary by replacing them by a permutation of $1,2,3$. Note also that if we set $a_5 = 4, a_6 = 5$ and choose a_1, a_2, a_3 in a way that $\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \left\lfloor \frac{a_3}{3} \right\rfloor = x_3$ we can obtain $x_3 + 1$. In other words, we can obtain a recursive formula for x_n. That led me to the following solution:

Solution.(Video Solution) Let x_n be the answer to this problem. We will show $x_n = \lceil \log_2(n+1) \rceil$.

First, we will prove that

$$x_n = \min\left\{ x_k + \left\lfloor \frac{n}{k+1} \right\rfloor \mid k = 0, \ldots, n-1 \right\}, \text{ where } x_0 = 0 \qquad (*)$$

Suppose $a_1, \ldots, a_n$ is a permutation of $1, \ldots, n$ with $a_{k+1} = n$ for some $k = 0, \ldots, n-1$. Since $a_1, \ldots, a_k$ are distinct integers, we have

$$\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_k}{k} \right\rfloor \geq x_k.$$

Note also that if we choose a permutation $b_1, \ldots, b_k$ of $1, \ldots, k$ for which $\left\lfloor \frac{b_1}{1} \right\rfloor + \left\lfloor \frac{b_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{b_k}{k} \right\rfloor = x_k$ the permutation $b_1, \ldots, b_k, n+1, k+1, \ldots, n-1$ satisfies

$$\left\lfloor \frac{b_1}{1} \right\rfloor + \left\lfloor \frac{b_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{b_k}{k} \right\rfloor + \left\lfloor \frac{n}{k+1} \right\rfloor + \left\lfloor \frac{k+1}{k+2} \right\rfloor + \cdots + \left\lfloor \frac{n-1}{n} \right\rfloor = x_k + \left\lfloor \frac{n}{k+1} \right\rfloor.$$

Therefore, $\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_n}{n} \right\rfloor \geq x_k + \left\lfloor \frac{n}{k+1} \right\rfloor$. Furthermore, we showed the equality holds for some permutation $a_1, \ldots, a_n$, which completes the proof of $(*)$.

Now, we will prove by induction that $x_n = \lfloor \log_2(n+1) \rfloor$.

$x_1 = 1 = \lceil \log_2 2 \rceil$.

Suppose $\lceil \log_2(n+1) \rceil = m$. Thus, $2^{m-1} < n+1 \leq 2^m$.

By inductive hypothesis $x_k = \lceil \log_2(k+1) \rceil$ for $k = 0, \ldots, n-1$. This implies $2^{x_k - 1} < k+1 \leq 2^{x_k}$. We have

$$x_k + \left\lfloor \frac{n}{k+1} \right\rfloor \geq x_k + \frac{2^{m-1}}{2^{x_k}} = x_k + 2^{m - x_k - 1} \geq m.$$

Here we used the fact that for every nonnegative integer r we have $2^{r-1} \geq r$. This is easy to prove by induction on r.

Therefore, $x_n \geq m$.

Since, by inductive hypothesis, $x_{2^{m-1}-1} = \lceil \log_2(2^{m-1}) \rceil = m-1$, there is a permutation $c_1, \ldots, c_{2^{m-1}-1}$ of $1, \ldots, 2^{m-1} - 1$ for which

$$\left\lfloor \frac{c_1}{1} \right\rfloor + \left\lfloor \frac{c_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{c_{2^{m-1}-1}}{2^{m-1}-1} \right\rfloor = m-1.$$

Now, the following shows $x_n = m$, as desired.

$$\underbrace{\left\lfloor \frac{c_1}{1} \right\rfloor + \cdots + \left\lfloor \frac{c_{2^{m-1}-1}}{2^{m-1}-1} \right\rfloor} + \left\lfloor \frac{n}{2^{m-1}} \right\rfloor + \left\lfloor \frac{2^{m-1}}{2^{m-1}+1} \right\rfloor + \left\lfloor \frac{2^{m-1}+1}{2^{m-1}+2} \right\rfloor + \cdots + \left\lfloor \frac{n-1}{n} \right\rfloor = \underbrace{m-1} + 1 + 0 = m.$$

He we use the fact that

$$\frac{n}{2^{m-1}} \geq \frac{2^{m-1}}{2^{m-1}} = 1, \text{ and } \frac{n}{2^{m-1}} < \frac{2^m}{2^{m-1}} = 2.$$

□

Example 7.8 (IMC 1995, Problem 2). *Suppose $f : [0,1] \to \mathbb{R}$ is a continuous function for which $\int_x^1 f(t)\, dt \geq \frac{1-x^2}{2}$ for all $x \in [0,1]$. Prove that $\int_0^1 (f(t))^2\, dt \geq \frac{1}{3}$.*

Solution. (Video Solution)

□

7.5 General Strategies

- Start with finding out when the equality holds. See what happens at extreme cases, when variables are equal, close or far away from one another.
- Check and make sure the intermediate inequalities that you are trying to prove are valid.
- Make sure you are only using inequalities that have the same equality condition.

7.6 Exercises

Exercise 7.1. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be positive real numbers such that $\sum_{k=1}^n a_k \geq \sum_{k=1}^n a_k b_k$. Prove that $\sum_{k=1}^n a_k \leq \sum_{k=1}^n \frac{a_k}{b_k}$.

Exercise 7.2 (VTRMC 1981). With k a positive integer, prove that $\left(1-k^{-2}\right)^k \geq 1 - 1/k$.

Exercise 7.3 (VTRMC 1982). Prove that $t^{n-1} + t^{1-n} < t^n + t^{-n}$ when $t \neq 1, t > 0$ and n is a positive integer.

Exercise 7.4 (VTRMC 1986). Given that $a > 0$ and $c > 0$, find a necessary and sufficient condition on b so that $ax^2 + bx + c > 0$ for all $x > 0$.

Exercise 7.5 (VTRMC 1991). Prove that if $x > 0$ and $n > 0$, where x is real and n is an integer, then

$$\frac{x^n}{(x+1)^{n+1}} \leq \frac{n^n}{(n+1)^{n+1}}$$

Exercise 7.6 (Putnam 1991, A5). Find the maximum value of

$$\int_0^y \sqrt{x^4 + (y-y^2)^2}\, dx$$

for $0 \leq y \leq 1$.

Exercise 7.7 (Putnam 1991, B6)**.** Let a and b be positive numbers. Find the largest number c, in terms of a and b, such that

$$a^x b^{1-x} \leq a\frac{\sinh(ux)}{\sinh u} + b\frac{\sinh(u1-ux)}{\sinh u}$$

for all u with $0 < |u| \leq c$ and for all x, $0 < x < 1$. (Note: $\sinh u = (e^u - e^{-u})/2$.)

Exercise 7.8 (VTRMC 1994)**.** Let f be continuous real function, strictly increasing in an interval $[0,a]$ such that $f(0) = 0$. Let g be the inverse of f, i.e., $g(f(x)) = x$ for all x in $[0,a]$. Show that for $0 \leq x \leq a, 0 \leq y \leq f(a)$, we have

$$xy \leq \int_0^x f(t)dt + \int_0^y g(t)dt$$

Exercise 7.9 (Putnam 1996, B2)**.** Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1\cdot 3\cdot 5\cdots(2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

Exercise 7.10 (Putnam 1996, B3)**.** Given that $\{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\}$, find, with proof, the largest possible value, as a function of n (with $n \geq 2$), of

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.$$

Exercise 7.11 (Putnam 1998, B1)**.** Find the minimum value of

$$\frac{(x+1/x)^6 - (x^6+1/x^6) - 2}{(x+1/x)^3 + (x^3+1/x^3)}$$

for $x > 0$.

Exercise 7.12 (Putnam 2000, A1)**.** Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

Exercise 7.13 (Putnam 2000, A3)**.** The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5, and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

Exercise 7.14 (Putnam 2002, B3)**.** Show that, for all integers $n > 1$,

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$

Exercise 7.15 (Putnam 2003, A2)**.** Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_1a_2\cdots a_n)^{1/n} + (b_1b_2\cdots b_n)^{1/n} \leq [(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)]^{1/n}.$$

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_1a_2\cdots a_n)^{1/n} + (b_1b_2\cdots b_n)^{1/n} \leq [(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)]^{1/n}.$$

Exercise 7.16 (Putnam 2003, A3)**.** Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

Exercise 7.17 (Putnam 2004, A6)**.** Suppose that $f(x,y)$ is a continuous real-valued function on the unit square $0 \le x \le 1, 0 \le y \le 1$. Show that

$$\int_0^1 \left(\int_0^1 f(x,y)dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)dy \right)^2 dx$$
$$\le \left(\int_0^1 \int_0^1 f(x,y)dx\,dy \right)^2 + \int_0^1 \int_0^1 [f(x,y)]^2\,dx\,dy.$$

Exercise 7.18 (Putnam 2004, B2)**.** Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

Exercise 7.19. Suppose x,y,z are positive real numbers for which $x+y+z \ge xyz$. Find the minimum value of $\dfrac{x^2+y^2+z^2}{xyz}$.

Exercise 7.20. Let $n \ge 3$ be an integer, and let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers for which $a_1+a_2+\cdots+a_n = 1$. Find the maximum of $a_1^2a_2 + a_2^2a_3 + \cdots + a_{n-1}^2a_n + a_n^2a_1$.

Exercise 7.21 (Putnam 2006, B5)**.** For each continuous function $f : [0,1] \to \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x)\,dx$ and $J(x) = \int_0^1 x\,(f(x))^2\,dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

Exercise 7.22 (VTRMC 2008)**.** Find the maximum value of $xy^3+yz^3+zx^3-x^3y-y^3z-z^3x$ where $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.

Exercise 7.23. Suppose that a,b,u,v are real numbers for which $av - bu = 1$. Prove that

$$a^2+b^2+u^2+v^2+au+bv \ge \sqrt{3}.$$

Exercise 7.24. Suppose $p(x) = ax^2+bx+c$ is a quadratic polynomial for which $|p(x)| \le 1$ for all $x \in [0,1]$. What is the maximum value of $|a|+|b|+|c|$?

Exercise 7.25. Let x be an irrational number. Prove that there are infinitely many rational numbers m/n where m,n are integers for which $|x - m/n| < /n^2$.

Exercise 7.26. Suppose $f(x) = \sum\limits_{k=1}^{n} a_k \sin(kx)$, where $a_1, \ldots, a_n$ are constants. Given that $|f(x)| \le |\sin x|$ for all $x \in \mathbb{R}$, prove

$$|\sum_{k=1}^{n} ka_k| \le 1.$$

Exercise 7.27 (VTRMC 2013)**.** Prove that

$$\frac{x}{\sqrt{1+x^2}} + \frac{y}{\sqrt{1+y^2}} + \frac{z}{\sqrt{1+z^2}} \le \frac{3\sqrt{3}}{2}$$

for any positive real numbers x,y,z such that $x+y+z = xyz$.

Exercise 7.28 (Putnam 2013, B2)**.** Let $C = \bigcup_{N=1}^{\infty} C_N$, where C_N denotes the set of those 'cosine polynomials' of the form

$$f(x) = 1 + \sum_{n=1}^{N} a_n \cos(2\pi nx)$$

for which:

(i) $f(x) \geq 0$ for all real x, and

(ii) $a_n = 0$ whenever n is a multiple of 3.

Determine the maximum value of $f(0)$ as f ranges through C, and prove that this maximum is attained.

Exercise 7.29 (Putnam 2013, B4)**.** For any continuous real-valued function f defined on the interval $[0,1]$, let

$$\mu(f) = \int_0^1 f(x)\,dx,\ \mathrm{Var}(f) = \int_0^1 (f(x) - \mu(f))^2\,dx,$$
$$M(f) = \max_{0 \leq x \leq 1} |f(x)|.$$

Show that if f and g are continuous real-valued functions defined on the interval $[0,1]$, then

$$\mathrm{Var}(fg) \leq 2\mathrm{Var}(f)M(g)^2 + 2\mathrm{Var}(g)M(f)^2.$$

Exercise 7.30 (Putnam 2014, B2)**.** Suppose that f is a function on the interval $[1,3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\,dx = 0$. How large can $\int_1^3 \frac{f(x)}{x}\,dx$ be?

Exercise 7.31 (Putnam 2015, A4)**.** For each real number x, let

$$f(x) = \sum_{n \in S_x} \frac{1}{2^n},$$

where S_x is the set of positive integers n for which $\lfloor nx \rfloor$ is even. What is the largest real number L such that $f(x) \geq L$ for all $x \in [0,1)$? (As usual, $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

Exercise 7.32 (Putnam 2016, A2)**.** Given a positive integer n, let $M(n)$ be the largest integer m such that

$$\binom{m}{n-1} > \binom{m-1}{n}.$$

Evaluate

$$\lim_{n\to\infty} \frac{M(n)}{n}.$$

Exercise 7.33 (VTRMC 2018)**.** For $n \in \mathbb{N}$, define $a_n = \frac{1+1/3+1/5+\cdots+1/(2n-1)}{n+1}$ and $b_n = \frac{1/2+1/4+1/6+\cdots+1/(2n)}{n}$. Find the maximum and minimum of $a_n - b_n$ for $1 \leq n \leq 999$

Exercise 7.34 (Putnam 2018, A3)**.** Determine the greatest possible value of $\sum_{i=1}^{10} \cos(3x_i)$ for real numbers $x_1, x_2, \ldots, x_{10}$ satisfying $\sum_{i=1}^{10} \cos(x_i) = 0$.

Exercise 7.35 (Putnam 2020, A6)**.** Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} (-1)^{\lfloor k(\sqrt{2}-1) \rfloor} \geq 0.$$

(As usual, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

Exercise 7.36 (Putnam 2021, B2)**.** Determine the maximum value of the sum

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} (a_1 a_2 \cdots a_n)^{1/n}$$

over all sequences $a_1, a_2, a_3, \cdots$ of nonnegative real numbers satisfying

$$\sum_{k=1}^{\infty} a_k = 1.$$

Chapter 8

Sequences

8.1 Basics

Definition 8.1. A **sequence** is a list of numbers, the n-th one of which is denoted by a_n. A **series** $\sum_{n=0}^{\infty} a_n$ is a sum of terms of a sequence.

Definition 8.2. A sequence a_n **converges** to a real number L if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ for which } |a_n - L| < \varepsilon \text{ for all } n \geq N,$$

in which case we write $\lim_{n\to\infty} a_n = L$. If no such real number L exists we say the sequence diverges.
A sequence a_n diverges to ∞ if

$$\forall M > 0\ \exists N \in \mathbb{N} \text{ for which } a_n > M \text{ for all } n \geq N,$$

in which case we write $\lim_{n\to\infty} a_n = \infty$. Similarly we may define $\lim_{n\to\infty} a_n = -\infty$.

8.1.1 Homogeneous Linear Recurrences

YouTube Video: https://youtu.be/YtZhJYcww1o

Definition 8.3. A **linear recurrence** (or **linear recursion**) is a recurrence given by

$$a_0, a_1, \ldots, a_{k-1} \text{ are given constants , and } a_{n+k} = \sum_{i=0}^{k-1} c_i(n) a_{n+i} + c_k(n) \text{ for all } n \geq 0 \qquad (*)$$

where $c_i(n)$'s are given but may depend on n. This recursion is called **homogeneous** if $c_k(n) = 0$.

Example 8.1. *The following are well-known examples of recurrence relations.*

(a) The Fibonacci sequence F_n is given by $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$.

(b) Every geometric sequence is given by $a_n = ra_{n-1}$ for all $n \geq 1$, where r and a_0 are given constants. Similarly every arithmetic sequence is also given by a recursion $a_n = a_{n-1} + d$.

(c) $a_0 = 1$, $a_n = n \cdot a_{n-1}$ *gives the sequence of factorials.*

It is worth noting that given initial values $a_0, \ldots, a_{k-1}$, and c_i's there is a unique sequence a_n satisfying $(*)$. Therefore if we can find *one* such a_n, that is *the* only solution of $(*)$.

Here we will explore some methods for solving some linear recurrences.

8.1.2 Homogeneous Case with Constant Coefficients

Definition 8.4. A linear recursion of the form $(*)$ is said to have **constant coefficients** if all c_i's are constants.

In this section we only consider homogeneous recursions with constant coefficients. That is those recurrences of form:

$$a_0 = d_0, a_1 = d_1, \ldots, a_{k-1} = d_{k-1} \text{ and } a_{n+k} = \sum_{i=0}^{k-1} c_i a_{n+i} \text{ for all } n \geq 0 \qquad (**),$$

where c_i's, d_i's and k are all constants.

Let's start with an example.

Example 8.2. *Find an explicit formula for the Fibonacci sequence F_n given by $F_0 = 0, F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$.*

Solution. (Video Solution) Since each term is the sum of the previous two terms, it is almost like each term is double the previous term. This motivates us to think F_n is may be nearly exponential. Let us try $F_n = r^n$. This yields

$$r^n = r^{n-1} + r^{n-2} \Rightarrow r^2 - r - 1 = 0 \Rightarrow r = \frac{1 \pm \sqrt{5}}{2}.$$

This means $\left(\frac{1+\sqrt{5}}{2}\right)^n$ and $\left(\frac{1-\sqrt{5}}{2}\right)^n$ both satisfy the recursion $F_n = F_{n-1} + F_{n-2}$, however, neither satisfies the initial conditions $F_0 = 0$ and $F_1 = 1$.

Note that if F_n and G_n satisfy the recursion, then any linear combination $H_n = aF_n + bG_n$ also satisfies the recursion $H_n = H_{n-1} + H_{n-2}$. Therefore, $F_n = a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n$ satisfies the recursion. If we select a and b such that $F_0 = 0$ and $F_1 = 1$, then we are done. For that we need to solve the system

$$\begin{cases} a + b = 0 \\ a\left(\frac{1+\sqrt{5}}{2}\right) + b\left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{cases}$$

This yields $a = -b = \frac{1}{\sqrt{5}}$. Therefore,

$$\boxed{F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]}.$$

This is an explicit formula for the n-th term of the Fibonacci sequence. $\square$

In general to solve the recurrence relation $(**)$ we follow the steps below:

- Solve the **characteristic equation** $r^k = c_0 + c_1 r + \cdots + c_{k-1} r^{k-1}$, which is obtained by substituting $a_n = r^n$. Let the roots of the characteristic equation be $r_1, r_2, \ldots, r_k$.
- If the roots are distinct then let $a_n = \alpha_1 r_1^n + \cdots + \alpha_k r_k^n$. If a root r is repeated instead of using r^n multiple times use $r^n, nr^n, n^2 r^n, \ldots$ as many times as the multiplicity of r. We call this a_n a **general solution** of this homogeneous recursion.
- Use the initial conditions to find α_i's.

Example 8.3. *Find an explicit formula for the sequence a_n given by $a_0 = 1$, $a_1 = 3$, $a_n = 4a_{n-1} - 4a_{n-2}$ for all $n \geq 2$.*

Solution. The characteristic equation is $r^2 = 4r - 4$. The roots are $r = 2, 2$. Thus $a_n = \alpha 2^n + \beta n 2^n$. The initial conditions give us

$$\begin{cases} \alpha = 1 \\ 2\alpha + 2\beta = 3 \end{cases}$$

Thus, $\beta = 0.5$, which gives $a_n = 2^n + n2^{n-1}$. $\square$

8.1.3 Nonhomogeneous Case

A recursion of form $(*)$ is called **nonhomogeneous** if $c_k \neq 0$. Assume h_n is a general solution to the homogeneous recursion $h_{n+k} = \sum_{i=0}^{k-1} c_i h_{n+i}$ and p_n is any solution (called a **particular solution**) to the nonhomogeneous recursion $p_{n+k} = \sum_{i=0}^{k-1} c_i p_{n+i} + c_k$, then $a_n = h_n + p_n$ is a general solution to the nonhomogeneous recursion $a_{n+k} = \sum_{i=0}^{k-1} c_i a_{n+i} + c_k$.

Example 8.4. *Find an explicit formula for the sequence a_n given by $a_0 = 2$, $a_1 = 1$, $a_{n+1} = 5a_n - 6a_{n-1} + 2$, for all $n \geq 1$.*

Solution. First, we will find a general solution to the homogeneous recursion $h_{n+1} = 5h_n - 6h_{n-1}$. The characteristic equation is $r^2 = 5r - 6$, which has roots $r = 2, 3$. Thus, a general solution to the homogeneous recursion is $h_n = \alpha 2^n + \beta 3^n$. Now, we need a particular solution to the nonhomogeneous equation. We guess a constant solution might work. Setting $a_n = c$ gives us $c = 5c - 6c + 2$, and hence $c = 1$. Therefore, $a_n = \alpha 2^n + \beta 3^n + 2$. Now, we use the initial conditions $a_0 = 2$ and $a_1 = 1$ to find $\alpha = 1$, $\beta = -1$. Therefore, $a_n = 2^n - 3^n + 2$. $\square$

Remark. Guessing a particular solution is not always easy. Start with constant, linear and quadratic polynomials.

8.2 Important Theorems

Theorem 8.1 (Monotone Convergence Theorem)**.** *Any bounded monotone sequence of real numbers converges.*

Theorem 8.2 (Stirling's Approximation of $n!$)**.** *For large integers n, $n!$ and $\left(\frac{n}{e}\right)^n\sqrt{2\pi n}$ are asymptotically the same. In other words*

$$\lim_{n\to\infty}\frac{n!}{\left(\frac{n}{e}\right)^n\sqrt{2\pi n}}=1.$$

8.3 Classical Examples

Example 8.5. *Let F_n be the Fibonacci sequence satisfying $F_1=F_2=1$, $F_{n+2}=F_{n+1}+F_n$ for all $n\geq 2$. Prove the following:*

(a) $F_n \mid F_m$ if and only if $n \mid m$.

(b) $F_{n+1}F_{n-1}=F_n^2+(-1)^n$.

Example 8.6. *Let α be irrational. Prove that the sequence of fractional parts $\{n\alpha\}$ with $n\in\mathbb{Z}^+$ is dense in $[0,1]$.*

Solution.(Video Solution)

□

8.4 Further Examples

Example 8.7 (IMC 2019, Problem 4)**.** *Define the sequence $a_0, a_1, \dots$ of numbers by the following recurrence:*

$$a_0=1,\quad a_1=2,\quad (n+3)a_{n+2}=(6n+9)a_{n+1}-na_n \quad \text{for } n\geq 0.$$

Prove that all terms of this sequence are integers.

Scratch: Here are a few thoughts:

- The first few terms are $1,2,6,22,90,394$. It is difficult to see a pattern other than the fact that this seems like having exponential growth.
- We need to show the right hand side is divisible by $n+3$.
- If we can come up with a different recurrence relation that writes a_{n+2} in terms of the previous terms without a denominator then we can solve the problem.
- We may also be able to find a_n using either ordinary generating function $\sum a_n x^n$ or exponential generating function $\sum a_n \frac{x^n}{n!}$.

Solution. Watch the video for a solution! □

Example 8.8. *Let p be a prime and let F_n be the Fibonacci sequence given by $F_1=F_2=1$, $F_{n+2}=F_n+F_{n+1}$ for all $n\geq 1$. Prove F_{p^2-1} is divisible by p.*

Exercise 8.1 (Putnam 2022, A3)**.** Let p be a prime number greater than 5. Let $f(p)$ denote the number of infinite sequences $a_1, a_2, a_3, \ldots$ such that $a_n \in \{1, 2, \ldots, p-1\}$ and $a_n a_{n+2} \equiv 1 + a_{n+1} \pmod{p}$ for all $n \geq 1$. Prove that $f(p)$ is congruent to 0 or 2 $\pmod{5}$.

Scratch: Here are a few ideas that come to mind:

- If we work in $\mathbb{F}_p$, the field of integers mod p, we notice that $a_{n+2} = (1 + a_{n+1})/a_n$. Thus, each sequence can be determined after we know a_1 and a_2.
- We notice that $1 + a_{n+1} \neq 0$ and thus, no term after the first can be -1.
- Perhaps we can start with small values of p and list all possible values of a_1, a_2 and see which ones yield a valild sequence. This may not be a great idea since the smallest value of p is 7 and that yields 6 possibilities for a_1 (since $a_1 \neq 0$) and 5 possibilities for a_2 (since $a_2 \neq 0, -1$). Which means we have to check 30 different sequences. So, while this is not a bad idea, it isn't something I will start with.
- Perhaps we can list the first few terms of the sequence and see what we get.

After listing the first few terms we see a neat pattern, so a solution is within our reach!

Solution.(Video Solution) We claim $f(p) = (p-2)(p-3)$.

First, note that working in $\mathbb{F}_p$, the field of integers modulo p, we obtain $a_{n+2} = \dfrac{1 + a_{n+1}}{a_n}$. Applying this repeatedly we obtain the following:

$$a_3 = \frac{1 + a_2}{a_1},$$

$$a_4 = \frac{1 + a_3}{a_2} = \frac{1 + \dfrac{1 + a_2}{a_1}}{a_2} = \frac{1 + a_1 + a_2}{a_1 a_2}$$

$$a_5 = \frac{1 + a_4}{a_3} = \frac{\dfrac{1 + a_1 + a_2 + a_1 a_2}{a_1 a_2}}{\dfrac{1 + a_2}{a_1}} = \frac{(1 + a_1)(1 + a_2) a_1}{a_1 a_2 (1 + a_2)} = \frac{1 + a_1}{a_2}$$

$$a_6 = \frac{1 + a_5}{a_4} = \frac{\dfrac{1 + a_1 + a_2}{a_2}}{\dfrac{1 + a_1 + a_2}{a_1 a_2}} = a_1$$

$$a_7 = \frac{1 + a_6}{a_5} = \frac{1 + a_1}{\dfrac{1 + a_1}{a_2}} = a_2$$

Since $a_6 = a_1$ and $a_7 = a_2$ and each term is given in terms of the previous two terms, the sequence is periodic. Thus, in order to count the number of such sequences we need to make sure the first five terms are all nonzero. This yields the following:

$$\begin{aligned}
&a_1 \neq 0 \\
&a_2 \neq 0 \\
&a_3 \neq 0 \Rightarrow 1 + a_2 \neq 0 \Rightarrow a_2 \neq -1 \\
&a_4 \neq 0 \Rightarrow 1 + a_1 + a_2 \neq 0 \Rightarrow a_2 \neq -a_1 - 1 \\
&a_5 \neq 0 \Rightarrow 1 + a_1 \neq 0 \Rightarrow a_1 \neq -1
\end{aligned}$$

To summarize, we have $a_1, a_2 \neq 0, -1$ and $a_2 \neq -a_1 - 1$. This means there are $p-2$ possible choices for a_1. Also note that $-a_1 - 1 = 0$ implies $a_1 = -1$ and $-a_1 - 1 = -1$ implies $a_1 = 0$. So, when $a_1 \neq 0, -1$, the three values $0, -1, -a_1 - 1$ are distinct. Thus, there are $p-3$ possible value for a_2. This means $f(p) = (p-2)(p-3)$.

Since $p > 5$ is prime, $p \equiv \pm 1, \pm 2 \mod 5$. When $p \equiv \pm 1 \mod 5$ we see that $f(p) \equiv 2 \mod 5$ and when $p \equiv \pm 2 \mod 5$, $f(p)$ is divisible by 5. This completes the proof! $\square$

Example 8.9. *Determine if there is a sequence of positive integers a_n, with $n \geq 1$, that satisfy both of the following conditions:*

(a) For every two positive integers m and n we have $a_{m+n} \leq a_m + a_n$ and $a_{mn} \leq a_m a_n$.

(b) For every positive integer k, there are infinitely many indices m with $a_m = k$.

Solution.(Video Solution)

$\square$

Example 8.10. *Prove that the following set is dense in $[0, \infty)$:*

$$\{2^m 3^n \mid m, n \in \mathbb{Z}\}.$$

Solution.(Video Solution) $\square$

Example 8.11 (Putnam 1985, A3)**.** *Let x be a real number. Define a double sequence a_{ij} as follows:*

$$a_{i0} = \frac{x}{2^i}, \textit{ for all integers } i \geq 0$$

and

$$a_{i(j+1)} = a_{ij}^2 + 2a_{ij}, \textit{ for all integers } i, j \geq 0$$

Find $\lim\limits_{n \to \infty} a_{nn}$.

Solution.(Video Solution)

$\square$

8.5 General Strategies

- Find the first few terms and see if there is a pattern.
- Can you prove the pattern?
- To study a sequence a_n it is enough to study the power series $\sum a_n x^n$ or $\sum \frac{a_n}{n!} x^n$. The former is called the ordinary generating function associated to a_n, while the latter is called the exponential generating function associated with a_n.

8.6 Exercises

Exercise 8.2 (VTRMC 1980)**.** Let $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$.

(a) Prove that $\lim_{n\to\infty} a_n$ exists.

(b) Show that $a_n = \dfrac{(1-(\frac{1}{2})^2)(1-(\frac{1}{4})^2)\cdots(1-(\frac{1}{2n})^2)}{(2n+1)a_n}$.

(c) Find $\lim_{n\to\infty} a_n$ and justify your answer.

Exercise 8.3 (VTRMC 1981)**.** Let $A = \{a_0, a_1, \ldots\}$ be a sequence of real numbers and define the sequence $A' = \{a'_0, a'_1, \ldots\}$ as follows for $n = 0, 1, \ldots : a'_{2n} = a_n, a'_{2n+1} = a_n + 1$. If $a_0 = 1$ and $A' = A$, find

(a) a_1, a_2, a_3 and a_4

(b) a_{1981}

(c) A simple general algorithm for evaluating a_n, for $n = 0, 1, \ldots$.

Exercise 8.4 (VTRMC 1981)**.** Let

(i) $0 < a < 1$,

(ii) $0 < M_{k+1} < M_k$, for $k = 0, 1, \ldots$,

(iii) $\lim_{k\to\infty} M_k = 0$.

If $b_n = \sum_{k=0}^{\infty} a^{n-k} M_k$, prove that $\lim_{n\to\infty} b_n = 0$.

Exercise 8.5 (VTRMC 1983)**.** A sequence f_n is generated by the recurrence formula

$$f_{n+1} = \frac{f_n f_{n-1} + 1}{f_{n-2}}$$

for $n = 2, 3, 4, \ldots$, with $f_0 = f_1 = f_2 = 1$. Prove that f_n is integer-valued for all integers $n \geq 0$.

Exercise 8.6 (VTRMC 1984)**.** A sequence $\{u_n\}$, $n = 0, 1, 2, \ldots$, is defined by $u_0 = 5$, $u_{n+1} = u_n + n^2 + 3n + 3$, for $n = 0, 1, 2, \ldots$. If u_n is expressed as a polynomial $u_n = \sum\limits_{k=0}^{d} c_k n^k$, where d is the degree of the polynomial, find the sum $\sum\limits_{k=0}^{d} c_k$.

Exercise 8.7 (VTRMC 1986)**.** Let $x_1 = 1, x_2 = 3$, and

$$x_{n+1} = \frac{1}{n+1}\sum_{i=1}^{n} x_i \quad \text{for } n = 2, 3, \ldots$$

Find $\lim_{n\to\infty} x_n$ and give a proof of your answer.

Exercise 8.8 (VTRMC 1987)**.** A sequence of integers $\{n_1, n_2, \ldots\}$ is defined as follows: n_1 is assigned arbitrarily and, for $k > 1$,

$$n_k = \sum_{j=1}^{j=k-1} z(n_j)$$

where $z(n)$ is the number of 0's in the binary representation of n (each representation should have a leading digit of 1 except for zero which has the representation 0). An example, with $n_1 = 9$, is $\{9, 2, 3, 3, 3, \ldots\}$, or in binary, $\{1001, 10, 11, 11, 11, \ldots\}$.

(a) Find n_1 so that $\lim_{k\to\infty} n_k = 31$, and calculate $n_2, n_3, \ldots, n_{10}$.

(b) Prove that, for every choice of n_1, the sequence $\{n_k\}$ converges.

Exercise 8.9 (Putnam 1990, A1)**.** Let

$$T_0 = 2, T_1 = 3, T_2 = 6,$$

and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

Exercise 8.10 (Putnam 1990, A2)**.** Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$ $(n, m = 0, 1, 2, \ldots)$?

Exercise 8.11 (VTRMC 1990)**.** The number of individuals in a certain population (in arbitrary real units) obeys, at discrete time intervals, the equation

$$y_{n+1} = y_n(2 - y_n) \text{ for } n = 0, 1, 2, \ldots,$$

where y_0 is the initial population.

(a) Find all "steady-state" solutions y^* such that, if $y_0 = y^*$, then $y_n = y^*$ for $n = 1, 2, \ldots$

(b) Prove that if y_0 is any number in $(0, 1)$, then the sequence $\{y_n\}$ converges monotonically to one of the steady-state solutions found in (a).

Exercise 8.12 (VTRMC 1991)**.** Let $a_0 = 1$ and for $n > 0$, let a_n be defined by

$$a_n = -\sum_{k=1}^{n} \frac{a_{n-k}}{k!}$$

Prove that $a_n = (-1)^n/n!$, for $n = 0, 1, 2, \ldots$

Exercise 8.13 (Putnam 1991, B1)**.** For each integer $n \geq 0$, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \geq 0$. For what positive integers A is this sequence eventually constant?

Exercise 8.14 (VTRMC 1992)**.** Let $\{t_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $t_1 = 1$ and $t_{n+1}^2 = 1 + t_n$, for $n \geq 1$. Show that t_n is increasing in n and find $\lim\limits_{n\to\infty} t_n$.

Exercise 8.15 (VTRMC 1992)**.** Find $\lim\limits_{n\to\infty} \dfrac{2\log 2 + 3\log 3 + \ldots + n\log n}{n^2 \log n}$.

Exercise 8.16 (Putnam 1993, A2)**.** Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

Exercise 8.17 (Putnam 1993, A6)**.** The infinite sequence of 2's and 3's

$$2,3,3,2,3,3,3,2,3,3,3,2,3,3,2,3,3,$$
$$3,2,3,3,3,2,3,3,3,2,3,3,2,3,3,3,2,\ldots$$

has the property that, if one forms a second sequence that records the number of 3's between successive 2's, the result is identical to the given sequence. Show that there exists a real number r such that, for any n, the nth term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m. (Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.)

Exercise 8.18 (Putnam 1995, B4)**.** Evaluate

$$\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2207 - \ldots}}}.$$

Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers.

Exercise 8.19 (VTRMC 1996)**.** Let us define

$$\begin{aligned} f_{n,0}(x) &= x + \frac{\sqrt{x}}{n} && \text{for } x > 0, n \geq 1 \\ f_{n,j+1}(x) &= f_{n,0}(f_{n,j}(x)) && j = 0, 1, \ldots, n-1 \end{aligned}$$

Find $\lim\limits_{n\to\infty} f_{n,n}(x)$ for $x > 0$.

Exercise 8.20 (VTRMC 1997)**.** Let J be the set of all sequences of real numbers, and let A, L and P be three mappings from J to J defined as follows. If $x = \{x_n\} = \{x_0, x_1, x_2, \ldots\} \in J$, then

$$Ax = \{x_n + 1\} = \{x_0 + 1, x_1 + 1, x_2 + 1, \ldots\}, Lx = \{1, x_0, x_1, x_2, \ldots\}, Px = \{\sum_{k=0}^{n} x_k\}.$$

Finally, define the composite mapping T on J by $Tx = L \circ A \circ Px$. In the following, let $y = \{1, 1, 1, \ldots\}$.

(a) Write down T^2y, giving the first eight terms of the sequence and a closed formula for the n-th term.

(b) Assuming that $z = \{z_n\} = \lim_{i\to\infty} T_i y$ exists, conjecture the general form for z_n, and prove your conjecture.

Exercise 8.21 (Putnam 1997, A6)**.** For a positive integer n and any real number c, define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k, $1 \leq k \leq n$.

Exercise 8.22 (Putnam 1999, A6)**.** The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 1, a_2 = 2, a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$

Show that, for all n, a_n is an integer multiple of n.

Exercise 8.23 (VTRMC 2000)**.** Let a_n ($n \geq 1$) be the sequence of numbers defined by the recurrence relation

$$a_1 = 1, \qquad a_n = a_{n-1}a_1 + a_{n-2}a_2 + \cdots + a_2 a_{n-2} + a_1 a_{n-1},$$

(so $a_2 = a_1^2 = 1, a_3 = 2a_1a_2 = 2$ etc.). Prove that $\sum_{n=1}^{\infty} \left(\frac{2}{9}\right)^n a_n = \frac{1}{3}$.

Exercise 8.24 (Putnam 2001, B6)**.** Assume that $(a_n)_{n\geq 1}$ is an increasing sequence of positive real numbers such that $\lim a_n/n = 0$. Must there exist infinitely many positive integers n such that $a_{n-i} + a_{n+i} < 2a_n$ for $i = 1, 2, \ldots, n-1$?

Exercise 8.25 (Putnam 2003, B2)**.** Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$, form a new sequence of $n-1$ entries $\frac{3}{4}, \frac{5}{12}, \ldots, \frac{2n-1}{2n(n-1)}$ by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n-2$ entries, and continue until the final sequence produced consists of a single number x_n. Show that $x_n < 2/n$.

Exercise 8.26 (VTRMC 2004)**.** A sequence of integers $\{f(n)\}$ for $n = 0, 1, 2, \ldots$ is defined as follows: $f(0) = 0$ and for $n > 0$,

$$f(n) = \begin{cases} f(n-1)+3, & \text{if } n \equiv 0 \text{ or } 1 \mod 6, \\ f(n-1)+1, & \text{if } n \equiv 2 \text{ or } 5 \mod 6, \\ f(n-1)+2, & \text{if } n \equiv 3 \text{ or } 4 \mod 6 \end{cases}$$

Derive an explicit formula for $f(n)$ when $n \equiv 0 \mod 6$, showing all necessary details in your derivation.

Exercise 8.27 (Putnam 2004, A3)**.** Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

Exercise 8.28 (Putnam 2006, A3)**.** Let $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

Exercise 8.29 (Putnam 2006, B6)**.** Let k be an integer greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$

Exercise 8.30 (Putnam 2007, B3)**.** Let $x_0 = 1$ and for $n \geq 0$, let $x_{n+1} = 3x_n + \lfloor x_n\sqrt{5}\rfloor$. In particular, $x_1 = 5$, $x_2 = 26$, $x_3 = 136$, $x_4 = 712$. Find a closed-form expression for x_{2007}. ($\lfloor a\rfloor$ means the largest integer $\leq a$.)

Exercise 8.31 (VTRMC 2008)**.** Let $f_1(x) = x$ and $f_{n+1}(x) = x^{f_n(x)}$ for n a positive integer. Thus $f_2(x) = x^x$ and $f_3(x) = x^{(x^x)}$. Now define $g(x) = \lim\limits_{n\to\infty} 1/f_n(x)$ for $x > 1$. Is g continuous on the open interval $(1,\infty)$? Justify your answer.

Exercise 8.32 (Putnam 2008, A4)**.** Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \leq e \\ xf(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?

Exercise 8.33 (Putnam 2008, B2)**.** Let $F_0(x) = \ln x$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)\,dt$. Evaluate

$$\lim_{n\to\infty} \frac{n!F_n(1)}{\ln n}.$$

Exercise 8.34 (Putnam 2009, B6)**.** Prove that for every positive integer n, there is a sequence of integers $a_0, a_1, \ldots, a_{2009}$ with $a_0 = 0$ and $a_{2009} = n$ such that each term after a_0 is either an earlier term plus 2^k for some nonnegative integer k, or of the form $b \bmod c$ for some earlier positive terms b and c. [Here $b \bmod c$ denotes the remainder when b is divided by c, so $0 \leq (b \bmod c) < c$.]

Exercise 8.35 (VTRMC 2010)**.** Define a sequence by $a_1 = 1, a_2 = 1/2$, and $a_{n+2} = a_{n+1} - \frac{a_n a_{n+1}}{2}$ for n a positive integer. Find $\lim_{n\to\infty} na_n$.

Exercise 8.36 (VTRMC 2011)**.** A sequence (a_n) is defined by $a_0 = -1, a_1 = 0$, and

$$a_{n+1} = a_n^2 - (n+1)^2 a_{n-1} - 1$$

for all positive integers n. Find a_{100}.

Exercise 8.37 (VTRMC 2012)**.** Determine whether the series $\sum\limits_{n=2}^{\infty} \dfrac{1}{\ln n} - \left(\dfrac{1}{\ln n}\right)^{(n+1)/n}$ is convergent.

Exercise 8.38 (Putnam 2012, B4)**.** Suppose that $a_0 = 1$ and that $a_{n+1} = a_n + e^{-a_n}$ for $n = 0, 1, 2, \ldots$. Does $a_n - \log n$ have a finite limit as $n \to \infty$? (Here $\log n = \log_e n = \ln n$.)

Exercise 8.39 (VTRMC 2013)**.** Define a sequence (a_n) for $n \geq 1$ by $a_1 = 2$ and $a_{n+1} = a_n^{1+n^{3/2}}$. Is (a_n) convergent (i.e. $\lim\limits_{n\to\infty} a_n < \infty$.)?

Exercise 8.40 (Putnam 2013, B1)**.** For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

Exercise 8.41 (Putnam 2015, A2)**.** Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015}.

Exercise 8.42 (VTRMC 2018)**.** For $n \in \mathbb{N}$, let $a_n = \int_0^{1/\sqrt{n}} \left|1 + e^{it} + e^{2it} + \cdots + e^{nit}\right| dt$. Determine whether the sequence $(a_n) = a_1, a_2, \ldots$ is bounded.

Exercise 8.43 (Putnam 2018, B4)**.** Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that if $x_n = 0$ for some n, then the sequence is periodic.

Exercise 8.44 (Putnam 2020, B4)**.** Let n be a positive integer, and let V_n be the set of integer $(2n+1)$-tuples $\mathbf{v} = (s_0, s_1, \cdots, s_{2n-1}, s_{2n})$ for which $s_0 = s_{2n} = 0$ and $|s_j - s_{j-1}| = 1$ for $j = 1, 2, \cdots, 2n$. Define

$$q(\mathbf{v}) = 1 + \sum_{j=1}^{2n-1} 3^{s_j},$$

and let $M(n)$ be the average of $\frac{1}{q(\mathbf{v})}$ over all $\mathbf{v} \in V_n$. Evaluate $M(2020)$.

Chapter 9

Linear Algebra

9.1 Basics

Definition 9.1. Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in a vector space V are said to be **linearly dependent** if there are scalars $c_1, \ldots, c_n$ not all zero for which $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$. If these vectors are not linearly dependent we say they are **linearly independent.** We say these vectors are **generating** or **spanning**, if every vector in V is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. They form a **basis** for V if they are linearly independent and generating. The size of a basis for V is called the **dimension** of V and is denoted by $\dim V$.

Definition 9.2. The **rank** of an $m \times n$ matrix A is the dimension of the subspace of $\mathbb{R}^m$ spanned by the columns of A. This number is denoted by rank A. The **null space** of A is the subspace of $\mathbb{R}^n$ given by $\text{Nul}\, A = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = \mathbf{0}\}$.

Definition 9.3. The **transpose** of a matrix A, denoted by A^T, is the matrix whose (i, j) entry is the (j, i) entry of A. The **conjugate** of A, denoted by $\overline{A}$, is the matrix whose (i, j) entry is the conjugate of the (i, j) entry of A. The **adjoint**, (**conjugate transpose** or **Hermitian**) of A is defined to be $\overline{A^T}$. The adjoint of A is denotes by $A^\star$.

Definition 9.4. A matrix is called **Hermitian** or **self-adjoint** if it is equal to its conjugate transpose.

Definition 9.5. A square matrix A is said to be **unitary** if $AA^\star = I$, i.e. $A^\star = A^{-1}$. A real unitary matrix is called **orthogonal**.

Definition 9.6. We say a nonzero vector $\mathbf{v}$ is an **eigenvector** for a square matrix A if $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ. The scalar λ is called an **eigenvalue**.

Definition 9.7. A square matrix A is said to be **diagonalizable** if it can be written as $A = PDP^{-1}$, where D is a diagonal matrix and P is an invertible matrix.

Definition 9.8. Let $A = (a_{ij})$ be an $n \times n$ matrix. The **determinant** of A is evaluated recursively by expanding along the i-th row as follows:

$$\det A = (-1)^{i+1} a_{i1} \det(A_{i1}) + (-1)^{i+2} a_{i2} \det(A_{i2}) + \cdots + (-1)^{i+n} a_{in} \det(A_{in}),$$

where A_{ij} is the matrix obtained by eliminating the i-th row and j-th column of A. This determinant can also be evaluated by expanding along a column.

9.2 Important Theorems

Theorem 9.1. *An $n \times n$ matrix A is diagonalizable if it has n linearly independent eigenvectors. In which case $A = PDP^{-1}$, where D is a diagonal matrix whose diagonal entries are eigenvalues of A and the columns of P are n linearly independent eigenvectors of A.*

Theorem 9.2. *Here are some properties of determinants:*

- $\det(AB) = (\det A)(\det B)$.
- *Swapping two rows negates the determinant.*
- *Adding a multiple of a row to another row does not change the determinant.*
- *Re-scalaing a row by c multiplies the determinant by c.*
- $\det A^T = \det A$.
- $\det A^* = \overline{\det A}$.

Theorem 9.3 (Rank-Nullity Theorem)**.** *For every $m \times n$ matrix A, we have*

$$\text{rank}\, A + \dim \text{Nul}\, A = n, \text{ and } \text{rank}\, A = \text{rank}\, A^T.$$

Theorem 9.4 (Cayley-Hamilton)**.** *Let A be a square matrix and let $p(x) = \det(xI - A)$ be a polynomial. Then $p(A) = 0$, the zero matrix.*

Theorem 9.5. *If A is a unitary matrix, then each eigenvalue λ of A satisfies $|\lambda| = 1$.*

Theorem 9.6. *Every Hermitian matrix is unitarily diagonalizable. In other words, if a matrix $A \in M_n(\mathbb{C})$ satisfies $A^* = A$, then, there is a unitary matrix U and a diagonal matrix D for which $A = UDU^{-1}$. In particular if $A \in M_n(\mathbb{R})$ is symmetric, then it is diagonalizable over $\mathbb{R}$.*

9.3 Classical Examples

Example 9.1. *If A is a real matrix and $\mathbf{v}$ a vector in the null space of A which is also in the image of $A^\star$, then $\mathbf{v} = \mathbf{0}$.*

Solution. We know $\mathbf{v} = A^\star\mathbf{w}$ and $A\mathbf{v} = \mathbf{0}$. This implies $AA^\star\mathbf{w} = \mathbf{0}$. Multiplying by $\mathbf{w}^\star$ we get $\mathbf{w}^\star AA^\star\mathbf{w} = 0$. Therefore, $||A^\star\mathbf{w}|| = 0$, thus $\mathbf{v} = \mathbf{0}$. □

9.4 Further Examples

Example 9.2 (Putnam 2019, B3)**.** *Let Q be an n-by-n real orthogonal matrix, and let $u \in \mathbb{R}^n$ be a unit column vector (that is, $u^T u = 1$). Let $P = I - 2uu^T$, where I is the n-by-n identity matrix. Show that if 1 is not an eigenvalue of Q, then 1 is an eigenvalue of PQ.*

Scratch: Here are my first thoughts:

- The given are $QQ^T = I$, $P = I - 2uu^T, u^T u = 1, \det(Q - I) \neq 0$, and we are trying to prove $\det(PQ - I) = 0$. Perhaps we could somehow show $(Q-I)(PQ-I)$ is not invertible.

- To be able to use the assumption maybe we should consider calculating $(PQ-I)(Q^T - I)$. Note that showing this matrix is singular is necessary for solving the problem, so it is probably a good idea to focus on proving this is singular.

- We could try some especial cases such as $n = 2$ and/or $u = e_1$.

- I do recall that real eigenvalues of orthogonal matrices are ± 1, so this *almost* solves the problem since Q and PQ can be shown to be both orthogonal!

We notice that $A = (PQ - I)(Q^T - I) = P - PQ - Q^T + I$.

We will start assuming $n = 2$ and $u = e_1$ just to see what we get. We let $Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$

After some computation we end up with $A = \begin{pmatrix} 0 & q_{12} - q_{21} \\ -q_{12} - q_{21} & -2q_{22} + 2 \end{pmatrix}$. To show this matrix is singular we need to prove its determinant is zero or $q_{12}^2 = q_{21}^2$. At this point I know how to do this because we know 2×2 orthogonal matrices are either rotation matrices or something similar. (In fact they are a reflection followed by a rotation.) However if I were to use this fact then we would not be able to generalize the idea. So, we need a better approach.

While working on the above case we realize that by a change of basis we can assume $u = e_1$. In that case P is a diagonal matrix with one -1 and the rest 1 on its main diagonal. This means $\det P = -1$. Thus, $\det(PQ) = -\det Q$. This yields the following solution:

Solution. Let $\mathscr{B} = \{u, u_2, \ldots, u_n\}$ be an orthonormal basis for $\mathbb{R}^n$. We know that Q in this basis is orthogonal and P in this basis is $I - 2e_1 e_1^T$ which is a diagonal matrix with -1 in the $(1,1)$ position and 1 in all other diagonal entries. Since by a change of basis the eigenvalues do not change without loss of generality we may assume $u = e_1$. Note that $PQ(PQ)^T = PQQ^T P^T = PP^T = I$, and thus PQ is orthogonal. Since $\det P = -1$, we have $\det(PQ) = -\det Q$. Note that all real eigenvalues of orthogonal matrices are ± 1. (This is because if A is orthogonal and $A\mathbf{v} = \lambda \mathbf{v}$, then $||A\mathbf{v}||^2 = \mathbf{v}^T A^T A \mathbf{v} = \lambda^2 ||\mathbf{v}||^2$, which implies $\lambda^2 = 1$ or $\lambda = \pm 1$.)

Suppose Q has no eigenvalue of 1. This means all eigenvalues of Q are either -1 or nonreal. Since nonreal roots come in conjugate pairs, and $\det Q$ is the product of its eigenvalues, the sign of $\det Q$ is the same as the sign of $(-1)^n$. Similarly the sign of $\det(PQ)$ is the same as $(-1)^n$. This is a contradiction since $\det(PQ) = -\det Q \neq 0$. $\square$

Example 9.3 (Putnam 1992, B5)**.** *Let D_n denote the value of the $(n-1)\times(n-1)$ determinant*

$$\begin{pmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{pmatrix}.$$

Is the set $\left\{\frac{D_n}{n!}\right\}_{n\geq 2}$ bounded?

Scratch: Here are our first thoughts:

- As usual trying a few examples is a good idea. Maybe we can find a pattern.
- To find a determinant we often use row reduction and then induction. Perhaps we could try to find the determinant first. This seems a bit of a long shot, but we can try!

We can see that $D_2 = 3$, $D_3 = 11$.

For D_4 by subtracting the second row from the last row we obtain the following¿

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & -3 & 4 \end{pmatrix}$$

Expanding along the last row we obtain $D_4 = 4D_3 + 3!$.

For D_4 we similarly row reduce:

$$\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 6 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 5 & 1 \\ 0 & 0 & -4 & 5 \end{pmatrix}$$

Expanding along the last row we get $D_5 = 5D_4 - 4E$, where E is the following determinant:

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Therefore, $D_5 = 5D_4 + 4!$. Putting these together we see that $\frac{D_4}{4!} = \frac{D_3}{3!} + \frac{1}{4}$, and $\frac{D_5}{5!} = \frac{D_4}{4!} - \frac{1}{5} = \frac{D_3}{3!} + \frac{1}{4} + \frac{1}{5}$.

This suggests the following solution:

Solution. The answer is no.

First we will prove that

$$\frac{D_n}{n!} = \frac{D_{n-1}}{(n-1)!} + \frac{1}{n}$$

By subtracting the $(n-1)$-st row from the n-th row we obtain the following:

$$\begin{pmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 4 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 5 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & n & 1 \\ 0 & 0 & 0 & \cdots & -(n-1) & n \end{pmatrix}$$

Expanding along the last row we obtain $D_n = nD_{n-1} + (n-1)E$, where E is the determinant of the following matrix, which we row reduce by subtracting the last row from all other rows.

$$\begin{pmatrix} 3 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 4 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 5 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & \cdots & 0 & 0 \\ 0 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & n-2 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

This matrix is lower triangular whose determinant is $(n-2)!$. Therefore, $D_n = nD_{n-1} + (n-1)(n-2)!$, and thus $\frac{D_n}{n!} = \frac{D_{n-1}}{(n-1)!} + \frac{1}{n}$. This proves the claim.

Next, we will prove by induction on n $\frac{D_n}{n!} = \sum_{k=1}^{n} \frac{1}{k}$.

Basis step: We know $D_2 = 3$. Thus, $\frac{D_2}{2!} = \frac{3}{2} = \frac{1}{1} + \frac{1}{2}$.

Inductive Step: Suppose $\frac{D_{n-1}}{(n-1)!} = \sum_{k=1}^{n-1} \frac{1}{k}$. By what we proved before

$$\frac{D_n}{n!} = \frac{D_{n-1}}{(n-1)!} + \frac{1}{n} = \sum_{k=1}^{n-1} \frac{1}{k} + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}.$$

Therefore, $\frac{D_n}{n!}$ is the n-th partial sum of the harmonic series, which diverges and thus is unbounded. □

Example 9.4 (IMC 2019, Problem 5). *Determine whether there exist an odd positive integer n and $n \times n$ matrices A and B with integer entries, that satisfy the following conditions:*

(1) $\det(B) = 1$*;*

(2) $AB = BA$*;*

(3) $A^4 + 4A^2B^2 + 16B^4 = 2019I$*.*

(Here I denotes the $n \times n$ identity matrix.

Scratch: First, I will try some special cases. The simplest case is when $B = I$. That gives us $B = 1$, and $A^4 + 4A^2 + 16I = 2019I$. Completing the square we get $(A^2 + 2I)^2 = 2007I$. Taking the determinant we obtain $\det(A^2 + 2I)^2 = 2007^n$. Since n is odd this is a contradiction. So, we are up to something! Let's try this in general: $(A^2 + 2B^2)^2 = 2019I - 12B^2$. Taking the determinat we conclude that $\det(2019I - 12B^2)$ must be a perfect square, but finding this determinant is not really possible! Could we maybe somehow get rid of the $12B^2$ part? That suggests taking everything $\mod 4$ or $\mod 3$. We can see that $\mod 4$ gives us a contradiction. This leads to a very simple solution.

Solution. There do not exist such matrices.

Taking everything modulo 4 we obtain $A^4 \equiv -I \mod 4$. Taking the determinat of both sides we get $(\det A)^4 \equiv (-1)^n = -1 \mod 4$, which is a contradiction since perfect fourth powers cannot be -1 modulo 4. □

Example 9.5 (IMC 2019, Problem 9)**.** *Determine all positive integers n for which there exist $n \times n$ real invertible matrices A and B that satisfy $AB - BA = B^2A$.*

Scratch: First playing with the identity we get $AB = (B + B^2)A$ or $ABA^{-1} = B(I + B)$. Conjugating with A we get $A^2BA^{-2} = ABA^{-1}(I + ABA^{-1}) = (B + B^2)(I + B + B^2$. This quickly gets out of hand.

We could also try it differently. $AB = (B + B^2)A$ means when commuting A and B, we change B to $B + B^2$. This suggests an identity of the form $AB^2 = (B + B^2)^2A$, and $AB^3 = (B + B^2)^3A$, and similar for 4, etc. So, we should get $AB^k = (B + B^2)^kA$. At this point I think we are up to something helpful as we can deduce $Ap(B) = p(B + B^2)A$ for any polynomial $p(x)$. If we take the characteristic polynomial we get $p(B + B^2) = 0$, which means all e-values satisfy $p(x + x^2) = 0$. So, this means we cannot have any real e-values, since these values keep getting larger: $\lambda + \lambda^2 > \lambda$. This suggests we cannot have any real eigenvalues, but if n is odd, they we will definitely have real e-values. Thus, n must be even.

Now that we know n cannot be odd, let's try $n = 2$. We just saw that if λ is an eigenvalue, then $\lambda + \lambda^2$ is also an eigenvalue. Since eigenvalues come in conjugate pairs we must have $\lambda + \lambda^2 = \overline{\lambda}$. If we write $\lambda = a + bi$, we get $a + a^2 - b^2 + (b + 2ab)i = a - bi$, which implies $a^2 = b^2$, and $2b + 2ab = 0$. This gives us a solution $a = b = -1$. Thus the eigenvalues are $-1 - i$ and $-1 + i$. So the trace of the matrix must be 2 and its determinant must be 2. One such matrix is $B = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ Putting all of these together we obtain the following solution:

Solution. We will claim that such matrices exist if and only if n is even.

First suppose n is odd. By what we are given $AB = (B + B^2)A$. Thus, $AB^2 = (B + B^2)AB = (B + B^2)^2A$, and $AB^3 = (B + B^2)^2AB = (B + B^2)^3A$. Repeating this we get $AB^k = (B + B^2)^kA$. Let $q(t)$ be the minimal polynomial of B. We see that $Aq(B) = q(B + B^2)A$. Since $q(B) = 0$, we have $q(B + B^2)A = 0$, and since A is invertible $q(B + B^2) = 0$. Since $q(x)$ is the minimal polynomial of B, we must have $q(t) \mid q(t + t^2)$. Thus, every eigenvalue λ of B must satisfy $q(\lambda + \lambda^2) = 0$. Since n is odd and every polynomial with odd degree has a real root B has a real eigenvalue. Since B is invertible all of its eigenvalues are nonzero. Suppose c is the largest real eigenvalue of B. By what we showed $c + c^2$ is an eigenvalue of B. However $c + c^2 > c$, which contradicts the fact that c is the largest eigenvalue of B. This proves when n is odd no such matrices exist.

Now, suppose $n = 2k$ is even, and let $B_1 = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$, and $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Note that both A_1 and B_1 are invertible and that $A_1B_1 - B_1A_1 = B_1^2A_1 = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$. Thus if we consider the $2k \times 2k$ matrices with k copies of A_1 and B_1 on the diagonal, we obtain an example of matrices A and B that satisfy $AB - BA = B^2A$.

$$A = \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & \cdots & A_1 \end{pmatrix}, \text{ and } B = \begin{pmatrix} B_1 & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & \cdots & B_1 \end{pmatrix}$$

$\square$

Example 9.6 (IMC 2018, Problem 6)**.** *Let k be a positive integer. Find the smallest positive integer n for which there exist k nonzero vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$ such that for every pair i, j of indices with $|i-j| > 1$ the vectors v_i and v_j are orthogonal.*

Scratch: Without the condition $|i-j| > 1$, we have a set of orthogonal vectors, which means they are linearly independent. That gives us a bound on k. So, to eliminate that condition we consider $v_1, v_3, v_5, \ldots$. That gives us the following solution:

Solution. We claim the answer is $\boxed{\left\lceil \frac{k}{2} \right\rceil}$.

Suppose $n < \lceil \frac{k}{2} \rceil$. Thus, $n+1 \leq \lceil \frac{k}{2} \rceil$ or $n+1 \leq \frac{k+1}{2}$, which implies $2n+1 \leq k$. By assumption the vectors $v_1, v_3, v_5, \ldots, v_{2n-1}, v_{2n+1}$ are $n+1$ orthogonal vectors and thus they are linearly independent. This contradicts the fact that $\dim \mathbb{R}^n = n < n+1$. This shows that $n \geq \lceil \frac{k}{2} \rceil$.

If $k = 2m$, then $m = \lceil k/2 \rceil$, and the vectors $e_1, e_1, e_2, e_2, \ldots, e_m, e_m$, where $\{e_1, e_2, \ldots, e_m\}$ is a basis for $\mathbb{R}^m$ satisfy the given condition.
If $k = 2m+1$, then $m+1 = \lceil k/2 \rceil$, and the vectors $e_1, e_1, e_2, e_2, \ldots, e_m, e_m, e_{m+1}$, where $\{e_1, e_2, \ldots, e_{m+1}\}$ is a basis for $\mathbb{R}^{m+1}$ satisfy the given condition.
Therefore, the answer is $\lceil k/2 \rceil$. $\square$

Example 9.7 (IMC 2022, Problem 2)**.** *Let n be a positive integer. Find all $n \times n$ real matrices A with only real eigenvalues satisfying*

$$A + A^k = A^T$$

for some integer $k \geq n$. (A^T denotes the transpose of A.)

Solution. (Video Solution) We claim the zero matrix is the only such $n \times n$ matrix.

Let $(\lambda, \mathbf{v})$ be an eigenpair of A. We know $A\mathbf{v} = \lambda\mathbf{v}$. By taking the transpose of both sides we obtain $\mathbf{v}^T A^T = \lambda \mathbf{v}^T$. Multiplying the given equality by $\mathbf{v}$ from the right and $\mathbf{v}^T$ from the left we obtain the following:

$$\mathbf{v}^T A\mathbf{v} + \mathbf{v}^T A^k \mathbf{v} = \mathbf{v}^T A^T \mathbf{v} \Rightarrow \mathbf{v}^T \lambda \mathbf{v} + \mathbf{v}^T \lambda^k \mathbf{v} = \lambda \mathbf{v}^T \mathbf{v} \Rightarrow \lambda^k ||\mathbf{v}||^2 = 0 \Rightarrow \lambda = 0.$$

Therefore, all eigenvalues of A are zero. Therefore, by the Cayley-Hamilton Theorem, $A^n = 0$. Since $k \geq n$ we have $A^k = 0$, and hence, $A = A^T$, which means A is symmetric. By a theorem, all symmetric real matrices can be diagonalized. Therefore, A can be diagonalized. Since all eigenvalues of A are zero, $A = P0P^{-1}$, for some invertible matrix P, which implies $A = 0$, as desired. □

Example 9.8 (IMC 2022, Problem 7). *Let $A_1, A_2, \ldots, A_k$ be $n \times n$ idempotent complex matrices such that*

$$A_i A_j = -A_j A_i \text{ for all } i \neq j.$$

Prove that at least one of the given matrices has rank $\leq \frac{n}{k}$.

(A matrix A is called idempotent if $A^2 = A$.)

Solution.(Video Solution)

□

Example 9.9. *Let A be a real $n \times n$ matrix such that $A^3 = 0$.*

(a) Prove that there is a unique real $n \times n$ matrix X that satisfies the equation $X + AX + XA^2 = A$.

(b) Express X in terms of A.

Solution.(Video Solution)

□

Example 9.10 (IMC 2023, Problem 2). *Let A, B and C be $n \times n$ matrices with complex entries satisfying*

$$A^2 = B^2 = C^2 \text{ and } B^3 = ABC + 2I.$$

Prove that $A^6 = I$.

Solution.(Video Solution)

□

Example 9.11 (IMC 2023, Problem 6). *Ivan writes the matrix* $\begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$ *on the board. Then he performs the following operation on the matrix several times:*

- *he chooses a row or a column of the matrix, and*
- *he multiplies or divides the chosen row or column entry-wise by the other row or column, respectively.*

Can Ivan end up with the matrix $\begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$ *after finitely many steps?*

Solution.(Video Solution)

□

9.5 General Strategies

- To find a determinant we typically use row reduction and then induction.
- Finding large powers of a square matrix is typically found by diagonalizing the matrix if possible and using the fact that $(PDP^{-1})^n = PD^nP^{-1}$.

9.6 Exercises

Exercise 9.1 (VTRMC 1979)**.** Let A be an $n \times n$ nonsingular matrix with complex elements, and let $\overline{A}$ be its complex conjugate. Let $B = A\overline{A} + I$, where I is the $n \times n$ identity matrix.

(a) Prove or disprove: $A^{-1}BA = \overline{B}$.

(b) Prove or disprove: the determinant of $A\overline{A} + I$ is real.

Exercise 9.2 (VTRMC 1982)**.** Let $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ be vectors such that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is linearly dependent. Show that

$$\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} = 0$$

Exercise 9.3 (VTRMC 1984)**.** A matrix is called excellent if it is square and the sum of its elements in each row and column equals the sum of its elements in every other row and column. Let V_n denote the collection of excellent $n \times n$ matrices.

(a) Show that V_n is a vector space under addition and scalar multiplication (by real numbers).

(b) Find the dimensions of V_2, V_3, and V_4.

(c) If $A \in V_n$ and $B \in V_n$, show that $AB \in V_n$.

Exercise 9.4 (VTRMC 1987)**.** A sequence of polynomials is given by $p_n(x) = a_{n+2}x^2 + a_{n+1}x - a_n$, for $n \geq 0$, where $a_0 = a_1 = 1$ and, for $n \geq 0, a_{n+2} = a_{n+1} + a_n$. Denote by r_n and s_n the roots of $p_n(x) = 0$, with $r_n \leq s_n$. Find $\lim_{n\to\infty} r_n$ and $\lim_{n\to\infty} s_n$. 7. Let $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ be $n \times n$ matrices such that A^{-1} exists. Define $A(t) = \{a_{ij}(t)\}$

and $B(t) = \{b_{ij}(t)\}$ by $a_{ij}(t) = a_{ij}$ for $i < n, a_{nj}(t) = ta_{nj}$, $b_{ij}(t) = b_{ij}$ for $i < n$, and $b_{nj}(t) = tb_{nj}$. For example, if $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then $A(t) = \begin{pmatrix} 1 & 2 \\ 3t & 4t \end{pmatrix}$. Prove that $A(t)^{-1}B(t) = A^{-1}B$ for $t > 0$ and any n. (Partial credit will be given for verifying the result for $n = 3$.)

Exercise 9.5 (VTRMC 1989)**.** Let A be a 3×3 matrix in which each element is either 0 or 1 but is otherwise arbitrary.

(a) Prove that $\det(A)$ cannot be 3 or -3 .

(b) Find all possible values of $\det(A)$ and prove your result.

Exercise 9.6 (VTRMC 1989)**.** The system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

has the solution $x_1 = -1, x_2 = 3, x_3 = 2$ when $b_1 = 1, b_2 = 0, b_3 = 1$ and it has the solution $x_1 = 2, x = -2, x_3 = 1$ when $b_1 = 0, b_2 = -1, b_3 = 1$. Find a solution of the system when $b_1 = 2, b_2 = -1, b_3 = 3$.

Exercise 9.7 (Putnam 1990, A5)**.** If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size such that $\mathbf{ABAB} = \mathbf{0}$, does it follow that $\mathbf{BABA} = \mathbf{0}$?

Exercise 9.8 (Putnam 1991, A2)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be different $n\times n$ matrices with real entries. If $\mathbf{A}^3 = \mathbf{B}^3$ and $\mathbf{A}^2\mathbf{B} = \mathbf{B}^2\mathbf{A}$, can $\mathbf{A}^2 + \mathbf{B}^2$ be invertible?

Exercise 9.9 (VTRMC 1992)**.** Let $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$. Find A^{100}. You have to find all four entries.

Exercise 9.10 (Putnam 1992, B6)**.** Let $\mathscr{M}$ be a set of real $n\times n$ matrices such that

(i) $I \in \mathscr{M}$, where I is the $n\times n$ identity matrix;

(ii) if $A \in \mathscr{M}$ and $B \in \mathscr{M}$, then either $AB \in \mathscr{M}$ or $-AB \in \mathscr{M}$, but not both;

(iii) if $A \in \mathscr{M}$ and $B \in \mathscr{M}$, then either $AB = BA$ or $AB = -BA$;

(iv) if $A \in \mathscr{M}$ and $A \neq I$, there is at least one $B \in \mathscr{M}$ such that $AB = -BA$.

Prove that $\mathscr{M}$ contains at most n^2 matrices.

Exercise 9.11 (VTRMC 1994)**.** Let A be an $n\times n$ matrix and let α be an n-dimensional vector such that $A\alpha = \alpha$. Suppose that all the entries of A and α are positive real numbers. Prove that α is the only linearly independent eigenvector of A corresponding to the eigenvalue 1.

Exercise 9.12 (Putnam 1994, A4)**.** Let A and B be 2×2 matrices with integer entries such that $A, A+B, A+2B, A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

Exercise 9.13 (Putnam 1994, B4)**.** For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that $\lim_{n\to\infty} d_n = \infty$.

Exercise 9.14 (VTRMC 1995)**.** Let $\mathbb{R}^2$ denote the xy-plane, and define $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by $\theta(x,y) = (4x - 3y + 1, 2x - y + 1)$. Determine $\theta^{100}(1,0)$, where θ^{100} indicates applying θ, 100 times.

Exercise 9.15 (Putnam 1995, A5)**.** Let $x_1, x_2, \ldots, x_n$ be differentiable (real-valued) functions of a single variable f which satisfy

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
\vdots & \qquad\qquad\qquad\qquad \vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}$$

for some constants $a_{ij} > 0$. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \ldots, x_n$ necessarily linearly dependent?

Exercise 9.16 (Putnam 1995, B3)**.** To each positive integer with n^2 decimal digits, we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det \begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50$. Find, as a function of n, the sum of all the determinants associated with n^2-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

Exercise 9.17 (Putnam 1996, B4)**.** For any square matrix A, we can define $\sin A$ by the usual power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Prove or disprove: there exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$

Exercise 9.18 (VTRMC 1999)**.** Let ε, M be positive real numbers, and let $A_1, A_2, \ldots$ be a sequence of matrices such that for all n,

(i) A_n is an $n \times n$ matrix with integer entries,

(ii) The sum of the absolute values of the entries in each row of A_n is at most M.

If δ is a positive real number, let $e_n(\delta)$ denote the number of nonzero eigenvalues of A_n which have absolute value less that δ. (Some eigenvalues can be complex numbers.) Prove that one can choose $\delta > 0$ so that $e_n(\delta)/n < \varepsilon$ for all n.

Exercise 9.19 (Putnam 1999, B5)**.** For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all j, k.

Exercise 9.20 (VTRMC 2000)**.** Let n be a positive integer and let A be an $n \times n$ matrix with real numbers as entries. Suppose $4A^4 + I = 0$, where I denotes the identity matrix. Prove that the trace of A (i.e. the sum of the entries on the main diagonal) is an integer.

Exercise 9.21 (VTRMC 2002)**.** Let S be a set of 2×2 matrices with complex numbers as entries, and let T be the subset of S consisting of matrices whose eigenvalues are ± 1 (so the eigenvalues for each matrix in T are $\{1, 1\}$ or $\{1, -1\}$ or $\{-1, -1\}$). Suppose there are exactly three matrices in T. Prove that there are matrices A, B in S such that AB is not a matrix in $S(A = B$ is allowed).

Exercise 9.22 (Putnam 2002, A4)**.** In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

Exercise 9.23 (VTRMC 2003)**.** Determine all invertible 2 by 2 matrices A with complex numbers as entries satisfying $A = A^{-1} = A'$, where A' denotes the transpose of A.

Exercise 9.24 (VTRMC 2004)**.** Let I denote the 2×2 identity matrix and let

$$M = \begin{pmatrix} I & A \\ B & C \end{pmatrix}, N = \begin{pmatrix} I & B \\ A & C \end{pmatrix},$$

where A, B, C are arbitrary 2×2 matrices which entries in $\mathbb{R}$, the real numbers. Thus M and N are 4×4 matrices with entries in $\mathbb{R}$. Is it true that M is invertible (i.e. there is a 4×4 matrix X such that $MX = XM =$ the identity matrix) implies N is invertible? Justify your answer.

Exercise 9.25 (VTRMC 2005)**.** Let A be a 5×10 matrix with real entries, and let A' denote its transpose (so A' is a 10×5 matrix, and the ij-th entry of A' is the ji-th entry of A). Suppose every 5×1 matrix with real entries (i.e. column vector in 5 dimensions) can be written in the form $A\mathbf{u}$ where $\mathbf{u}$ is a 10×1 matrix with real entries. Prove that every 5×1 matrix with real entries can be written in the form $AA'\mathbf{v}$ where $\mathbf{v}$ is a 5×1 matrix with real entries.

Exercise 9.26 (Putnam 2005, A4)**.** Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

Exercise 9.27 (Putnam 2006, B4)**.** Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are the vertices of a unit hypercube in $\mathbb{R}^n$.) Let k be a given integer with $0 \leq k \leq n$. Find the maximum number of points in $V \cap Z$, where V ranges over all subspaces of $\mathbb{R}^n$ of dimension k.

Exercise 9.28 (VTRMC 2007)**.** Let n be a positive integer, let A, B be square symmetric $n \times n$ matrices with real entries (so if a_{ij} are the entries of A, the a_{ij} are real numbers and $a_{ij} = a_{ji}$.) Suppose there are $n \times n$ matrices X, Y (with complex entries) such that $\det(AX + BY) \neq 0$. Prove that $\det\left(A^2 + B^2\right) \neq 0$ (det indicates the determinant).

Exercise 9.29 (Putnam 2008, A2)**.** Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

Exercise 9.30 (VTRMC 2009)**.** Let $\mathbb{C}$ denote the complex numbers and let $M_3(\mathbb{C})$ denote the 3 by 3 matrices with entries in $\mathbb{C}$. Suppose $A, B \in M_3(\mathbb{C}), B \neq 0$, and $AB = 0$ (where 0 denotes the 3 by 3 matrix with all entries zero). Prove that there exists $0 \neq D \in M_3(\mathbb{C})$ such that $AD = DA = 0$.

Exercise 9.31 (Putnam 2009, A3)**.** Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$.

Exercise 9.32 (VTRMC 2010)**.** Let d be a positive integer and let A be a $d \times d$ matrix with integer entries. Suppose $I + A + A^2 + \cdots + A^{100} = 0$ (where I denotes the identity $d \times d$ matrix, so I has 1 's on the main diagonal, and 0 denotes the zero matrix, which has all entries 0). Determine the positive integers $n \leq 100$ for which $A^n + A^{n+1} + \cdots + A^{100}$ has determinant ± 1.

Exercise 9.33 (Putnam 2010, B6)**.** Let A be an $n \times n$ matrix of real numbers for some $n \geq 1$. For each positive integer k, let $A^{[k]}$ be the matrix obtained by raising each entry to the kth power. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \ldots, n+1$, then $A^k = A^{[k]}$ for all $k \geq 1$.

Exercise 9.34 (Putnam 2011, A4)**.** For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?

Exercise 9.35 (Putnam 2011, B4)**.** In a tournament, 2011 players meet 2011 times to play a multiplayer game. Every game is played by all 2011 players together and ends with each of the players either winning or losing. The standings are kept in two 2011×2011 matrices, $T = (T_{hk})$ and $W = (W_{hk})$. Initially, $T = W = 0$. After every game, for every (h, k) including for $h = k$), if players h and k tied (that is, both won or both lost), the entry T_{hk} is increased by 1, while if player h won and player k lost, the entry W_{hk} is increased by 1 and W_{kh} is decreased by 1.
Prove that at the end of the tournament, $\det(T + iW)$ is a non-negative integer divisible by 2^{2010}.

Exercise 9.36 (VTRMC 2012)**.** Let A_1, A_2, A_3 be 2×2 matrices with entries in $\mathbb{C}$ (the complex numbers). Let tr denote the trace of a matrix (so $\operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$). Suppose $\{A_1, A_2, A_3\}$ is closed under matrix multiplication (i.e. given i, j, there exists k such that $A_i A_j = A_k$), and $\operatorname{tr}(A_1 + A_2 + A_3) \neq 3$. Prove that there exists i such that $A_i A_j = A_j A_i$ for all j (here i, j are $1, 2$ or 3).

Exercise 9.37 (Putnam 2012, A5)**.** Let $\mathbb{F}_p$ denote the field of integers modulo a prime p, and let n be a positive integer. Let v be a fixed vector in $\mathbb{F}_p^n$, let M be an $n \times n$ matrix with entries of $\mathbb{F}_p$, and define $G : \mathbb{F}_p^n \to \mathbb{F}_p^n$ by $G(x) = v + Mx$.

Let $G^{(k)}$ denote the k-fold composition of G with itself, that is, $G^{(1)}(x) = G(x)$ and $G^{(k+1)}(x) = G(G^{(k)}(x))$. Determine all pairs p,n for which there exist v and M such that the p^n vectors $G^{(k)}(0)$, $k = 1,2,\dots,p^n$ are distinct.

Exercise 9.38 (VTRMC 2013)**.** Let

$$X = \begin{pmatrix} 7 & 8 & 9 \\ 8 & -9 & -7 \\ -7 & -7 & 9 \end{pmatrix}, Y = \begin{pmatrix} 9 & 8 & -9 \\ 8 & -7 & 7 \\ 7 & 9 & 8 \end{pmatrix}$$

Let A = $Y^{-1} - X$ and let B be the inverse of $X^{-1} + A^{-1}$. Find a matrix M such that $M^2 = XY - BY$ (you may assume that A and $X^{-1} + A^{-1}$ are invertible).

Exercise 9.39 (VTRMC 2014)**.** Let S denote the set of 2 by 2 matrices with integer entries and determinant 1, and let T denote those matrices of S which are congruent to the identity matrix I mod 3 (so $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in T$ means that $a,b,c,d \in \mathbb{Z}, ad - bc = 1$, and 3 divides $b,c,a-1,d-1$;; " $\in$ " means "is in").

(a) Let $f : T \to \mathbb{R}$ (the real numbers) be a function such that for every $X,Y \in T$ with $Y \neq I$, either $f(XY) > f(X)$ or $f\left(XY^{-1}\right) > f(X)$ (or both). Show that given two finite nonempty subsets A,B of T, there are matrices $a \in A$ and $b \in B$ such that if $a' \in A, b' \in B$ and $a'b' = ab$, then $a' = a$ and $b' = b$.

(b) Show that there is no $f : S \to \mathbb{R}$ such that for every $X,Y \in S$ with $Y \neq \pm I$, either $f(XY) > f(X)$ or $f\left(XY^{-1}\right) > f(X)$.

Exercise 9.40 (Putnam 2014, A2)**.** Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is

$$\frac{1}{\min(i,j)}$$

for $1 \leq i,j \leq n$. Compute $\det(A)$.

Exercise 9.41 (Putnam 2014, A6)**.** Let n be a positive integer. What is the largest k for which there exist $n \times n$ matrices $M_1,\dots,M_k$ and $N_1,\dots,N_k$ with real entries such that for all i and j, the matrix product M_iN_j has a zero entry somewhere on its diagonal if and only if $i \neq j$?

Exercise 9.42 (Putnam 2014, B3)**.** Let A be an $m \times n$ matrix with rational entries. Suppose that there are at least $m+n$ distinct prime numbers among the absolute values of the entries of A. Show that the rank of A is at least 2.

Exercise 9.43 (VTRMC 2015)**.** Let $(a_i)_{1 \leq i \leq 2015}$ be a sequence consisting of 2015 integers, and let $(k_i)_{1 \leq i \leq 2015}$ be a sequence of 2015 positive integers (positive integer excludes 0). Let

$$A = \begin{pmatrix} a_1^{k_1} & a_1^{k_2} & \cdots & a_1^{k_{2015}} \\ a_2^{k_1} & a_2^{k_2} & \cdots & a_2^{k_{2015}} \\ \vdots & \vdots & \cdots & \vdots \\ a_{2015}^{k_1} & a_{2015}^{k_2} & \cdots & a_{2015}^{k_{2015}} \end{pmatrix}$$

Prove that 2015! divides $\det A$.

Exercise 9.44 (Putnam 2015, A6)**.** Let n be a positive integer. Suppose that A, B, and M are $n \times n$ matrices with real entries such that $AM = MB$, and such that A and B have the same characteristic polynomial. Prove that $\det(A - MX) = \det(B - XM)$ for every $n \times n$ matrix X with real entries.

Exercise 9.45 (Putnam 2015, B3)**.** Let S be the set of all 2×2 real matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose entries a, b, c, d (in that order) form an arithmetic progression. Find all matrices M in S for which there is some integer $k > 1$ such that M^k is also in S.

Exercise 9.46 (VTRMC 2016)**.** Let n be a positive integer and let $M_n(\mathbb{Z}_2)$ denote the n by n matrices with entries from the integers modulo 2. If $n \geq 2$, prove that the number of matrices A in $M_n(\mathbb{Z}_2)$ satisfying $A^2 = 0$ (the matrix with all entries zero) is an even positive integer.

Exercise 9.47 (VTRMC 2016)**.** Let A, B, P, Q, X, Y be square matrices of the same size. Suppose that

$$\begin{aligned} A + B + AB &= XY \quad & AX &= XQ \\ P + Q + PQ &= YX \quad & PY &= YB. \end{aligned}$$

Prove that $AB = BA$.

Exercise 9.48 (Putnam 2016, B4)**.** Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability $1/2$. Find the expected value of $\det(A - A^t)$ (as a function of n), where A^t is the transpose of A.

Exercise 9.49 (IMC 2018, Problem 3)**.** Determine all rational numbers a for which the matrix

$$\begin{pmatrix} a & -a & -1 & 0 \\ a & -a & 0 & -1 \\ 1 & 0 & a & -a \\ 0 & 1 & a & -a \end{pmatrix}$$

is the square of a matrix with all rational entries.

Exercise 9.50. Suppose $A \in M_n(\mathbb{C})$ satisfies $A^k = I$ for some positive integer k. Assume $\operatorname{tr} A = n$. Prove that $A = I$.

Exercise 9.51 (VTRMC 2018)**.** Let $A, B \in \mathrm{M}_6(\mathbb{Z})$ such that $A \equiv I \equiv B \bmod 3$ and $A^3B^3A^3 = B^3$. Prove that $A = I$. Here $\mathrm{M}_6(\mathbb{Z})$ indicates the 6 by 6 matrices with integer entries, I is the identity matrix, and $X \equiv Y \bmod 3$ means all entries of $X - Y$ are divisible by 3.

Exercise 9.52 (Putnam 2018, A2)**.** Let $S_1, S_2, \ldots, S_{2^n-1}$ be the nonempty subsets of $\{1, 2, \ldots, n\}$ in some order, and let M be the $(2^n - 1) \times (2^n - 1)$ matrix whose (i, j) entry is

$$m_{ij} = \begin{cases} 0 & \text{if } S_i \cap S_j = \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

Calculate the determinant of M.

Exercise 9.53 (Putnam 2021, B5)**.** Say that an n-by-n matrix $A = (a_{ij})_{1\le i,j\le n}$ with integer entries is *very odd* if, for every nonempty subset S of $\{1,2,\dots,n\}$, the $|S|$-by-$|S|$ submatrix $(a_{ij})_{i,j\in S}$ has odd determinant. Prove that if A is very odd, then A^k is very odd for every $k \ge 1$.

Exercise 9.54 (VTRMC 2022)**.** Let A be an invertible $n\times n$ matrix with complex entries. Suppose that for each positive integer m, there exists a positive integer k_m and an $n\times n$ invertible matrix B_m such that $A^{k_m m} = B_m A B_m^{-1}$. Show that all eigenvalues of A are equal to 1.

Exercise 9.55 (Putnam 2023, B6)**.** Let n be a positive integer. For i and j in $\{1,2,\dots,n\}$, let $s(i,j)$ be the number of pairs (a,b) of nonnegative integers satisfying $ai+bj=n$. Let S be the n-by-n matrix whose (i,j)-entry is $s(i,j)$.

For example, when $n=5$, we have $S = \begin{bmatrix} 6 & 3 & 2 & 2 & 2 \\ 3 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 2 \end{bmatrix}$.

Compute the determinant of S.

Chapter 10

Series

10.1 Basics

Definition 10.1. A series $\sum\limits_{n=1}^{\infty} a_n$ converges if its partial sums $\sum\limits_{i=1}^{n} a_i$ approach a real number as n approaches infinity.

10.2 Important Theorems

Theorem 10.1 (Geometric and Arithmetic Sums)**.** *Let g_n be a geometric sequence and a_n be an arithmetic sequence. Then,*

(a) $a_1 + a_2 + \cdots + a_n = \left(\frac{a_1 + a_n}{2}\right) n$ *=(Average of first and last terms) $\times$ (The number of terms).*

(b) $g_1 + g_2 + \cdots + g_n = \frac{g_1 - g_{n+1}}{1-r} = \frac{\textit{first} - \textit{after last}}{1 - \textit{common ratio}}$.

(c) $\sum\limits_{n=1}^{\infty} g_n = \frac{g_1}{1-r} = \frac{\textit{first}}{1 - \textit{common ratio}}$*, if the common ratio r satisfies $|r| < 1$.*

Theorem 10.2 (Comparison Test)**.** *Suppose $a_n \leq b_n$ are two sequences with nonnegative terms.*

- *If $\sum a_n$ diverges, then $\sum b_n$ diverges.*
- *If $\sum b_n$ converges, then $\sum a_n$ converges.*

Theorem 10.3 (Absolute Convergence Test)**.** *If $\sum |a_n|$ converges, then $\sum a_n$ converges.*

Definition 10.2. A series $\sum a_n$ is said to be **absolutely convergent** if $\sum |a_n|$ converges. If $\sum a_n$ converges, but $\sum |a_n|$ diverges, we say the series $\sum a_n$ converges **conditionally**.

Theorem 10.4 (Ratio Test)**.** *Suppose a_n is a sequence of nonzero real numbers. Let $\ell = \lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.*

- *If $\ell < 1$, then $\sum a_n$ converges absolutely.*
- *If $\ell > 1$, then $\sum a_n$ diverges.*

Theorem 10.5 (Root Test)**.** *Suppose a_n is a sequence of nonzero real numbers. Let $\ell = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.*

- *If $\ell < 1$, then $\sum a_n$ converges absolutely.*
- *If $\ell > 1$, then $\sum a_n$ diverges.*

Theorem 10.6 (Limit Comparison Test)**.** *Suppose a_n and b_n are two positive sequences such that $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is a nonzero real number. Then either both $\sum a_n$ and $\sum b_n$ converge or both diverge.*

Theorem 10.7 (Integral Test)**.** *Suppose f is a function that is continuous, decreasing and nonnegative over an interval (a,∞) for some real number a. Then $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_a^{\infty} f(x)\,\mathrm{d}x$ converges.*

Theorem 10.8 (p-Test)**.** *The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is a constant, converges if and only if $p > 1$.*

Theorem 10.9 (Alternating Series Test)**.** *Suppose a_n is a decreasing sequence of nonnegative real numbers that approaches zero. Then the alternating series $\sum (-1)^n a_n$ converges.*

10.3 Classical Examples

Example 10.1. *Let S be the set consisting of all positive integers each of which has no prime factor other than possibly 2 or 5. For example $1 \in S, 2 \in S$, and $500 \in S$, but $6 \notin S$. Evaluate*

$$\sum_{s\in S} \frac{1}{s^2}.$$

Solution. (Video Solution) Every element $s \in S$ can uniquely be written as $2^a 5^b$ for some integers $a, b \geq 0$. Therefore, every term of the form $\frac{1}{s^2}$ can uniquely be written as $\frac{1}{2^{2a}} \cdot \frac{1}{5^{2b}}$. Therefore, the given sum is equal to

$$\left(\sum_{a=0}^{\infty} \frac{1}{2^{2a}}\right)\left(\sum_{a=0}^{\infty} \frac{1}{5^{2a}}\right) = \frac{1}{1-1/4} \cdot \frac{1}{1-1/25} = \frac{25}{18}.$$

□

Example 10.2. *Find a formula in closed form for each of the following sums:*

(a) $\sum_{k=1}^{n} \binom{n}{k}$.

(b) $\sum_{k=1}^{n} k\binom{n}{k}$.

(c) $\sum_{k=1}^{n} k^2\binom{n}{k}$.

Solution 1. (Video Solution) (a) The answer is $2^n - 1$. We will use the Binomial Theorem:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \qquad (*)$$

Substituting $x = 1$ we obtain

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Subtracting $\binom{n}{0}$ we obtain the result.

(b) The answer is $n\ 2^{n-1}$. Differentiating both sides of $(*)$, we obtain:

$$n(1+x)^{n-1} = \sum_{k=1}^{n} k \binom{n}{k} x^{k-1} \qquad (**)$$

Substituting $x = 1$ we obtain $n\ 2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}$.

(c) The answer is $n\ 2^{n-1} + n(n-1)\ 2^{n-2}$.

Multiplying both sides of $(**)$ by x and then differentiating we obtain:

$$nx(1+x)^{n-1} = \sum_{k=1}^{n} k \binom{n}{k} x^{k} \Rightarrow n(1+x)^{n-1} + n(n-1)x(1+x)^{n-2} = \sum_{k=1}^{n} k^2 \binom{n}{k} x^{k-1}.$$

Substituting $x = 1$ we obtain $\sum_{k=1}^{n} k^2 \binom{n}{k} = n\ 2^{n-1} + n(n-1)\ 2^{n-2}$. $\square$

Solution 2. We will provide a combinatorial proof. We will employ the Two-Way Counting method. Let $A = \{1,2,\ldots,n\}$.

(a) We will find the number of subsets of A is two ways.

To form a subset of A we need to decide if each integer k with $1 \le k \le n$ belongs to this subset or does not. Thus, there are 2^n subsets of A.

On the other hand, the number of subsets of A of size k is $\binom{n}{k}$. Thus, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Subtracting $\binom{n}{0}$ we conclude the given sum is $2^n - 1$.

(b) Define the set B by

$$B = \{(a,S) \mid a \in S, \text{ and } S \subseteq A\}.$$

There are n ways to choose an element $a \in A$. After selecting a, we need to decide if each element of A except for a belongs to S or does not belong to S. Therefore, there are 2^{n-1} possible subsets S with $a \in S$. Thus, the size of B is $n\ 2^{n-1}$.

Now, we will find the size of B in a different way. There are $\binom{n}{k}$ subsets of A of size k. For each one, there are k possible elements a that belong to S. Thus, the size of B is $\sum_{k=1}^{n} k \binom{n}{k}$. Therefore, $\sum_{k=1}^{n} k \binom{n}{k} = n\ 2^{n-1}$.

(c) Similar to (b), define a set C as follows:

$$B = \{(a,b,S) \mid a,b \in S, \text{ and } S \subseteq A\}.$$

We will find the size of C in two different ways.

If $a=b$, then there are n possibilities for $a=b$ and 2^{n-1} possible subsets S with $a \in S \subseteq A$.

If $a \neq b$, then there are $n(n-1)$ possibilities for (a,b). The set S must consist both a and b, and hence, there are 2^{n-2} possible subsets S. Thus, the size of C is $n\ 2^{n-1}+n(n-1)\ 2^{n-2}$.

Now, we will find the size of C in a differently. We may first select a subset S of size k and then select elements a,b. This can be done in $\binom{n}{k}k^2$ ways.

Therefore, $\displaystyle\sum_{k=1}^{n} k^2 \binom{n}{k} = n\ 2^{n-1}+n(n-1)\ 2^{n-2}$. $\square$

Example 10.3. *Evaluate each sum:*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\binom{n}{3}}$

Scratch: We will use partial fractions to break up the terms into simpler terms and be able to evaluate the sum.

$$\frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \Rightarrow 1 = A(n+1)+Bn$$

Setting $n=-1$, we obtain $B=-1$, and setting $n=0$, we obtain $A=1$. This yields a telescoping sum that allows the sum to be evaluated. Similarly, we write

$$\frac{1}{\binom{n}{3}} = \frac{6}{n(n-1)(n-2)} = \frac{A}{n} + \frac{B}{n-1} + \frac{C}{n-2} \Rightarrow 6 = A(n-1)(n-2)+Bn(n-2)+Cn(n-1)$$

Substituting $n=0,1,2$ we obtain $A=3, B=-6, C=3$.

Solution. (Video Solution) In order to make our solution rigorous we will first fine the partial sums and then take the limit.

(a) $\displaystyle\sum_{n=1}^{N} \frac{1}{n^2+n} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{1} - \frac{1}{N+1}$. Taking the limit we can see that the sum is 1.

(b)

$$\sum_{n=3}^{N} \frac{1}{\binom{n}{3}} = \sum_{n=3}^{N} \left(\frac{3}{n} + \frac{-6}{n-1} + \frac{3}{n-2}\right) = \sum_{n=3}^{N} \frac{3}{n} - \sum_{n=2}^{N-1} \frac{6}{n} + \sum_{n=1}^{N-2} \frac{3}{n} = \frac{3}{N-1} + \frac{3}{N} - \frac{6}{2} - \frac{6}{N-1} + \frac{3}{1} + \frac{3}{2}$$

Taking the limit, we conclude that the sum is $\boxed{\frac{3}{2}}$. $\square$

Example 10.4. *Evaluate each of the following sums:*

$$\sum_{n=1}^{\infty} \frac{n}{3^n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n^2}{3^n}$$

Solution 1. (Video Solution) We differentiate the geometric series, for $|x| < 1$, as follows:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} \Rightarrow \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n \qquad (*)$$

Substituting $x = 1/3$ we obtain $\sum\limits_{n=1}^{\infty} \frac{n}{3^n} = \frac{3}{4}$.

Differentiating $(*)$ and later multiplying by x we obtain the following:

$$\frac{(1-x)^2 + 2(1-x)x}{(1-x)^4} = \sum_{n=1}^{\infty} n^2 x^{n-1} \Rightarrow \frac{1+x}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^{n-1} \Rightarrow \frac{x(1+x)}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^n.$$

Substituting $x = 1/3$ we conclude that $\sum\limits_{n=1}^{\infty} \frac{n^2}{3^n} = \frac{3}{2}$. □

The following is a pre-calculus solution to the same problem.

Solution 2. Let $A = \sum\limits_{n=1}^{\infty} \frac{n}{3^n}$. Multiplying both sides by 3 we obtain the following:

$$3A = \sum_{n=1}^{\infty} \frac{n}{3^{n-1}} = \sum_{n=0}^{\infty} \frac{n+1}{3^n} = \sum_{n=1}^{\infty} \frac{n}{3^n} + \sum_{n=0}^{\infty} \frac{1}{3^n} = A + \frac{1}{1-1/3} = A + \frac{3}{2} \Rightarrow 2A = \frac{3}{2} \Rightarrow \sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{3}{4}.$$

For the second sum we will do the same: Set $S = \sum\limits_{n=1}^{\infty} \frac{n^2}{3^n}$. Multiplying both sides by 3 we obtain the following:

$$3A = \sum_{n=1}^{\infty} \frac{n^2}{3^{n-1}} = \sum_{n=0}^{\infty} \frac{(n+1)^2}{3^n} = \sum_{n=0}^{\infty} \frac{n^2+2n+1}{3^n} = A + 2\sum_{n=0}^{\infty} \frac{n}{3^n} + \sum_{n=0}^{\infty} \frac{1}{3^n} = A + \frac{3}{2} + \frac{1}{1-1/3} = A + 3.$$

Therefore, $\sum\limits_{n=1}^{\infty} \frac{n^2}{3^n} = \frac{3}{2}$. □

10.4 Further Examples

Example 10.5 (VTRMC 1980)**.** *The sum of the first n terms of the sequence*

$$1, (1+2), (1+2+2^2), \ldots, (1+2+\cdots+2^{k-1}), \ldots$$

is of the form $2^{n+R} + Sn^2 + Tn + U$ *for all* $n > 0$*. Find* R, S, T *and* U*.*

Solution. The answer is $\boxed{R = 1, S = 0, T = -1, \text{ and } U = -2.}$

The k-th term of the sequence is $a_k = \sum\limits_{j=0}^{k-1} 2^j$, which is a geometric series, and is equal to $2^k - 1$, by the geometric sum formula. The sum of the first n terms is $\sum\limits_{k=1}^{n} (2^k - 1) = \sum\limits_{k=1}^{n} 2^k - \sum\limits_{k=1}^{n} 1$. The first sum is a geometric sum and the second sum is n. Thus, the answer is $2^{n+1} - 2 - n$, as desired. □

Example 10.6 (Putnam 2019, B2)**.** *For all $n \geq 1$, let*

$$a_n = \sum_{k=1}^{n-1} \frac{\sin\left(\frac{(2k-1)\pi}{2n}\right)}{\cos^2\left(\frac{(k-1)\pi}{2n}\right)\cos^2\left(\frac{k\pi}{2n}\right)}.$$

Determine

$$\lim_{n\to\infty} \frac{a_n}{n^3}.$$

Scratch: Here are a few ideas that come to mind:

- Perhaps we could find an explicit formula for a_n and then take the limit. This seems to be a long shot, but given this is B2 it may be a possibility. If we were to evaluate this sum it probably has to be a telescoping sum. We could also substitute those trig functions in terms of complex numbers and see if we can simplify anything.
- Finding a few terms of the sequence is always a good idea.

I started evaluating the first couple of terms of the sequence, but it got fairly complicated quickly. Using complex numbers did not make things much easier, although in retrospect they may work as well! So, I tried to write the sum as a telescoping sum. This means we are looking for some A and B that may depend on n and k which satisfy

$$\frac{\sin\left(\frac{(2k-1)\pi}{2n}\right)}{\cos^2\left(\frac{(k-1)\pi}{2n}\right)\cos^2\left(\frac{k\pi}{2n}\right)} = \frac{A}{\cos^2\left(\frac{(k-1)\pi}{2n}\right)} + \frac{B}{\cos^2\left(\frac{k\pi}{2n}\right)}.$$

Clearing the denominators gives us $\sin\left(\frac{(2k-1)\pi}{2n}\right) = A\cos^2\left(\frac{k\pi}{2n}\right) + B\cos^2\left(\frac{(k-1)\pi}{2n}\right)$. Using half angle formula we obtain $2\sin\left(\frac{(2k-1)\pi}{2n}\right) = A(\cos\left(\frac{k\pi}{n}\right)+1) + B(\cos\left(\frac{(k-1)\pi}{n}\right)+1)$. If we can make $A+B=0$ and $2\sin\left(\frac{(2k-1)\pi}{2n}\right) = A\cos\left(\frac{k\pi}{n}\right) - A\cos\left(\frac{(k-1)\pi}{n}\right)$, then we should be good. Using difference to sum formulas we obtain $\cos\left(\frac{k\pi}{n}\right) - \cos\left(\frac{(k-1)\pi}{n}\right) = -2\sin\left(\frac{\pi}{2n}\right)\sin\left(\frac{(2k-1)\pi}{2n}\right)$. Putting these together we obtain the following solution:

Solution. The answer is $\boxed{\dfrac{8}{\pi^3}.}$

First we will show that

$$\frac{\sin\left(\frac{(2k-1)\pi}{2n}\right)}{\cos^2\left(\frac{(k-1)\pi}{2n}\right)\cos^2\left(\frac{k\pi}{2n}\right)} = \frac{1}{\sin\left(\frac{\pi}{2n}\right)}\left(\frac{1}{\cos^2\left(\frac{k\pi}{2n}\right)} - \frac{1}{\cos^2\left(\frac{(k-1)\pi}{2n}\right)}\right).$$

For simplicity let $\theta = \frac{\pi}{2n}$. This is equivalent to $2\sin((2k-1)\theta)\sin\theta = 2\cos^2((k-1)\theta) - 2\cos^2(k\theta)$. Applying the half angle formula, we see the right hand side is $\cos(2(k-1)\theta) - \cos(2k\theta)$. Using the product to sum formulas we see the left hand side equals $\cos((2k-1)\theta - \theta) - \cos((2k-1)\theta + \theta)$. This proves the claim.

Therefore,

$$a_n = \frac{1}{\sin\left(\frac{\pi}{2n}\right)} \sum_{k=1}^{n-1}\left(\frac{1}{\cos^2\left(\frac{k\pi}{2n}\right)} - \frac{1}{\cos^2\left(\frac{(k-1)\pi}{2n}\right)}\right) = \frac{1}{\sin\left(\frac{\pi}{2n}\right)}\left(\frac{1}{\cos^2\left(\frac{(n-1)\pi}{2n}\right)} - \frac{1}{\cos^2\left(\frac{(1-1)\pi}{2n}\right)}\right).$$

This can be simplified to

$$a_n = \frac{1}{\sin\left(\frac{\pi}{2n}\right)}\left(\frac{1-\cos^2\left(\frac{(n-1)\pi}{2n}\right)}{\cos^2\left(\frac{(n-1)\pi}{2n}\right)}\right) = \frac{\sin^2\left(\frac{(n-1)\pi}{2n}\right)}{\sin\left(\frac{\pi}{2n}\right)\cos^2\left(\frac{(n-1)\pi}{2n}\right)}.$$

Since $\frac{\pi}{2n}+\frac{(n-1)\pi}{2n}=\frac{\pi}{2}$, we have $\cos\left(\frac{(n-1)\pi}{2n}\right)=\sin\left(\frac{\pi}{2n}\right)$, and thus

$$a_n=\frac{\sin^2\left(\frac{(n-1)\pi}{2n}\right)}{\sin^3\left(\frac{\pi}{2n}\right)}.$$

As $n\to\infty$, the fraction $\frac{(n-1)\pi}{2n}$ approaches $\pi/2$ and thus, the numerator of a_n approaches 1. Thus,

$$\lim_{n\to\infty}\frac{a_n}{n^3}=\lim_{n\to\infty}\frac{1}{n^3\sin^3(\frac{\pi}{2n})}=\lim_{n\to\infty}\left(\frac{\frac{\pi}{2n}}{\sin\left(\frac{\pi}{2n}\right)}\right)^3\cdot\frac{8}{\pi^3}=\frac{8}{\pi^3}.$$

□

Example 10.7 (IMC 2018, Problem 7)**.** *Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers such that $a_0=0$ and $a_{n+1}^3=a_n^2-8$ for $n=0,1,2,\ldots$. Prove that the following series is convergent:*

$$\sum_{n=0}^{\infty}|a_{n+1}-a_n|.$$

Scratch: Here are my initial thoughts:

- We evaluate the first few terms. They do get complicated quickly, though, but we do observe that all terms are negative.
- We will use the information above to find some bounds for the sequence.
- We notice that each term is evaluated by applying a certain function to the previous term. This reminds us of the Fixed Point Theorem.
- We employ a proof similar to that of the Fixed Point Theorem.

Solution. We will prove by induction that $-2\le a_n<-1$ for all $n\ge 1$.
Basis step: $a_1=\sqrt[3]{a_0^2-8}=-2$, which proves the basis step.
Inductive step: Suppose $-2\le a_n<-1$. We have $1-8<a_n^2-8\le 4-8$, and thus

$$-2<\sqrt[3]{-7}<a_{n+1}\le\sqrt[3]{-4}<-1,$$

as desired.
Claim: For every $n\ge 1$, $|a_{n+1}-a_n|\le\left(\dfrac{2}{3\sqrt[3]{2}}\right)^{n-1}|a_2-a_1|$ Let $f(x)=\sqrt[3]{x^2-3}$.
Basis step: $|a_2-a_1|\le\left(\dfrac{2}{3\sqrt[3]{2}}\right)^{1-1}|a_2-a_1|$ is trivial.
Inductive step: Suppose $|a_{n+1}-a_n|\le\left(\dfrac{2}{3\sqrt[3]{2}}\right)^{n-1}|a_2-a_1|$. The recursion can be written as $a_{n+1}=f(a_n)$. We see that $|a_{n+2}-a_{n+1}|=|f(a_{n+1})-f(a_n)|=|f'(c)||a_{n+1}-a_n|$, for some c between a_n and a_{n+1} by the Mean Value Theorem. $|f'(c)|=\left|\dfrac{2c}{3\sqrt[3]{(c^2-8)^2}}\right|\le\dfrac{4}{3\sqrt[3]{(4-8)^2}}=\dfrac{2}{3\sqrt[3]{2}}$, since c is between -2 and -1. Therefore, $|a_{n+2}-a_{n+1}|\le\dfrac{2}{3\sqrt[3]{2}}|a_{n+1}-a_n|\le\left(\dfrac{2}{3\sqrt[3]{2}}\right)^{n}|a_2-a_1|$, as desired.
Since $\dfrac{2}{3\sqrt[3]{2}}<1$, by Comparison test $\sum|a_{n+1}-a_n|$ converges. □

Example 10.8 (IMC 2019, Problem 1)**.** *Evaluate the product*

$$\prod_{n=3}^{\infty}\frac{(n^3+3n)^2}{n^6-64}.$$

Scratch: Here are my first thoughts:

- Typically we need to telescope infinite products like this.
- I see a bunch of factorizations.
- Evaluating the first few terms could help.

Factoring we get

$$\frac{(n^3+3n)^2}{n^6-64}=\frac{n^2(n^2+3)^2}{(n-2)(n+2)(n^2+2n+4)(n^2-2n+4)}.$$

We will separately list the sequences that appear in the numerator and denominator:

$$\begin{aligned}
&n: 3,4,5,6,\dots\\
&n-2: 1,2,3,4,\dots\\
&n+2: 5,6,7,8,\dots\\
&n^2+3: 12,19,28,39,\dots\\
&n^2-2n+4: 7,12,19,28,\dots\\
&n^2+2n+4: 19,28,39,52,\dots
\end{aligned}$$

The first three sequences and the last three sequences seem to be the same sequences. This leads us to the following solution:

Solution. The answer is $\boxed{\frac{72}{7}}$.

For every $k \geq 3$ we have

$$\begin{aligned}
\prod_{n=3}^{k}\frac{(n^3+3n)^2}{n^6-64} &= \prod_{n=3}^{k}\frac{n^2(n^2+3)^2}{(n^3-8)(n^3+8)}\\
&= \prod_{n=3}^{k}\frac{n^2}{(n-2)(n+2)}\frac{(n^2+3)}{(n^2-2n+4)(n^2+2n+4)}\\
&= \left(\textstyle\prod_{n=3}^{k}\frac{n}{(n-2)}\right)\left(\textstyle\prod_{n=3}^{k}\frac{n}{n+2}\right)\left(\textstyle\prod_{n=3}^{k}\frac{n^2+3}{(n-1)^2+3}\right)\left(\textstyle\prod_{n=3}^{k}\frac{n^2+3}{(n+1)^2+3}\right)\\
&= \frac{k(k-1)}{1\cdot 2}\frac{3\cdot 4}{(k+1)(k+2)}\frac{k^2+3}{2^2+3}\frac{3^2+3}{(k+1)^2+3}\\
&= \frac{72}{7}\cdot\frac{(1-\frac{1}{k})}{(1+\frac{1}{k})(1+\frac{2}{k})}\cdot\frac{1+\frac{3}{k^2}}{(1+\frac{1}{k})^2+\frac{3}{k^2}}
\end{aligned}$$

Since $\lim\limits_{k\to\infty}\frac{1}{k}=0$, the answer is $72/7$, as desired. $\square$

Example 10.9 (IMC 2019, Problem 7)**.** *Let $C = \{4, 6, 8, 9, 10, \dots\}$ be the set of composite positive integers. For each $n \in C$ let a_n be the smallest positive integer k such that $k!$ is divisible by n. Determine whether the following series converges:*

$$\sum_{n \in C} \left(\frac{a_n}{n}\right)^n.$$

Scratch: Here are my initial thoughts.

- The exponent of n suggests using the Root test.
- Why did they exclude primes? I see! That is because if n is prime then $a_n = n$, which makes the series clearly divergent.
- When n is not prime, then $n = ab$, which means $b!$ usually has both a factor of a and b. OK. I think I know how to do the problem now.

Solution. The series converges.

We will prove that for every $n \in C$, $\frac{a_n}{n} \le \frac{2}{3}$, unless $n = 4$. Suppose $a_n = rs$, for two integers r, s with $1 < r \le s$. We will take two cases:

Case I: $r < s$. Then $s! = 1 \cdots r \cdots s$, which is divisible by $rs = n$. Thus $a_n \le s$, which implies $\frac{a_n}{n} \le \frac{1}{r} \le \frac{2}{3}$.
Case II: $r = s > 2$, then $(2s)! = 1 \cdots s \cdots (2s)$ which is divisible by $s^2 = n$. Thus, $a_n \le 2s$, and thus $\frac{a_n}{n} \le \frac{2}{s} \le \frac{2}{3}$, as desired.

Therefore, $\left(\frac{a_n}{n}\right)^n \le \left(\frac{2}{3}\right)^n$ for all $n \in C$, with $n > 4$. Since $\frac{2}{3} < 1$, the series $\sum (2/3)^n$ converges and thus by comparison test, the desired series also converges. $\square$

Example 10.10 (Putnam 2015, B4)**.** *Let T be the set of all triples (a, b, c) of positive integers for which there exist triangles with side lengths a, b, c. Express*

$$\sum_{(a,b,c) \in T} \frac{2^a}{3^b 5^c}$$

as a rational number in lowest terms.

Solution. (Video Solution) The answer is $\frac{17}{21}$.
Note that for a, b, c to form sides of a triangle, we need $|b - c| < a < b + c$. So, we can write down the given sum as a triple sum:

$$\sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \sum_{a=|b-c|+1}^{b+c} \frac{2^a}{3^b 5^c} = \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \frac{1}{3^b 5^c} \left(\frac{2^{|b-c|+1} - 2^{b+c}}{1 - 2}\right) = \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \frac{1}{3^b 5^c} \left(2^{b+c} - 2^{|b-c|+1}\right)$$

We can break the above sum into two sums. The first of which can be evaluated as follows:

$$\sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \frac{2^{b+c}}{3^b 5^c} = \left(\sum_{b=1}^{\infty} \left(\frac{2}{3}\right)^b\right) \left(\sum_{c=1}^{\infty} \left(\frac{2}{5}\right)^c\right) = \frac{2/3}{1 - 2/3} \frac{2/5}{1 - 2/5} = \frac{4}{3}.$$

We will now consider the cases where $b \geq c$ and where $b < c$ separately. The first case yields the sum

$$\sum_{c=1}^{\infty}\sum_{b=c}^{\infty}\frac{2^{b-c+1}}{3^b5^c} = \sum_{c=1}^{\infty}\left(\frac{2}{3^c5^c(1-\frac{2}{3})}\right) = 3\left(\frac{\frac{2}{15}}{1-\frac{1}{15}}\right) = \frac{3}{7}.$$

When $b < c$ we obtain the following double sum:

$$\sum_{b=1}^{\infty}\sum_{c=b+1}^{\infty}\frac{2^{c-b+1}}{3^b5^c} = \sum_{b=1}^{\infty}\frac{4}{3^b5^{b+1}(1-2/5)} = \frac{20}{3}\frac{1/75}{1-1/15} = \frac{20}{3}\cdot\frac{1}{70} = \frac{2}{21}.$$

Therefore, the answer is

$$\frac{4}{3}-\frac{3}{7}-\frac{2}{21} = \frac{17}{21}$$

□

Example 10.11 (Putnam 1976, B5). *Evaluate the following sum in closed form:*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)^n.$$

Scratch. First, we find the sum for small values of n.

$$n=1 \Rightarrow x-(x-1)=1$$
$$n=2 \Rightarrow x^2-2(x-1)^2+(x-2)^2=2$$
$$n=3 \Rightarrow x^3-3(x-1)^3+3(x-2)^3-(x-3)^3=6$$
$$n=4 \Rightarrow x^4-4(x-1)^4+6(x-2)^4-4(x-3)^4+(x-4)^4=24$$

At this point it is clear that the answer is very likely $n!$ But, how do we prove it? Induction would be an obvious choice. Let's do it!

Solution 1.(Video Solution) We will prove by induction on n that the answer is $n!$.
Let $p_n(x) = \sum\limits_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)^n$.
Basis step. $p_1(x) = x-(x-1) = 1!$

Inductive step. Suppose $p_n(x) = n!$ for all x.

$$\begin{aligned} p_{n+1}(x) &= \sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}(x-k)^{n+1} \\ &= x\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}(x-k)^n - \sum_{k=0}^{n+1}(-1)^k k\binom{n+1}{k}(x-k)^n \qquad (*) \end{aligned}$$

Using the identity $\binom{n+1}{k} = \binom{n}{k}+\binom{n}{k-1}$, the first sum can be evaluated as:

$$\begin{aligned} x\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}(x-k)^n &= x\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)^n + \sum_{k=1}^{n+1}(-1)^k\binom{n}{k-1}(x-k)^n\right) \\ &= x(p_n(x)-p_n(x-1)) = 0, \end{aligned}$$

where we used the inductive hypothesis in the last step.

We will simplify the second sum using the following identity:

$$k\binom{n+1}{k} = \frac{(n+1)!}{(k-1)!(n+1-k)!} = (n+1)\binom{n}{k-1}.$$

$$\sum_{k=0}^{n+1}(-1)^k k\binom{n+1}{k}(x-k)^n = \sum_{k=1}^{n+1}(-1)^k(n+1)\binom{n}{k-1}(x-k)^n = -(n+1)p_n(x-1).$$

Combining that with $(*)$ we conclude $p_{n+1}(x) = (n+1)p_n(x) = (n+1)!$, as desired. $\square$

Solution 2. (Video Solution) Let $p(x)$ be the given polynomial. We have,

$$p(x) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{j=0}^{n}\binom{n}{j}x^j(-k)^{n-j} = \sum_{j=0}^{n}\binom{n}{j}x^j(-1)^{n-j}\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^{n-j}$$

For every $0 \le j \le n$, we will count the number of surjective functions $f:\{1,2,\dots,n-j\}\to\{1,2,\dots,n\}$. Note that if $j>0$, then the size of the domain is larger than the size of the co-domain, which implies there are no such surjective functions. If $j=0$, then there are $n!$ such surjective functions. Now, we will count this using the Principal of Inclusion-Exclusion.

The number of all functions $f:\{1,2,\dots,n-j\}\to\{1,2,\dots,n\}$ is n^{n-j} since each element in the domain can be mapped to n elements. If the number 1 is not in the range of a function, then each element in the domain can be mapped into $n-1$ elements. Therefore, there are $(n-1)^{n-j}$ functions where 1 is not in the range. Similar argument works for $2,3,\dots,n$. If 1 and 2 are not in the range of a function, then each element in the domain can be mapped to $n-2$ elements. Thus, there are $(n-2)^{n-j}$ such functions. Repeating this and using PIE, we obtain the following sum:

$$\sum_{i=0}^{n}(-1)^k\binom{n}{k}(n-k)^{n-j} = \sum_{i=0}^{n}(-1)^{n-k}\binom{n}{n-k}k^{n-j} = (-1)^n\sum_{i=0}^{n}(-1)^k\binom{n}{k}k^{n-j}$$

Comparing what we found here and what we have above, we conclude:

$$(-1)^n\sum_{i=0}^{n}(-1)^k\binom{n}{k}k^{n-j} = \begin{cases} 0 & \text{if } j=1,2,\dots,n \\ n! & \text{if } j=n \end{cases}$$

Therefore, $p(x) = \binom{n}{0}x^0(-1)^n(-1)^n n! = n!$, as desired. $\square$

Example 10.12 (AMC 12A, 2018, Problem 19)**.** *Evaluate the infinite sum*

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{12}+\frac{1}{15}+\frac{1}{16}+\frac{1}{18}+\frac{1}{20}+\cdots$$

of the reciprocals of positive integers that have no prime factors other than 2, 3*, or* 5.

Scratch: The denominators look like $2^a3^b5^c$, where a,b,c are nonnegative integers. Setting $a = 0,1,2,\ldots$, we get the following sums:

$$\begin{aligned}
a = 0 &\Rightarrow \sum_{b,c\ge 0} \frac{1}{3^b5^c}\\
a = 1 &\Rightarrow \sum_{b,c\ge 0} \frac{1}{2\cdot 3^b5^c} = \frac{1}{2}\sum_{b,c\ge 0}\frac{1}{3^b5^c}\\
a = 2 &\Rightarrow \sum_{b,c\ge 0} \frac{1}{4\cdot 3^b5^c} = \frac{1}{4}\sum_{b,c\ge 0}\frac{1}{3^b5^c}\\
&\vdots
\end{aligned}$$

Since the term $\sum_{b,c\ge 0} \frac{1}{3^b5^c}$ is common in all of the above sums, we can rewite the sum as following:

$$\left(\sum_{a=0}^{\infty}\frac{1}{2^a}\right)\left(\sum_{b,c\ge 0}\frac{1}{3^b5^c}\right)$$

Similar to above, the second sum can be written as:

$$\left(\sum_{b=0}^{\infty}\frac{1}{3^b}\right)\left(\sum_{c=0}^{\infty}\frac{1}{5^c}\right)$$

This yields the following solution:

Solution. (Video Solution) The denominator of each term of the given sum can be written as $2^a3^b5^c$ where a,b,c are nonnegative integers. Therefore, each term has a unique representation as $\frac{1}{2^a}\frac{1}{3^b}\frac{1}{5^c}$. This means the given sum is equal to

$$\left(\sum_{a=0}^{\infty}\frac{1}{2^a}\right)\left(\sum_{b=0}^{\infty}\frac{1}{3^b}\right)\left(\sum_{c=0}^{\infty}\frac{1}{5^c}\right)$$

Using the geometric series sum we conclude the answer is $\frac{1}{1-\frac{1}{2}}\frac{1}{1-\frac{1}{3}}\frac{1}{1-\frac{1}{5}} = 2\cdot\frac{3}{2}\cdot\frac{5}{4} = \boxed{\frac{15}{4}}$. □

Example 10.13 (VTRMC 2007)**.** *Find the exact values of*

(a) $\frac{1}{1!}+\frac{2}{3!}+\frac{3}{5!}+\cdots+\frac{n}{(2n-1)!}+\cdots$ *and*

(b) $\frac{1}{3!}+\frac{2}{5!}+\frac{3}{7!}+\cdots+\frac{n}{(2n+1)!}+\cdots$

Scratch: The first sum reminds us of the Taylor series expansion for e^x. We know $e^x = 1+x+\frac{x^2}{2!}+\cdots$. Substituting $x=\pm 1$, we obtain $e^{\pm 1} = 1\pm\frac{1}{1!}+\frac{1}{2!}\pm\cdots$. The denominators are not quite what we would want, since the given sum is missing all the even numbered terms. Let us combine the consecutive terms to see what we get:

$$1+\frac{1}{1!} = 2$$

$$\frac{1}{2!}+\frac{1}{3!} = \frac{4}{3!}$$

$$\frac{1}{4!}+\frac{1}{5!} = \frac{6}{5!}$$

$$\frac{1}{6!}+\frac{1}{7!} = \frac{8}{7!}$$

These seem to generate double the given sum. So, we obtain the following solution:

Solution. (Video Solution) (a) By Taylor series for e^x we know $e = \sum_{n=0}^{\infty} \frac{1}{n!}$. For every $n \geq 0$ we have

$$\frac{1}{(2n)!} + \frac{1}{(2n+1)!} = \frac{2n+1+1}{(2n+1)!} = \frac{2n+2}{(2n+1)!} \Rightarrow e = \sum_{n=0}^{\infty} \frac{2(n+1)}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{2n}{(2n-1)!}.$$

Therefore, the answer to part (a) is $\boxed{\frac{e}{2}}$.
(b) Note that using the Taylor series for e^x we have

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{1}{(2n)!} - \frac{1}{(2n+1)!} \right) = \sum_{n=0}^{\infty} \frac{2n+1-1}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{2n}{(2n+1)!}$$

Therefore, the given sum is $\boxed{\frac{1}{2e}}$. □

Example 10.14 (Putnam 1999, A4)**.** *Evaluate*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n3^m + m3^n)}.$$

Scratch:

Solution. (Video Solution) Let S be the given sum. Swapping n and m and using the symmetry in the sums we obtain the following:

$$S = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{m^2 n}{3^m (n3^m + m3^n)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n^2 m}{3^n (m3^n + n3^m)}.$$

Adding up the above sum to the original sum we obtain

$$2S = \sum_{m,n=1}^{\infty} \left(\frac{m^2 n}{3^m (n3^m + m3^n)} + \frac{n^2 m}{3^n (m3^n + n3^m)} \right) = \sum_{m,n=1}^{\infty} \frac{m^2 n3^n + n^2 m3^m}{3^{n+m} (m3^n + n3^m)} = \sum_{m,n=1}^{\infty} \frac{mn(m3^n + n3^m)}{3^{n+m} (m3^n + n3^m)} = \sum_{m,n=1}^{\infty} \frac{mn}{3^{m+n}}$$

Writing the above as a double sum and factoring the terms that are independent of the variable n we obtain the following:

$$\sum_{m,n=1}^{\infty} \frac{mn}{3^{m+n}} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{mn}{3^{m+n}} = \sum_{m=1}^{\infty} \frac{m}{3^m} \sum_{n=1}^{\infty} \frac{n}{3^n} = \left(\sum_{n=1}^{\infty} \frac{n}{3^n} \right)^2.$$

The sum $\sum_{n=1}^{\infty} \frac{n}{3^n}$ has been evaluated in Example 10.3 as $\frac{3}{4}$. The answer is $\boxed{\frac{9}{32}}$. □

Example 10.15. *Evaluate the infinite sum*

$$\frac{1}{\binom{3}{3}} + \frac{1}{\binom{4}{3}} + \frac{1}{\binom{5}{3}} + \cdots.$$

Solution.(Video Solution) We will use partial fractions in order to obtain a telescoping sum.

$$\frac{1}{\binom{n}{3}} = \frac{6}{n(n-1)(n-2)} = \frac{A}{n} + \frac{B}{n-1} + \frac{C}{n-2}$$

$$6 = A(n-1)(n-2) + Bn(n-2) + Cn(n-1)$$

$$n = 0 \Rightarrow 6 = A(-1)(-2) \Rightarrow A = 3.$$

$$n = 1 \Rightarrow 6 = B(1)(1-2) \Rightarrow B = -6$$

$$n = 2 \Rightarrow 6 = C(2)(2-1) \Rightarrow C = 3$$

Therefore,

$$\begin{aligned}
\sum_{n=3}^{m} \frac{1}{\binom{n}{3}} &= \sum_{n=3}^{m} \frac{3}{n} - \frac{6}{n-1} + \frac{3}{n-2} \\
&= \sum_{n=3}^{m} \frac{3}{n} - \sum_{n=3}^{m} \frac{6}{n-1} + \sum_{n=3}^{m} \frac{3}{n-2} \\
&= \sum_{n=3}^{m} \frac{3}{n} - \sum_{n=2}^{m-1} \frac{6}{n} + \sum_{n=1}^{m-2} \frac{3}{n} \\
&= \frac{3}{m-1} + \frac{3}{m} - \frac{6}{2} - \frac{6}{m-1} + \frac{3}{1.} + \frac{3}{2}
\end{aligned}$$

Taking the limit as $m \to \infty$ we obtain $\boxed{\frac{3}{2}}$. □

Example 10.16. *Evaluate the sum:*

$$\cot^2\left(\frac{\pi}{2m+1}\right) + \cot^2\left(\frac{2\pi}{2m+1}\right) + \cdots + \cot^2\left(\frac{m\pi}{2m+1}\right).$$

Solution. (Video Solution)

□

Example 10.17. *Use the previous example to prove:*

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots = \frac{\pi^2}{6}$$

Solution. (Video Solution)

□

Example 10.18. *Determine if the series converges:*

$$\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1).$$

Solution. (Video Solution) Using difference of n-th powers we obtain

$$\sqrt[n]{2}-1=\frac{(\sqrt[n]{2}-1)(\sqrt[n]{2^{n-1}}+\sqrt[n]{2^{n-2}}+\cdots+1)}{\sqrt[n]{2^{n-1}}+\sqrt[n]{2^{n-2}}+\cdots+1}=\frac{2-1}{\sqrt[n]{2^{n-1}}+\sqrt[n]{2^{n-2}}+\cdots+1}>\frac{1}{2n}.$$

Since each term $\sqrt[n]{2^k}$ is less than 2. Therefore, by the Comparison Test, the given series diverges. □

Example 10.19. *Let S be the set of all positive integers that have no digit 7 when written in base 10. Prove that the following series converges:*

$$\sum_{n\in S}\frac{1}{n}.$$

Solution. (Video Solution)

□

Example 10.20 (A π-Series)**.** *Let S be the set consisting of all positive integers for which the ten-digit block* 3141592653 *appears somewhere in their base 10 representation. For example* 3141592653 *and* 1314159265389 *are elements of S, but* 1 *and* 31415929653 *are not in S. Does the series* $\sum_{n\in S}\frac{1}{n}$ *converge?*

Solution. (Video Solution) □

Example 10.21 (Putnam 1960, B2)**.** *Evaluate* $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2^{-3m-n-(m+n)^2}$.

Solution. (Video Solution)

□

Example 10.22. *Evaluate the sum:*

$$\sum_{r=0}^{n}\sum_{k=0}^{r}(-1)^k(k+1)(k+2)\binom{n+5}{r-k}.$$

Solution. (Video Solution) □

10.5 General Strategies

For convergence of series:

- Estimate the general term of the series and get a feeling of what the answer might be.
- You need to now compare the series to one of the known series or use one of the tests mentioned above.

For evaluating series there are typically four different techniques:

- Relate the series to a known series such as:
  - a geometric series, or an arithmetic sum.

- a Taylor series of a known function ($e^x, \ln(1-x), \sin x, \cos x, (1+x)^\alpha$.)

- Write the series $\sum a_n$ as a telescoping sum. For that write $a_n = b_n - b_{n-1}$ and see if there is a sequence b_n satisfying this.

- Write the series as a double sum and swap the order of summation. This is similar to what you might have seen in multivariable calculus. Some examples of swapping the summations are as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(n,m) \text{ and } \sum_{n=1}^{\infty}\sum_{m=n}^{\infty} f(m,n) = \sum_{m=1}^{\infty}\sum_{n=1}^{m} f(m,n)$$

- Find a polynomial that resembles the product.

- Use Two-Way Counting.

10.6 Exercises

Exercise 10.1 (VTRMC 1982)**.** For $n \geq 2$, define S_n by $S_n = \sum\limits_{k=n}^{\infty} \dfrac{1}{k^2}$.

(a) Prove or disprove that $1/n < S_n < 1/(n-1)$.

(b) Prove or disprove that $S_n < 1/(n-3/4)$.

Exercise 10.2 (VTRMC 1989)**.** Let g be defined on $(1,\infty)$ by $g(x) = x/(x-1)$, and let $f^k(x)$ be defined by $f^0(x) = x$ and for $k > 0, f^k(x) = g\left(f^{k-1}(x)\right)$. Evaluate $\sum\limits_{k=0}^{\infty} 2^{-k} f^k(x)$ in the form $\frac{ax^2+bx+c}{dx+e}$

Exercise 10.3 (VTRMC 1990)**.** Determine all real values of p for which the following series converge.

(a) $\sum_{n=1}^{\infty} \left(\sin \frac{1}{n}\right)^p$

(b) $\sum_{n=1}^{\infty} |\sin n|^p$

Exercise 10.4 (VTRMC 1992)**.** Let $f_n(x)$ be defined recursively by $f_0(x) = x, f_1(x) = f(x), f_{n+1}(x) = f(f_n(x))$, for $n \geq 0$, where $f(x) = 1 + \sin(x-1)$.

(i) Show that there is a unique point x_0 such that $f_2(x_0) = x_0$.

(ii) Find $\sum\limits_{n=0}^{\infty} \dfrac{f_n(x_0)}{3^n}$ with the above x_0.

Exercise 10.5 (VTRMC 1993)**.** Find $\sum\limits_{n=1}^{\infty} \dfrac{3^{-n}}{n}$.

Exercise 10.6 (VTRMC 1994)**.** Let a sequence $\{x_n\}_{n=0}^{\infty}$ of rational numbers be defined by $x_0 = 10, x_1 = 29$ and $x_{n+2} = \frac{19x_{n+1}}{94x_n}$ for $n \geq 0$. Find $\sum_{n=0}^{\infty} \frac{x_{6n}}{2^n}$.

Exercise 10.7 (Putnam 1994, A1)**.** Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Exercise 10.8 (Putnam 1997, B1)**.** Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$F_n = \sum_{m=1}^{6n-1} \min(\{\frac{m}{6n}\}, \{\frac{m}{3n}\}).$$

(Here $\min(a,b)$ denotes the minimum of a and b.)

Exercise 10.9 (VTRMC 1998)**.** Let a_n be sequence of positive numbers ($n = 1,2,\ldots,a_n \neq 0$ for all n), and let $b_n = (a_1 + \cdots + a_n)/n$, the average of the first n numbers of the sequence. Suppose $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is a convergent series. Prove that $\sum_{n=1}^{\infty} \frac{1}{b_n}$ is also a convergent series.

Exercise 10.10 (Putnam 1998, B4)**.** Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$

Exercise 10.11 (Putnam 2000, B3)**.** Let $f(t) = \sum_{j=1}^{N} a_j \sin(2\pi jt)$, where each a_j is real and a_N is not equal to 0. Let N_k denote the number of zeroes (including multiplicities) of $\frac{d^k f}{dt^k}$ that lie in the interval $[0,1)$. Prove that

$$N_0 \leq N_1 \leq N_2 \leq \cdots, \text{ and } \lim_{k\to\infty} N_k = 2N.$$

Exercise 10.12 (Putnam 2001, B3)**.** For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

Exercise 10.13 (VTRMC 2002)**.** Let $\{a_n\}_{n\geq 1}$ be an infinite sequence with $a_n \geq 0$ for all n. For $n \geq 1$, let b_n denote the geometric mean of $a_1,\ldots,a_n$, that is $(a_1 \ldots a_n)^{1/n}$. Suppose $\sum_{n=1}^{\infty} a_n$ is convergent. Prove that $\sum_{n=1}^{\infty} b_n^2$ is also convergent.

Exercise 10.14 (Putnam 2002, A6)**.** Fix an integer $b \geq 2$. Let $f(1) = 1$, $f(2) = 2$, and for each $n \geq 3$, define $f(n) = nf(d)$, where d is the number of base-b digits of n. For which values of b does

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?

Exercise 10.15 (VTRMC 2003)**.** Find $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = \frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \cdots$ for $|x| < 1$.

Exercise 10.16 (Putnam 2004, B5)**.** Evaluate

$$\lim_{x\to 1^-} \prod_{n=0}^{\infty} \left(\frac{1+x^{n+1}}{1+x^n}\right)^{x^n}.$$

Exercise 10.17 (VTRMC 2004)**.** Let $\{a_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$. Prove that $\sum_{n=1}^{\infty} \left|1 - \frac{a_{n+1}}{a_n}\right|$ is divergent.

Exercise 10.18 (Putnam 2005, B6)**.** Let S_n denote the set of all permutations of the numbers $1,2,\ldots,n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if π is an even permutation and $\sigma(\pi) = -1$ if π is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of π. Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi)+1} = (-1)^{n+1} \frac{n}{n+1}.$$

Exercise 10.19 (VTRMC 2006)**.** Let $\{a_n\}$ be a monotonic decreasing sequence of positive real numbers with limit 0 (so $a_1 \geq a_2 \geq \cdots \geq 0$). Let $\{b_n\}$ be a rearrangement of the sequence such that for every non-negative integer m, the terms $b_{3m+1}, b_{3m+2}, b_{3m+3}$ are a rearrangement of the terms $a_{3m+1}, a_{3m+2}, a_{3m+3}$ (e.g. the first 6 terms of the sequence $\{b_n\}$ could be $a_3, a_2, a_1, a_4, a_6, a_5$.) Prove or give a counterexample to the following statement: the series $\sum_{n=1}^{\infty}(-1)^n b_n$ is convergent.

Exercise 10.20 (Putnam 2006, A5)**.** Let n be a positive odd integer and let θ be a real number such that θ/π is irrational. Set $a_k = \tan(\theta + k\pi/n)$, $k = 1, 2, \ldots, n$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{a_1 a_2 \cdots a_n}$$

is an integer, and determine its value.

Example 10.23 (VTRMC 2007)**.** *Find the exact values of*

(a) $\frac{1}{1!} + \frac{2}{3!} + \frac{3}{5!} + \cdots + \frac{n}{(2n-1)!} + \cdots$ *and*

(b) $\frac{1}{3!} + \frac{2}{5!} + \frac{3}{7!} + \cdots + \frac{n}{(2n+1)!} + \cdots$

Exercise 10.21 (VTRMC 2007)**.** Determine whether the series $\sum_{n=2}^{\infty} n^{-\left(1+(\ln(\ln n))^{-2}\right)}$ is convergent or divergent (ln denotes natural log).

Exercise 10.22 (VTRMC 2010)**.** Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms (so $a_i > 0$ for all i) and set $b_n = \frac{1}{n a_n^2}$ for $n \geq 1$. Prove that $\sum_{n=1}^{\infty} \frac{n}{b_1 + b_2 + \cdots + b_n}$ is convergent.

Exercise 10.23 (Putnam 2010, B1)**.** Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer m?

Exercise 10.24 (VTRMC 2011)**.** Find $\sum_{k=1}^{\infty} \frac{k^2-2}{(k+2)!}$.

Exercise 10.25 (Putnam 2011, A2)**.** Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1} a_n - 2$ for $n = 2, 3, \ldots$. Assume that the sequence (b_j) is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \ldots a_n}$$

converges, and evaluate S.

Exercise 10.26 (Putnam 2011, B5)**.** Let $a_1, a_2, \ldots$ be real numbers. Suppose that there is a constant A such that for all n,

$$\int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} \frac{1}{1 + (x - a_i)^2} \right)^2 dx \leq An.$$

Prove there is a constant $B > 0$ such that for all n,

$$\sum_{i,j=1}^{n} (1 + (a_i - a_j)^2) \geq Bn^3.$$

Exercise 10.27 (VTRMC 2013)**.** Find $\sum_{n=1}^{\infty} \frac{n}{(2^n+2^{-n})^2} + \frac{(-1)^n n}{(2^n-2^{-n})^2}$.

Exercise 10.28 (VTRMC 2014)**.** Find $\sum_{n=2}^{n=\infty} \frac{n^2-2n-4}{n^4+4n^2+16}$.

Exercise 10.29 (Putnam 2014, A3)**.** Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute

$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k}\right)$$

in closed form.

Exercise 10.30 (Putnam 2015, B6)**.** For each positive integer k, let $A(k)$ be the number of odd divisors of k in the interval $[1, \sqrt{2k})$. Evaluate

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{A(k)}{k}.$$

Exercise 10.31 (VTRMC 2016)**.** Determine all the numbers k such that $\sum_{n=1}^{\infty} \left(\frac{(2n)!}{4^n n! n!}\right)^k$ is convergent.

Exercise 10.32 (Putnam 2016, B1)**.** Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.

Exercise 10.33 (Putnam 2016, B6)**.** Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n+1}.$$

Exercise 10.34. Let a_n be a sequence of positive real numbers for which $\sum_{n=1}^{\infty} a_n$ diverges. Must the series $\sum_{n=1}^{\infty} \frac{a_n}{2020+a_n}$ diverge?

Exercise 10.35 (IMC 2016, Problem 6)**.** Let $(x_1, x_2, \ldots)$ be a sequence of positive real numbers satisfying $\sum_{n=1}^{\infty} \frac{x_n}{2n-1} =$ 1. Prove that

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{x_n}{k^2} \leq 2.$$

Exercise 10.36. Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{(7n+32)3^n}{(n^2+2n)4^n}.$$

Exercise 10.37. Evaluate

$$\sum_{k=1}^{n} \frac{1}{(k+1)\sqrt{k}+k\sqrt{k+1}}.$$

Exercise 10.38 (Putnam 2017, B3)**.** Suppose that $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1. Show that if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

Exercise 10.39 (Putnam 2017, B4)**.** Evaluate the sum

$$\sum_{k=0}^{\infty}\left(3\cdot\frac{\ln(4k+2)}{4k+2}-\frac{\ln(4k+3)}{4k+3}-\frac{\ln(4k+4)}{4k+4}-\frac{\ln(4k+5)}{4k+5}\right)$$
$$=3\cdot\frac{\ln 2}{2}-\frac{\ln 3}{3}-\frac{\ln 4}{4}-\frac{\ln 5}{5}+3\cdot\frac{\ln 6}{6}-\frac{\ln 7}{7}$$
$$-\frac{\ln 8}{8}-\frac{\ln 9}{9}+3\cdot\frac{\ln 10}{10}-\cdots.$$

(As usual, $\ln x$ denotes the natural logarithm of x.)

Exercise 10.40 (VTRMC 2019)**.** Let S denote the positive integers that have no 0 in their decimal expansion. Determine whether $\sum_{n\in S} n^{-99/100}$ is convergent.

Exercise 10.41 (Putnam 2020, A2)**.** Let k be a nonnegative integer. Evaluate

$$\sum_{j=0}^{k} 2^{k-j}\binom{k+j}{j}.$$

Exercise 10.42 (Putnam 2020, A3)**.** Let $a_0=\pi/2$, and let $a_n=\sin(a_{n-1})$ for $n\geq 1$. Determine whether

$$\sum_{n=1}^{\infty} a_n^2$$

converges.

Exercise 10.43 (VTRMC 2022)**.** Calculate the exact value of the series $\sum_{n=2}^{\infty}\log(n^3+1)-\log(n^3-1)$ and provide justification.

Exercise 10.44. Determine all constants p for which the following series converges:

$$\sum_{n=1}^{\infty}(\sqrt[n]{2}-1)^p.$$

Video Solution

Exercise 10.45. Let S be the set of all positive integers that have no digit 7 when written in base 10. Find all constants p for which the following series converges:

$$\sum_{n\in S}\frac{1}{n^p}.$$

Chapter 11

Polynomials

11.1 Basics

11.2 Important Theorems

Theorem 11.1 (Division Algorithm)**.** *Let F be a field. For every two polynomials $f(x), g(x) \in F[x]$ with $g(x) \neq 0$, there are unique polynomials $q(x), r(x) \in F[x]$ satisfying both of the following conditions:*

- $f(x) = g(x)q(x) + r(x)$*, and*
- $r(x) = 0$ *or* $\deg r(x) < \deg g(x)$.

Theorem 11.2 (Factor Theorem)**.** *Let F be a field and $f(x) \in F[x]$. Suppose $r_1, r_2, \ldots, r_n \in F$ are n distinct roots of $f(x)$. Then, there is a polynomial $g(x) \in F[x]$ for which*

$$f(x) = (x - r_1)(x - r_2) \cdots (x - r_n) g(x).$$

Theorem 11.3 (Rational Root Theorem)**.** *Consider the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ of degree n. If r is a rational root of the equation $f(x) = 0$, then $r = \dfrac{p}{q}$, for some integers p, q, where $p \mid a_0$ and $q \mid a_n$.*

11.3 Classical Examples

Example 11.1. *(Symmetric Polynomials) Solve the following equations:*

(a) $x^4 - 2x^3 - x^2 - 2x + 1 = 0.$

(b) $x^6 - 15x^4 + 20x^3 - 30x^2 + 8 = 0.$

Solution. (Video Solution) (a) Since the coefficients are symmetric we can

□

11.4 Further Examples

Example 11.2 (Putnam 1992, B4). *Let $p(x)$ be a nonzero polynomial of degree less than 1992 having no nonconstant factor in common with x^3-x. Let*

$$\frac{d^{1992}}{dx^{1992}}\left(\frac{p(x)}{x^3-x}\right)=\frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$.

Scratch: Here is my first thoughts: We could break up the denominator into linear factors and use Partial fractions. That makes evaluation of the derivative easier.

Solution. The answer is $\boxed{3984}$.

First we will prove the following claim by induction on n:

Claim: For every constant c, the n-th derivative of $\dfrac{1}{x-c}$ is $\dfrac{(-1)^n n!}{(x-c)^{n+1}}$.

Basis step: The first derivative of $(x-c)^{-1}$ is $-(x-c)^{-2}=\dfrac{(-1)^1 1!}{(x-c)^2}$.

Inductive step: The derivative of $\dfrac{(-1)^n n!}{(x-c)^{n+1}}=(-1)^n n!(x-c)^{-(n+1)}$ is $(-1)^n n!(-(n+1))(x-c)^{-(n+1)-1}=\dfrac{(-1)^{n+1}(n+1)!}{(x-c)^{n+2}}$, which completes the proof of the claim.

Note that by the method of partial fractions we see that $\dfrac{1}{x^3-x}=\dfrac{-1}{x}+\dfrac{0.5}{x-1}+\dfrac{0.5}{x+1}$. Multiplying by $p(x)$ and then dividing $p(x)$ by $x, x-1$, and $x+1$, and noting that $p(0), p(1)$, and $p(-1)$ are all nonzero, we obtain

$$\frac{p(x)}{x^3-x}=q(x)+\frac{a}{x}+\frac{b}{x-1}+\frac{c}{x+1},$$

where $q(x)$ is a polynomial whose degree is less than the degree of $p(x)$. Thus the 1992nd derivative of $q(x)$ is zero. By the claim above, the 1992nd derivative of $\dfrac{p(x)}{x^3-x}$ becomes

$$\frac{a1992!}{x^{1993}}+\frac{b1992!}{(x-1)^{1993}}+\frac{c1992!}{(x+1)^{1993}}=1992!\cdot\frac{a(x^2-1)^{1993}+bx^{1993}(x+1)^{1993}+cx^{1993}(x-1)^{1993}}{(x^3-x)^{1993}}.$$

Since $abc\neq 0$ the numerator is nonzero at $x=0, x=1$, and $x=-1$. Therefore, the fraction is in reduced form. We will show the degree of the numerator is no less than 3984.

The coefficient of x^{3986} in the numerator is $a+b+c$. The coefficient of x^{3985} is $1993b-1993c$, and the coefficient of x^{3984} is $-1993a+\binom{1993}{2}b+\binom{1993}{2}$. If the degree of the numerator were less than 3984, then

$$a+b+c=1993b-1993c=-1993a+\binom{1993}{2}b+\binom{1993}{2}c=0,$$

which implies $b=c$, $a+2b=0$, and $-a+1992b=0$. This implies $b=0$, which is a contradiction. Therefore, the degree of $f(x)$ is not less than 3984. If we take $p(x)=3x^2-2$, then $p(0)=-2$, $p(1)=p(-1)=1$. Thus, $a+2b=0$ and $b=c$, which means the degree of $f(x)$ is at most 3984. This completes the proof. □

Example 11.3 (Putnam 2019, B5). *Let F_m be the m-th Fibonacci number, defined by $F_1=F_2=1$ and $F_m=F_{m-1}+F_{m-2}$ for all $m\geq 3$. Let $p(x)$ be the polynomial of degree* 1008 *such that $p(2n+1)=F_{2n+1}$ for $n=0,1,2,\dots,1008$. Find integers j and k such that $p(2019)=F_j-F_k$.*

Scratch: Let's start listing a few ideas that come to mind:

- This is a recursion, so induction might help.
- Instead of solving the problem for 2019, how about testing some small cases?
- Since the value of this polynomial is given at 1009 points and the degree is 1008, the polynomial can be uniquely determined.
- We know an explicit formula for the Fibonacci sequence. Could that help?

At this point it is unclear how any of these ideas might help, but we can certainly check a few examples to see if there is a pattern.

$\deg p = 0$ gives us $p(1) = F_1 = 1$, and thus $p(3) = 1 = F_3 - F_1 = F_3 - F_2 = F_4 - F_3$. This unfortunately has multiple possible values, but that could be an exception.

$\deg p = 1$ gives us $p(1) = F_1 = 1$, and $p(3) = F_3 = 2$, which gives us $p(x) = \dfrac{x+1}{2}$, and thus $p(5) = 3 = F_5 - F_3$. It is still too early to see a pattern.

$\deg p = 2$ gives us $p(1) = 1, p(3) = 2, p(5) = 5$. Finding $p(x)$ is fairly computational. One might ask if there is a way to find $p(7)$ *without* finding $p(x)$ first! In fact there is. This can be obtained using the method of finite differences: $p(1) - 3p(3) + 3p(5) - p(7) = 0$, which implies $p(7) = 10 = F_7 - F_4$. You may see a pattern at this point, but to be sure, let's find one more term.

$\deg p = 3$ gives us $p(1) = 1, p(3) = 2, p(5) = 5, p(7) = 13$. Similar to what we did above we obtain $p(1) - 4p(3) + 6p(5) - 4p(7) + p(9) = 0$, which implies $p(9) = 29 = F_9 - F_5$. Well, this seems to fit into the above pattern really well. So, we conjecture that the answer must be $F_{2m+1} - F_{m+1}$, where $\deg p = m - 1$.

Now, how do we prove this? Following what we did above we need to prove the following identity:

$$\binom{m}{0}F_1 - \binom{m}{1}F_3 + \binom{m}{2}F_5 - \cdots + (-1)^{m-1}\binom{m}{m-1}F_{2m-1} + (-1)^m(F_{2m+1} - F_{m+1}) = 0.$$

Now, recall that we know a relatively simple formula for F_n. In fact we know $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$, where $\alpha > \beta$ are roots of $x^2 - x - 1 = 0$. Substituting that and using the Binomial Theorem should complete the solution. So, let's now write the solution.

Solution. We will prove

$$\boxed{p(2019) = F_{2019} - F_{1010}}$$

For simplicity set $m = 1009$. Since the degree of $p(x)$ is $m - 1$, by finite differences we know

$$\binom{m}{0}p(1) - \binom{m}{1}p(3) + \cdots + (-1)^{m-1}\binom{m}{m-1}p(2m-1) + (-1)^m\binom{m}{m}p(2m+1) = 0. \qquad (*)$$

Note that $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$, where $\alpha > \beta$ are roots of $x^2 - x - 1 = 0$. Using that we obtain

$$\begin{aligned}
\sum_{j=0}^{m}(-1)^j\tbinom{m}{j}F_{2j+1} &= \frac{1}{\sqrt{5}}\sum_{j=0}^{m}(-1)^j\tbinom{m}{j}(\alpha^{2j+1} - \beta^{2j+1}) \\
&= \frac{1}{\sqrt{5}}(\alpha(1-\alpha^2)^m - \beta(1-\beta^2)^m) \\
&= \frac{1}{\sqrt{5}}(-1)^m(\alpha^{m+1} - \beta^{m+1}) = (-1)^m F_{m+1}
\end{aligned}$$

This implies $\sum_{j=0}^{m-1}\binom{m}{j}(-1)^j p(2j+1)+(-1)^m F_{2m+1}=(-1)^m F_{m+1}$. Comparing this and $(*)$ yields $p(2m+1)=F_{2m+1}-F_{m+1}$, as desired. $\square$

If you do not remember the finite difference formula that I used above, you could still do the problem by using induction. Note that if $\deg p=m-1$, then $p(x+2)-p(x)$ would have degree $m-2$. Also, $F_{2n+1}-F_{2n-1}=F_{2n}$. Repeating this again we obtain the polynomial $p(x+4)-2p(x+2)+p(x)$ whose values are $F_{2n+2}-F_{2n}=F_{2n+1}$, and then we can use induction to prove our claim.

Example 11.4 (Putnam 2022, A2)**.** *Let n be an integer with $n\geq 2$. Over all real polynomials $p(x)$ of degree n, what is the largest possible number of negative coefficients of $p(x)^2$?*

Scratch: First, we will try some examples. For $n=2$, we have $p(x)=a_2x^2+a_1x+a_0$. The coefficients of $p(x)^2$ are as follows:

$$a_2^2, 2a_2a_1, 2a_2a_0+a_1^2, 2a_1a_0, a_0^2$$

The first and last coefficients are never negative. Let's see if we can make sure all the other coefficients are negative. $2a_2a_1<0$ implies a_2 and a_1 have different signs. So, for simplicity let's assume $a_2>0$. Thus, $a_1<0$. The inequality $2a_2a_0+a_1^2<0$ implies $a_2a_0<0$, which implies $a_0<0$. However, $2a_1a_0$ is positive, since both a_1 and a_0 are both negative. So, we can make at most 2 of the terms negative.

For $n=3$, we have $p(x)=a_3x^3+a_2x^2+a_1x+a_0$. Assume $a_3>0$. Similar to above, the first and last coefficients are squares and thus cannot be negative. The rest of the coefficients are as follows:

$$2a_32a_2, 2a_3a_1+a_2^2, 2a_3a_0+2a_2a_1, 2a_2a_0+a_1^2, 2a_1a_0$$

For $2a_32a_2$ to be negative, we need $a_2<0$. The inequality $2a_3a_1+a_2^2<0$ implies $a_3a_1<0$ and thus $a_1<0$. The inequality $2a_3a_0+2a_2a_1<0$ implies $a_0<0$. However, this means the last two coefficients are not negative. But, if we start from the other side and assume $a_0>0$, we can make sure more terms are negative. In other words, make sure a_3 and a_0 are large positive numbers, and a_1, a_2 are negative. That way, all coefficients except for the middle one are negative. So, the answer in this case is 4.

In general, there are $2n+1$ coefficients in $p(x)^2$. The first and last coefficients are not negative. From the remaining $2n-1$ coefficients one has to be positive or zero. Thus, the answer seems to be $2n-2$.

Solution.(Video Solution) The answer is $2n-2$.

Let $p(x)=\sum_{k=0}^{n}a_kx^k$. Note that since $p(x)^2=(-p(x))^2$, we may assume the leading coefficient of $p(x)$ is positive. The leading coefficient and the constant term of $p(x)^2$ are a_n^2 and a_0^2 which are not negative. We will also show the remaining $2n-1$ coefficients cannot all be negative. Assume on the contrary they are all negative. We will show that $a_k<0$ for $k=0,\dots,n-1$. The coefficient of x^{2n+1} in $p(x)^2$ is $2a_na_{n-1}$. For this to be negative we need $a_{n-1}<0$. Suppose $a_{n-1}, a_{n-2},\dots,a_i$ are all negative. For the coefficient of x^{n+i-1} which is $a_na_{i-1}+a_{n-1}a_i+\cdots$, to be negative we need $a_n a_{i-1}$ to be negative. This is because each of the remaining products in this coefficient is positive. Thus, a_{i-1} is negative. This completes the proof by induction. Now, consider the coefficient of x. This coefficient is $2a_1a_0$ which is positive. Therefore, there is at least one of the remaining coefficients that is not negative. So, there is no more than $2n-1$ negative coefficients when $p(x)^2$ is expanded.

It is left to give an example of a polynomial $p(x)$ where there are precisely $2n-2$ negative coefficients in $p(x)^2$.

Consider a positive constant c and let $a_n = a_0 = c$ and $a_1 = \cdots = a_{n-1} = -1$. Let $p(x) = \sum_{k=0}^{n} a_k x^k$. For $\ell < n$, a positive integer, the coefficient of x^ℓ in $p(x)^2$ is

$$a_\ell a_0 + a_{\ell-1}a_1 + \cdots + a_0 a_\ell = -c + \underbrace{1 + \cdots + 1}_{\ell-1 \text{ times}} - c = -2c + \ell - 1 < -2c + n.$$

So, if we make sure $c > n/2$ all of these coefficients are negative. Also, note that by symmetry of the coefficients of $p(x)$, the coefficient of $x^{2n-\ell}$ and the coefficient of x^ℓ in $p(x)^2$ are the same. Therefore, the coefficient of x^m in $p(x)^2$ is negative for $m = 1, \ldots, n-1, n+1, \ldots, 2n-1$. Thus, $p(x)^2$ has $2n-2$ negative coefficients. $\square$

Example 11.5 (IMO 2019, Shortlisted Problem, A5)**.** *Let $x_1, \ldots, x_n$ be distinct real numbers. Prove that*

$$\sum_{k=1}^{n} \prod_{j \neq k} \frac{1 - x_k x_j}{x_k - x_j} = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

Solution.(Video Solution)

$\square$

Example 11.6 (IMO 2019, Shortlisted Problem, A6)**.** *A polynomial of three variables $P(x,y,z)$ with real coefficients satisfies the identities:*

$$P(x,y,z) = P(yz - x, y, z) = P(x, xz - y, z) = P(x, y, xy - z).$$

Prove that there exists a polynomial $F(t)$ in one variable for which

$$P(x,y,z) = F(x^2 + y^2 + z^2 - xyz).$$

Solution.(Video Solution)

$\square$

Example 11.7 (IMC 2023, Problem 3)**.** *Find all polynomials P in two variables with real coefficients satisfying the identity*

$$P(x,y)P(z,t) = P(xz - yt, xt + yz).$$

Solution.(Video Solution)

$\square$

11.5 General Strategies

11.6 Exercises

Exercise 11.1 (VTRMC 1979)**.** Let S be a finite set of polynomials in two variables, x and y. For n a positive integer, define $\Omega_n(S)$ to be the collection of all expressions $p_1p_2\dots p_k$, where $p_i \in S$ and $1 \le k \le n$. Let $d_n(S)$ indicate the maximum number of linearly independent polynomials in $\Omega_n(S)$. For example, $\Omega_2(\{x^2,y\}) = \{x^2,y,x^2y,x^4,y^2\}$ and $d_2(\{x^2,y\}) = 5$.

(a) Find $d_2(\{1,x,x+1,y\})$.

(b) Find a closed formula in n for $d_n(\{1,x,y\})$.

(c) Calculate the least upper bound over all such sets of $\overline{\lim_{n\to\infty}} \dfrac{\log d_n(S)}{\log n}$.
($\overline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty}(\sup\{a_n,a_{n+1},\dots\})$, where sup means supremum or least upper bound.)

Exercise 11.2 (VTRMC 1982)**.** Let $p(x)$ be a polynomial of the form $p(x) = ax^2+bx+c$, where a,b and c are integers, with the property that $1 < p(1) < p(p(1)) < p(p(p(1)))$. Show that $a \ge 0$.

Exercise 11.3 (VTRMC 1983, Modified)**.** Suppose a and b are real numbers for which the equation $x^4+ax+b=0$ only has real roots. Prove $a=b=0$.

Exercise 11.4 (VTRMC 1985)**.** Let $p(x) = a_0+a_1x+\cdots+a_nx^n$, where the coefficients a_i are real. Prove that $p(x)=0$ has at least one root in the interval $0 \le x \le 1$ if $a_0+a_1/2+\cdots+a_n/(n+1)=0$.

Exercise 11.5 (VTRMC 1987)**.** Let $p(x)$ be given by $p(x) = a_0+a_1x+a_2x^2+\cdots+a_nx^n$ and let $|p(x)| \le |x|$ on $[-1,1]$.

(a) Evaluate a_0.

(b) Prove that $|a_1| \le 1$.

Exercise 11.6 (VTRMC 1988)**.** Find positive real numbers a and b such that $f(x) = ax-bx^3$ has four extrema on $[-1,1]$, at each of which $|f(x)|=1$.

Exercise 11.7 (Putnam 1990, B5)**.** Is there an infinite sequence $a_0,a_1,a_2,\dots$ of nonzero real numbers such that for $n \ge 1$ the polynomial

$$p_n(x) = a_0+a_1x+a_2x^2+\cdots+a_nx^n$$

has exactly n distinct real roots.

Exercise 11.8 (VTRMC 1990)**.** Suppose that $P(x)$ is a polynomial of degree 3 with integer coefficients and that $P(1)=0, P(2)=0$. Prove that at least one of its four coefficients is equal to or less than -2.

Exercise 11.9 (VTRMC 1991)**.** Let $f(x) = x^5-5x^3+4x$. In each part (i)-(iv), prove or disprove that there exists a real number c for which $f(x)-c=0$ has a root of multiplicity (i) one, (ii) two, (iii) three, (iv) four.

Exercise 11.10 (VTRMC 1991)**.** Prove that if α is a real root of $\left(1-x^2\right)\left(1+x+x^2+\cdots+x^n\right)-x=0$ which lies in $(0,1)$, with $n=1,2,\dots$, then α is also a root of $\left(1-x^2\right)\left(1+x+x^2+\cdots+x^{n+1}\right)-1=0$

Exercise 11.11 (Putnam 1991, A3)**.** Find all real polynomials $p(x)$ of degree $n \geq 2$ for which there exist real numbers $r_1 < r_2 < \cdots < r_n$ such that

(a) $p(r_i) = 0, \qquad i = 1, 2, \ldots, n$, and

(b) $p'\left(\frac{r_i + r_{i+1}}{2}\right) = 0 \qquad i = 1, 2, \ldots, n-1$,

where $p'(x)$ denotes the derivative of $p(x)$.

Exercise 11.12 (VTRMC 1992)**.** Let $p(x)$ be the polynomial $p(x) = x^3 + ax^2 + bx + c$. Show that if $p(r) = 0$ then

$$\frac{p(x)}{x-r} - 2\frac{p(x+1)}{x+1-r} + \frac{p(x+2)}{x+2-r} = 2,$$

for all x except $x = r, r-1$ and $r-2$.

Exercise 11.13 (VTRMC 1994)**.** Consider the polynomial equation $ax^4 + bx^3 + x^2 + bx + a = 0$, where a and b are real numbers, and $a > 1/2$. Find the maximum possible value of $a + b$ for which there is at least one positive real root of the above equation.

Exercise 11.14 (VTRMC 1995)**.** Let $n \geq 2$ be a positive integer and let $f(x)$ be the polynomial

$$1 - \left(x + x^2 + \cdots + x^n\right) + \left(x + x^2 + \cdots + x^n\right)^2 - \cdots + (-1)^n \left(x + x^2 + \cdots + x^n\right)^n.$$

If r is an integer such that $2 \leq r \leq n$, show that the coefficient of x^r in $f(x)$ is zero.

Exercise 11.15 (VTRMC 1996)**.** Let $a_i, i = 1, 2, 3, 4$, be real numbers such that $a_1 + a_2 + a_3 + a_4 = 0$. Show that for arbitrary real numbers $b_i, i = 1, 2, 3$, the equation

$$a_1 + b_1 x + 3a_2 x^2 + b_2 x^3 + 5a_3 x^4 + b_3 x^5 + 7a_4 x^6 = 0$$

has at least one real root which is on the interval $-1 \leq x \leq 1$.

Exercise 11.16 (VTRMC 1997)**.** Suppose that $r_1 \neq r_2$ and $r_1 r_2 = 2$. If r_1 and r_2 are roots of

$$x^4 - x^3 + ax^2 - 8x - 8 = 0,$$

find r_1, r_2 and a. (Do not assume that they are real numbers.)

Exercise 11.17 (Putnam 1997, B4)**.** Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1 + x + x^2)^m$. Prove that for every integer $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor \frac{2k}{3} \rfloor} (-1)^i a_{k-i,i} \leq 1.$$

Exercise 11.18 (Putnam 1999, A1)**.** Find polynomials $f(x)$,$g(x)$, and $h(x)$, if they exist, such that for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

Exercise 11.19 (Putnam 1999, A2)**.** Let $p(x)$ be a polynomial that is nonnegative for all real x. Prove that for some k, there are polynomials $f_1(x),\ldots,f_k(x)$ such that

$$p(x)=\sum_{j=1}^{k}(f_j(x))^2.$$

Exercise 11.20 (Putnam 1999, A5)**.** Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)|\le C\int_{-1}^{1}|p(x)|\,dx.$$

Exercise 11.21 (Putnam 1999, B2)**.** Let $P(x)$ be a polynomial of degree n such that $P(x)=Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots.

Exercise 11.22 (Putnam 2001, A3)**.** For each integer m, consider the polynomial

$$P_m(x)=x^4-(2m+4)x^2+(m-2)^2.$$

For what values of m is $P_m(x)$ the product of two non-constant polynomials with integer coefficients?

Exercise 11.23 (Putnam 2002, A1)**.** Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

Exercise 11.24 (Putnam 2003, A4)**.** Suppose that a,b,c,A,B,C are real numbers, $a\neq 0$ and $A\neq 0$, such that

$$|ax^2+bx+c|\le|Ax^2+Bx+C|$$

for all real numbers x. Show that

$$|b^2-4ac|\le|B^2-4AC|.$$

Exercise 11.25 (Putnam 2003, B4)**.** Let $f(z)=az^4+bz^3+cz^2+dz+e=a(z-r_1)(z-r_2)(z-r_3)(z-r_4)$ where a,b,c,d,e are integers, $a\neq 0$. Show that if r_1+r_2 is a rational number and $r_1+r_2\neq r_3+r_4$, then r_1r_2 is a rational number.

Exercise 11.26 (Putnam 2003, B1)**.** Do there exist polynomials $a(x),b(x),c(y),d(y)$ such that

$$1+xy+x^2y^2=a(x)c(y)+b(x)d(y)$$

holds identically?

Exercise 11.27 (Putnam 2004, B1)**.** Let $P(x)=c_nx^n+c_{n-1}x^{n-1}+\cdots+c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r)=0$. Show that the n numbers

$$c_nr,\ c_nr^2+c_{n-1}r,\ c_nr^3+c_{n-1}r^2+c_{n-2}r,$$
$$\ldots,\ c_nr^n+c_{n-1}r^{n-1}+\cdots+c_1r$$

are integers.

Exercise 11.28 (Putnam 2004, A4)**.** Show that for any positive integer n there is an integer N such that the product $x_1x_2\cdots x_n$ can be expressed identically in the form

$$x_1x_2\cdots x_n = \sum_{i=1}^{N} c_i(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)^n$$

where the c_i are rational numbers and each a_{ij} is one of the numbers $-1,0,1$.

Exercise 11.29 (Putnam 2005, B2)**.** Find a nonzero polynomial $P(x,y)$ such that $P(\lfloor a\rfloor, \lfloor 2a\rfloor) = 0$ for all real numbers a. (Note: $\lfloor \nu \rfloor$ is the greatest integer less than or equal to ν.)

Exercise 11.30 (Putnam 2007, A4)**.** A *repunit* is a positive integer whose digits in base 10 are all ones. Find all polynomials f with real coefficients such that if n is a repunit, then so is $f(n)$.

Exercise 11.31 (Putnam 2007, B1)**.** Let f be a non-constant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$.

Exercise 11.32 (Putnam 2007, B4)**.** Let n be a positive integer. Find the number of pairs P,Q of polynomials with real coefficients such that

$$(P(X))^2 + (Q(X))^2 = X^{2n} + 1$$

and $\deg P > \deg Q$.

Exercise 11.33 (Putnam 2008, A5)**.** Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1),g(1)),(f(2),g(2)),\ldots,(f(n),g(n))$ in $\mathbb{R}^2$ are the vertices of a regular n-gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n-1$.

Exercise 11.34 (Putnam 2009, B4)**.** Say that a polynomial with real coefficients in two variables, x,y, is *balanced* if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space V over $\mathbb{R}$. Find the dimension of V.

Exercise 11.35 (VTRMC 2010)**.** Prove that $\cos(\pi/7)$ is a root of the equation $8x^3 - 4x^2 - 4x + 1 = 0$, and find the other two roots.

Exercise 11.36 (VTRMC 2011)**.** Let $P(x) = x^{100} + 20x^{99} + 198x^{98} + a_{97}x^{97} + \cdots + a_1x + 1$ be a polynomial where the $a_i (1 \leq i \leq 97)$ are real numbers. Prove that the equation $P(x) = 0$ has at least one complex root (i.e. a root of the form $a+bi$ with a,b real numbers and $b \neq 0$).

Exercise 11.37 (Putnam 2011, B2)**.** Let S be the set of all ordered triples (p,q,r) of prime numbers for which at least one rational number x satisfies $px^2 + qx + r = 0$. Which primes appear in seven or more elements of S?

Exercise 11.38 (Putnam 2014, A5)**.** Let

$$P_n(x) = 1 + 2x + 3x^2 + \cdots + nx^{n-1}.$$

Prove that the polynomials $P_j(x)$ and $P_k(x)$ are relatively prime for all positive integers j and k with $j \neq k$.

Exercise 11.39 (Putnam 2014, B4)**.** Show that for each positive integer n, all the roots of the polynomial

$$\sum_{k=0}^{n} 2^{k(n-k)} x^k$$

are real numbers.

Exercise 11.40 (Putnam 2016, A1)**.** Find the smallest positive integer j such that for every polynomial $p(x)$ with integer coefficients and for every integer k, the integer

$$p^{(j)}(k) = \frac{d^j}{dx^j} p(x) \bigg|_{x=k}$$

(the j-th derivative of $p(x)$ at k) is divisible by 2016.

Exercise 11.41 (Putnam 2016, A6)**.** Find the smallest constant C such that for every real polynomial $P(x)$ of degree 3 that has a root in the interval $[0,1]$,

$$\int_0^1 |P(x)|\, dx \leq C \max_{x\in[0,1]} |P(x)|\,.$$

Exercise 11.42 (IMC 2017, Problem 7)**.** Let $p(x)$ be a nonconstant polynomial with real coefficients. For every positive integer n, let

$$q_n(x) = (x+1)^n p(x) + x^n p(x+1).$$

Prove that there are only finitely many numbers n such that all roots of $q_n(x)$ are real.

Exercise 11.43 (VTRMC 2017)**.** Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients such that $f(1) = -1, f(4) = 2$ and $f(8) = 34$. Suppose $n \in \mathbb{Z}$ is an integer such that $f(n) = n^2 - 4n - 18$. Determine all possible values for n.

Exercise 11.44 (Putnam 2017, A2)**.** Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

Exercise 11.45 (IMC 2018, Problem 9)**.** Determine all pairs $P(x), Q(x)$ of complex polynomials with leading coefficient 1 such that $P(x)$ divides $Q(x)^2 + 1$ and $Q(x)$ divides $P(x)^2 + 1$.

Exercise 11.46 (VTRMC 2019)**.** Let n be a nonnegative integer and let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ be a polynomial with real coefficients a_i. Suppose that

$$\frac{a_n}{(n+1)(n+2)} + \frac{a_{n-1}}{n(n+1)} + \cdots + \frac{a_1}{6} + \frac{a_0}{2} = 0.$$

Prove that $f(x)$ has a real zero.

Exercise 11.47 (IMC 2020, Problem 4)**.** A polynomial $p(x)$ with real coefficients satisfies the equation $p(x+1) - p(x) = x^{100}$ for all $x \in \mathbb{R}$. Prove that $p(1-t) \geq p(t)$ for $0 \leq t \leq 1/2$.

Exercise 11.48 (Putnam 2021, A6)**.** Let $P(x)$ be a polynomial whose coefficients are all either 0 or 1. Suppose that $P(x)$ can be written as a product of two nonconstant polynomials with integer coefficients. Does it follow that $P(2)$ is a composite integer?

Exercise 11.49 (Putnam 2023, A2)**.** Let n be an even positive integer. Let p be a monic, real polynomial of degree $2n$; that is to say, $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1x + a_0$ for some real coefficients $a_0, \ldots, a_{2n-1}$. Suppose that $p(1/k) = k^2$ for all integers k such that $1 \leq |k| \leq n$. Find all other real numbers x for which $p(1/x) = x^2$.

Exercise 11.50. Does there exist a sequence of real numbers a_n with $n \geq 0$ such that for every $n > 0$ the polynomial $p_n(x) = a_0 + a_1x + \cdots + a_nx^n$ has n simple real roots?

Exercise 11.51 (Putnam 2023, A5)**.** For a nonnegative integer k, let $f(k)$ be the number of ones in the base 3 representation of k. Find all complex numbers z such that

$$\sum_{k=0}^{3^{1010}-1} (-2)^{f(k)}(z+k)^{2023} = 0.$$

Chapter 12

Multi-variable Functions

12.1 Important Theorems

Theorem 12.1 (Fubini's Theorem)**.** *Let $f(x,y)$ be a function that is bounded on a bounded subset D of the xy-plane.*

(a) Suppose

$$D = \{(x,y) \mid a \le x \le b, \delta_1(x) \le y \le \delta_2(x)\},$$

where δ_1, δ_2 are continuous over $[a,b]$. Then $\iint\limits_D f \, \mathrm{d}A = \int_a^b \int_{\delta_1(x)}^{\delta_2(x)} f(x,y) \, \mathrm{d}y \, \mathrm{d}x.$

(b) Suppose

$$D = \{(x,y) \mid \gamma_1(y) \le x \le \gamma_2(y), c \le y \le d\},$$

where γ_1, γ_2 are continuous over $[c,d]$. Then $\iint\limits_D f \, \mathrm{d}A = \int_c^d \int_{\gamma_1(y)}^{\gamma_2(y)} f(x,y) \, \mathrm{d}x \, \mathrm{d}y.$

A similar result holds in $\mathbb{R}^3$.

Theorem 12.2 (Change of Coordinates)**.** *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function given by $T(u,v) = (x(u,v), y(u,v))$. Suppose D and $T(D)$ are regions in the uv- and xy-planes, respectively. Assume f is integrable over $T(D)$. Then*

$$\iint\limits_{T(D)} f(x,y) \, \mathrm{d}A = \iint\limits_D f(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \mathrm{d}A.$$

A similar result holds in $\mathbb{R}^3$.

To every point P in $\mathbb{R}^3$ we assign a triple (r, θ, z), called the **cylindrical coordinates of** P, where (r, θ) are the polar coordinates of the point (x,y). Similarly we assign a triple (ρ, φ, θ), called the **spherical coordinates of** P, where ρ is the distance to the origin, φ is the angle that the vector $\overrightarrow{OP}$ makes with the positive direction of the z-axis, and θ is the same angle as in the polar coordinates of (x,y). We have the following useful formulas:

$$x = r\cos\theta, y = r\sin\theta.$$

$$r = \rho\sin\varphi, \, x = \rho\sin\varphi\cos\theta, \, y = \rho\sin\varphi\sin\theta, \, z = \rho\cos\varphi.$$

$$\mathrm{d}x \, \mathrm{d}y = r \, \mathrm{d}r \, \mathrm{d}\theta, \text{ and } \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \rho^2 \sin\varphi \, \mathrm{d}\rho \, \mathrm{d}\varphi \, \mathrm{d}\theta.$$

Theorem 12.3 (Green's Theorem). *Let D be a closed bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple, closed, piecewise continuously differentiable curves. Suppose ∂D is oriented in such a way that D lies on the left as one traverses ∂D. Let $F = M\mathbf{i} + N\mathbf{j}$ be a continuously differentiable vector field on D. Then*

$$\int_{\partial D} M \,\mathrm{d}x + N \,\mathrm{d}y = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \,\mathrm{d}A$$

Theorem 12.4. *Let C be a simple closed curve in $\mathbb{R}^2$, and D be the closed region bounded by C. Suppose $M(x,y)$ and $N(x,y)$ are continuously differentiable functions over D for which $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 1$. Then*

$$\textit{Area of } D = \oint_C M \,\mathrm{d}x + N \,\mathrm{d}y,$$

where C oriented counterclockwise. In particular

$$\textit{Area of } D = \oint_C x \,\mathrm{d}y = -\oint_C y \,\mathrm{d}x = \frac{1}{2} \oint_C x \,\mathrm{d}y - y \,\mathrm{d}x.$$

Theorem 12.5 (Gauss' Theorem). *Let D be a bounded region whose boundary ∂D consists of finitely many piecewise smooth closed orientable surfaces, each of which is oriented by unit normal vectors away from D. Let $\mathbf{F}$ be a continuously differentiable vector field whose domain contains D. Then*

$$\iint_{\partial D} \mathbf{F} \cdot \,\mathrm{d}\mathbf{S} = \iiint_D \operatorname{div} \mathbf{F} \,\mathrm{d}V.$$

12.2 Classical Examples

Example 12.1 (Pascal's Identity). *Prove that for any two integers $0 < k \leq n$, we have $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.*

12.3 Further Examples

Example 12.2 (Putnam 2019, B4). *Let $\mathscr{F}$ be the set of functions $f(x,y)$ that are twice continuously differentiable for $x \geq 1$, $y \geq 1$ and that satisfy the following two equations (where subscripts denote partial derivatives):*

$$x f_x + y f_y = xy \ln(xy),$$
$$x^2 f_{xx} + y^2 f_{yy} = xy.$$

For each $f \in \mathscr{F}$, let

$$m(f) = \min_{s \geq 1} \left(f(s+1, s+1) - f(s+1, s) - f(s, s+1) + f(s,s) \right).$$

Determine $m(f)$, and show that it is independent of the choice of f.

Scratch: Here are some ideas:

- Can we find *any* function that satisfies the given system?
- Are we able to guess the minimum using this function?
- Can we find *all* such functions? This may be too ambitious but is worth having an eye on.

The first equality motivates $f = xy\ln(xy)$, but that does not work. However substituting we get $xf_x + yf_y = 2xy\ln(xy) + 2xy$, which is pretty close. At this point one would see that $f = xy(\ln(xy) - 1)/2$ is a solution to the first equation and fortunately it happens to satisfy the second equation as well. But is this the only solution? Clearly not: adding a constant would give us other solutions, but maybe all other solutions differ f only by a constant. Since the system is linear we only need to solve the homogeneous system $xf_x + yf_y = x^2 f_{xx} + y^2 f_{yy} = 0$. As seen below I tried proving this but then I realized that is not true. Instead I got something which was good enough.

Solution. The answer is $\boxed{2\ln 2 - \frac{1}{2}}$.

First we will prove the following claim:

Claim: If a twice continuously differentiable function $g(x,y)$ with $x, y \geq 1$ satisfies

$$xg_x + yg_y = x^2 g_{xx} + y^2 g_{yy} = 0 \qquad (*),$$

then $g = c\ln(x/y) + d$, where c and d are two constants.

Differentiating the first one with respect to both x and y gives us $xg_{xx} + g_x + yg_{xy} = xg_{xy} + yg_{yy} + g_y = 0$. Multiplying the first one by x and the second one by y and adding the two we obtain

$$x^2 g_{xx} + xg_x + xyg_{xy} + yxg_{xy} + y^2 g_{yy} + yg_y = 0$$

Using the assumption we obtain $g_{xy} = 0$. Thus, $g_x = h(x)$ is a function of x, and $g(x) = H(x) + G(y)$ for two differentiable functions G and H. Since $xg_x + yg_y = 0$, we obtain $xH'(x) + yG'(y) = 0$, for all $x, y \geq 1$, which implies $xH'(x) = -yG'(y)$. Since one side depends on x and the other depends on y, they must both be constants. Thus, $G(y) = -c\ln y + d_1$ and $H(x) = c\ln x + d_2$ for constants c, d_1, d_2. This means $g = c\ln(x/y) + d_1 + d_2$, as desired.

Note that $f_1(x,y) = \frac{1}{2}xy(\ln(xy) - 1)$ satisfies the given system. (You must fully write this up.) Assume f is another solution. Then since the system is linear $f - f_1$ satisfies $(*)$, and thus by the claim proved above $f - f_1 = c\ln(x/y) + d$ for some constants c, d.

Note that $c\ln((s+1)/(s+1)) + d - c\ln((s+1)/s) - d - c\ln(s/(s+1)) - d + c\ln(s/s) + d = 0$. Thus, $m(f) = m(f_1)$.

$$\begin{aligned}
&f_1(s+1,s+1) - f_1(s+1,s) - f_1(s,s+1) + f_1(s,s)\\
&= \tfrac{1}{2}((s+1)^2(\ln((s+1)^2) - 1) - 2s(s+1)(\ln(s(s+1)) - 1) + s^2(\ln(s^2) - 1))\\
&= \tfrac{1}{2}((2(s+1)^2 - 2s(s+1))\ln(s+1) + (-2s(s+1) + 2s^2)\ln s - (s+1)^2 + 2s(s+1) - s^2)\\
&= (s+1)\ln(s+1) - s\ln s - \tfrac{1}{2}
\end{aligned}$$

The derivative of this function is $1 + \ln(s+1) - 1 - \ln s = \ln(1 + 1/s)$ which is positive. So, the minimum is obtained for $s = 1$. This minimum is equal to $2\ln 2 - 0.5$, as desired. □

Example 12.3 (Putnam 2019, A4). *Let f be a continuous real-valued function on $\mathbb{R}^3$. Suppose that for every sphere S of radius 1, the integral of $f(x,y,z)$ over the surface of S equals 0. Must $f(x,y,z)$ be identically 0?*

Scratch: Here are a few thoughts about this problem:

- Okay, this is tough. We don't even know what the answer is. Maybe it is yes, which means we have to prove it, but maybe there is a counterexample. This makes the problem much more difficult.

- The problem is for $f(x,y,z)$. Can we solve it first for $f(x,y)$ or even $f(x)$? This is not necessarily useful as the answer may be different in different dimensions, but it is worth trying.

In 1 dimension, we need a function $f(x)$, with $f(x+1) = -f(x-1)$. In other words, we need $f(x+2) = -f(x)$. The function $f(x) = \sin(\pi x/2)$ is a good example. For a function of two variables $f(x,y)$, the x-values on a unit circle travel an interval of length 1 exactly twice, but that would be problematic, since the angle $\pi x/2$ could remain in the first and second quadrant and thus giving only nonnegative values. We could fix that by considering $\sin(\pi x)$ or $\sin(\pi y)$. This suggests the same example might work in dimension three. We will use $\sin(\pi z)$ since integrating that would be easier, $z = \rho\cos\phi$ is simpler than the formulas of x and y in spherical coordinates.

Solution. (Video Solution) The answer is no.

We will prove that the continuously differentiable function $f(x,y,z) = \sin(\pi z)$ satisfies the given conditions. Let S be a unit sphere centered at (x_0,y_0,z_0). Using a change of coordinates $(u,v,w) = (x-x_0, y-y_0, z-z_0)$, we need to find the integral of $\sin(\pi(w+z_0)) = \sin(\pi w)\cos(\pi z_0) + \cos(\pi w)\sin(\pi z_0)$ over the unit sphere $\rho = 1$. It is enough to show the integrals of $\sin(\pi w)$ and $\cos(\pi w)$ over S are zero. Since S is symmetric about the origin and sin is odd, the first integral is clearly zero. For the second one S can be parametrized with ϕ and θ.

$$\iint_S \cos(\pi w)\, dS = \int_0^{2\pi}\int_0^{\pi} \cos(\pi\cos\phi)\sin\phi\; d\phi d\theta$$

The inner integral is equal to $\dfrac{-\sin(\pi\cos\phi)}{\pi}\Big]_{\phi=0}^{\phi=\pi} = 0$, as desired. □

Example 12.4 (Putnam 2010, A3)**.** *Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation*

$$h(x,y) = a\frac{\partial h}{\partial x}(x,y) + b\frac{\partial h}{\partial y}(x,y)$$

for some constants a,b. Prove that if there is a constant M such that $|h(x,y)| \le M$ for all $(x,y) \in \mathbb{R}^2$, then h is identically zero.

Scratch: My first thought is: Can we solve a single variable version of this problem? Indeed, a natural single variable version of this problem can be stated as follows:

"Suppose $h : \mathbb{R} \to \mathbb{R}$ has continuous derivative and $h(x) = ah'(x)$ for some constant a. Assume there is a constant M for which $|h(x)| \le M$ for all $x \in \mathbb{R}$. Is it true that h must be identically zero?"

The differential equation $h(x) = ah'(x)$ is a linear one and can be solve by multiplying both sides by the integrating factor e^{-ax}. The solution to this equation is $h(x) = ce^{ax}$ for some constant c. Since $h(x)$ is bounded, either $c = 0$ or $a = 0$. In both cases h must be constant. Thus, $h'(x) = 0$ and hence $h(x) = 0$. This shows that the single variable version of this problem is in fact true!

We now realize we can turn h into a single variable function by restricting the domain of h to a path $(x(t), y(t))$. Substiuting this into the given equality we obtain

$$h(x(t), y(t)) = a\frac{\partial h}{\partial x}(x(t), y(t)) + b\frac{\partial h}{\partial y}(x(t), y(t)) \qquad (*)$$

We would like the left hand side to be a multiple of the derivative of $h(x(t), y(t))$ in order to be able to use the single variable version of the problem. Note that by the Chain Rule:

$$\frac{dh}{dt}(x(t), y(t)) = x'(t)\frac{\partial h}{\partial x}(x(t), y(t)) + y'(t)\frac{\partial h}{\partial y}(x(t), y(t)) \qquad (**)$$

Comparing $(*), (**)$ we would need to have $x'(t) = a$ and $y'(t) = b$, which means $x = at + x_0$ and $y = bt + y_0$. This yields the following solution.

Solution. (Video Solution) For a given (x_0, y_0), define $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = h(at + x_0, bt + y_0)$. By the Chain Rule and assumption:

$$g'(t) = ah_x + bh_y = g(t)$$

Therefore, $g(t) = ce^t$ for some constant c. Since $h(x, y)$ is bounded, so is g, and hence $c = 0$. Therefore,

$$g(t) = 0 \Rightarrow g(0) = 0 \Rightarrow h(x_0, y_0) = 0.$$

Since this is true for every $(x_0, y_0) \in \mathbb{R}^2$, the function h is identically zero. $\square$

12.4 Exercises

Exercise 12.1 (VTRMC 1993)**.** Prove that $\displaystyle\int_0^1 \int_{x^2}^1 e^{y^3/2} dydx = \frac{2e-2}{3}$.

Exercise 12.2 (Putnam 1993, B4)**.** The function $K(x, y)$ is positive and continuous for $0 \le x \le 1, 0 \le y \le 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \le x \le 1$. Suppose that for all x, $0 \le x \le 1$,

$$\int_0^1 f(y)K(x, y)\, dy = g(x)$$

and

$$\int_0^1 g(y)K(x, y)\, dy = f(x).$$

Show that $f(x) = g(x)$ for $0 \le x \le 1$.

Exercise 12.3 (VTRMC 1994)**.** Evaluate $\int_0^1 \int_0^x \int_0^{1-x^2} e^{(1-z)^2} dzdydx$.

Exercise 12.4 (VTRMC 1995)**.** Evaluate $\displaystyle\int_0^3 \int_0^2 \frac{1}{1 + (\max(3x, 2y))^2 dxdy}$.

Exercise 12.5 (VTRMC 1996)**.** Evaluate $\displaystyle\int_0^1 \int_{\sqrt{y-y^2}}^{\sqrt{1-y^2}} xe^{(x^4+2x^2y^2+y^4)}\, \mathrm{d}x\, \mathrm{d}y$.

Exercise 12.6 (VTRMC 1997)**.** Evaluate $\displaystyle\iint_D \frac{x^3}{x^2+y^2} dA$, where D is the half disk given by

$$(x-1)^2 + y^2 \le 1, y \ge 0.$$

Exercise 12.7 (VTRMC 1998)**.** Let $f(x,y) = \ln\left(1-x^2-y^2\right) - \frac{1}{(y-x)^2}$ with domain $D = \left\{(x,y) \mid x \neq y, x^2 + \ y^2 < 1\right\}$. Find the maximum value M of $f(x,y)$ over D. You have to show that $M \geq f(x,y)$ for every $(x,y) \in D$. Here $\ln(\cdot)$ is the natural logarithm function.

Exercise 12.8 (Putnam 1998, B3)**.** let H be the unit hemisphere $\{(x,y,z) : x^2+y^2+z^2 = 1, z \geq 0\}$, C the unit circle $\{(x,y,0) : x^2+y^2 = 1\}$, and P the regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A\sin\alpha + B\cos\beta$, where A, B, α, β are real numbers.

Exercise 12.9 (VTRMC 1999)**.** Let G be the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$, satisfying the following properties.

(i) $f(x) = f(x+1)$ for all x,

(ii) $\int_0^1 f(x)dx = 1999$.

Show that there is a number α such that $\alpha = \int_0^1 \int_0^x f(x+y)dydx$ for all $f \in G$.

Exercise 12.10 (Putnam 1999, B3)**.** Let $A = \{(x,y) : 0 \leq x, y < 1\}$. For $(x,y) \in A$, let

$$S(x,y) = \sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n,$$

where the sum ranges over all pairs (m,n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{(x,y)\to(1,1),(x,y)\in A} (1-xy^2)(1-x^2y)S(x,y).$$

Exercise 12.11 (VTRMC 2001)**.** Three infinitely long circular cylinders each with unit radius have their axes along the x, y and z-axes. Determine the volume of the region common to all three cylinders. (Thus one needs the volume common to $\{y^2+z^2 \leq 1\}$, $\{z^2+x^2 \leq 1\}$, $\{x^2+y^2 \leq 1\}$.)

Exercise 12.12 (Putnam 2003, B6)**.** Let $f(x)$ be a continuous real-valued function defined on the interval $[0,1]$. Show that

$$\int_0^1 \int_0^1 |f(x)+f(y)|\,dx\,dy \geq \int_0^1 |f(x)|\,dx.$$

Exercise 12.13. Is there a function $f : \mathbb{R}^2 \to \mathbb{R}$ for which $\displaystyle\int_0^1 \int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ exists but f is not integrable over $[0,1]\times[0,1]$?

Exercise 12.14. Given that $\displaystyle\int_0^1 \frac{\ln(1+x)}{x}\,\mathrm{d}x = \frac{\pi^2}{12}$, evaluate $\displaystyle\int_0^1 \int_0^y \frac{\ln(1+x)}{x}\,\mathrm{d}x\,\mathrm{d}y$.

Exercise 12.15. Let C be a given unit circle and D be a fixed diagonal of this circle. Find the area of the region consisting of all points inside or on the circle C that are closer to D than to the circumference of C.

Exercise 12.16. Let R be the region in $\mathbb{R}^3$ consisting of all triples (x,y,z) of nonnegative real numbers satisfying $x+y+z \leq 1$. Evaluate

$$\iiint_R xy^8z^9(1-x-y-z)^4\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Exercise 12.17. Let $x(t)$ and $y(t)$ be real-valued functions of the real variable t satisfying the differential equations

$$\begin{cases} \dfrac{dx}{dt} = -xt + 3yt - 2t^2 + 1 \\ \dfrac{dy}{dt} = xt + yt + 2t^2 - 1 \end{cases}$$

with the initial conditions $x(0) = y(0) = 1$. Find $x(1) + 3y(1)$.

Exercise 12.18. Let $f(x,y,z) = x^2 + y^2 + z^2 + xyz$. Let $p(x,y,z), q(x,y,z)$, and $r(x,y,z)$ be polynomials satisfying

$$f(p(x,y,z), q(x,y,z), r(x,y,z)) = f(x,y,z).$$

Prove or disprove: (p,q,r) consists of some permutation of $(\pm x, \pm y, \pm z)$, where the number of minus signs is even.

Exercise 12.19. Suppose

$$\frac{1}{1 - x - y - z - 6(xy + yz + zx)} = \sum_{r,s,t=0}^{\infty} f(r,s,t) x^r y^s z^t,$$

for when $|x|, |y|$, and $|z|$ are sufficiently small. Find the largest real number R for which the power series

$$F(u) = \sum_{n=0}^{\infty} f(n,n,n) u^n$$

converges for all u with $|u| < R$.

Exercise 12.20. Let a, b be positive real numbers. Evaluate $\int_0^a \int_0^b e^{\max(b^2x^2, a^2y^2)}\, \mathrm{d}y\, \mathrm{d}x$

Exercise 12.21 (Putnam 2005, B5)**.** Let $P(x_1, \ldots, x_n)$ denote a polynomial with real coefficients in the variables $x_1, \ldots, x_n$, and suppose that

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) P(x_1, \ldots, x_n) = 0 \quad \text{(identically)}$$

and that

$$x_1^2 + \cdots + x_n^2 \text{ divides } P(x_1, \ldots, x_n).$$

Show that $P = 0$ identically.

Exercise 12.22 (VTRMC 2006)**.** Three spheres each of unit radius have centers P, Q, R with the property that the center of each sphere lies on the surface of the other two spheres. Let C denote the cylinder with cross-section PQR (the triangular lamina with vertices P, Q, R) and axis perpendicular to PQR. Let M denote the space which is common to the three spheres and the cylinder C, and suppose the mass density of M at a given point is the distance of the point from PQR. Determine the mass of M.

Exercise 12.23 (Putnam 2006, A1)**.** Find the volume of the region of points (x,y,z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

Exercise 12.24 (VTRMC 2008)**.** Find the area of the region of points (x,y) in the xy-plane such that $x^4 + y^4 \leq x^2 - x^2y^2 + y^2$.

Exercise 12.25 (Putnam 2009, A6)**.** Let $f:[0,1]^2\to\mathbb{R}$ be a continuous function on the closed unit square such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on the interior $(0,1)^2$. Let $a=\int_0^1 f(0,y)\,dy$, $b=\int_0^1 f(1,y)\,dy$, $c=\int_0^1 f(x,0)\,dx$, $d=\int_0^1 f(x,1)\,dx$. Prove or disprove: There must be a point (x_0,y_0) in $(0,1)^2$ such that

$$\frac{\partial f}{\partial x}(x_0,y_0)=b-a \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0,y_0)=d-c.$$

Exercise 12.26 (Putnam 2011, A5)**.** Let $F:\mathbb{R}^2\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be twice continuously differentiable functions with the following properties:

- $F(u,u)=0$ for every $u\in\mathbb{R}$;
- for every $x\in\mathbb{R}$, $g(x)>0$ and $x^2g(x)\le 1$;
- for every $(u,v)\in\mathbb{R}^2$, the vector $\nabla F(u,v)$ is either $\mathbf{0}$ or parallel to the vector $\langle g(u),-g(v)\rangle$.

Prove that there exists a constant C such that for every $n\ge 2$ and any $x_1,\dots,x_{n+1}\in\mathbb{R}$, we have

$$\min_{i\neq j}|F(x_i,x_j)|\le\frac{C}{n}.$$

Exercise 12.27 (Putnam 2012, A6)**.** Let $f(x,y)$ be a continuous, real-valued function on $\mathbb{R}^2$. Suppose that, for every rectangular region R of area 1, the double integral of $f(x,y)$ over R equals 0. Must $f(x,y)$ be identically 0?

Exercise 12.28 (VTRMC 2015)**.** Let $(a_1,b_1),\dots,(a_n,b_n)$ be n points in $\mathbb{R}^2$ (where $\mathbb{R}$ denotes the real numbers), and let $\varepsilon>0$ be a positive number. Can we find a real-valued function $f(x,y)$ that satisfies the following three conditions?

(a) $f(0,0)=1$

(b) $f(x,y)\neq 0$ for only finitely many $(x,y)\in\mathbb{R}^2$;

(c) $\sum_{r=1}^{r=n}|f(x+a_r,y+b_r)-f(x,y)|<\varepsilon$ for every $(x,y)\in\mathbb{R}^2$.

Justify your answer.

Exercise 12.29 (Putnam 2018, B5)**.** Let $f=(f_1,f_2)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ with continuous partial derivatives $\frac{\partial f_i}{\partial x_j}$ that are positive everywhere. Suppose that

$$\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}-\frac{1}{4}\left(\frac{\partial f_1}{\partial x_2}+\frac{\partial f_2}{\partial x_1}\right)^2>0$$

everywhere. Prove that f is one-to-one.

Exercise 12.30 (Putnam 2021, A4)**.** Let

$$I(R)=\iint_{x^2+y^2\le R^2}\left(\frac{1+2x^2}{1+x^4+6x^2y^2+y^4}-\frac{1+y^2}{2+x^4+y^4}\right)dx\,dy.$$

Find

$$\lim_{R\to\infty}I(R),$$

or show that this limit does not exist.

Exercise 12.31 (Putnam 2021, B3)**.** Let $h(x,y)$ be a real-valued function that is twice continuously differentiable throughout $\mathbb{R}^2$, and define

$$\rho(x,y) = yh_x - xh_y.$$

Prove or disprove: For any positive constants d and r with $d > r$, there is a circle $\mathscr{S}$ of radius r whose center is a distance d away from the origin such that the integral of ρ over the interior of $\mathscr{S}$ is zero.

Chapter 13

Combinatorics

13.1 Basics

Definition 13.1. A **partially ordered set** (poset) is a set S along with a binary relation $\leq$ satisfying the following:

(a) $\leq$ is reflexive: $s \leq s$ for all $s \in S$.

(b) $\leq$ is transitive: If $s \leq t$ and $t \leq r$, then $s \leq r$.

(c) $\leq$ is anti-symmetric: If $s \leq t$ and $t \leq s$, then $s = t$.

Definition 13.2. A **chain** in a poset is a set of elements every two of which are comparable. In other words, a subset T for which for all $x, y \in T$ either $x \leq y$ or $y \leq x$.

Definition 13.3. An **anti-chain** in a poset is a set of elements no two of which are comparable.

13.2 Important Theorems

Theorem 13.1. *The number of subsets of* $\{1, 2, \ldots, n\}$ *is* 2^n.

Theorem 13.2 (Pigeonhole Principle)**.** *Let* $A_1, A_2, \ldots, A_n$ *be n sets and r be a positive integer such that*

$$|A_1 \cup A_2 \cup \cdots \cup A_n| > rn.$$

Then, there exists j for which $|A_j| \geq r + 1$.

Definition 13.4. Let A and B be two lattice points in the xy-plane. A **northeastern lattice path** from A to B is a list of lattice points $A = A_0, A_1, A_2, \ldots, A_n = B$ for which for each i, $A_{i+1} = A_i + (1,0)$ or $A_{i+1} = A_i + (0,1)$.

Definition 13.5. Let n be a non-negative integer. The number of northeastern lattice paths from $(0,0)$ to (n,n), for which no lattice point in the path is above the line $y = x$ is the n-th **Catalan number** and is denoted by c_n.

Theorem 13.3. *The sequence of Catalan numbers satisfies the recursion:*

$$c_0 = 1, \text{ and } c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k} \text{ for all } n \geq 0.$$

Furthermore, $c_n = \dfrac{1}{n+1}\dbinom{2n}{n}$.

Theorem 13.4 (Dilworth)**.** *Suppose $(S, \leq)$ is a finite poset. Then, the maximum length of an anti-chain in S is the same as the smallest number of chains needed to partition S. Furthermore, the maximum length of a chain is equal to the smallest number of anti-chains needed to partition S.*

13.3 Further Examples

Example 13.1 (Putnam 1992, B1)**.** *Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of elements in A_S?*

Scratch: As usual let's try some small cases.
For $n = 2$, the answer is 1.
For $n = 3$, given $a < b < c$, the averages are $\frac{a+b}{2} < \frac{a+c}{2} < \frac{b+c}{2}$, so the answer is 3.
For $n = 4$, given $a < b < c < d$, the sums of pairs that are certainly distinct (we realize there is no need for the 2's in the denominators.) are $a+b < a+c < a+d < b+d < c+d$, so the answer is at least 5. We see that three of them are already the ones that we had from before. So, we want to only allow two new ones. One example is 1, 2, 3, 4. Similar technique works for $n = 5$. Let's put these together in the following solution:
Solution. The answer is $\boxed{2n-3}$.

Suppose $S = \{a_1, a_2, \ldots, a_n\}$ with $a_1 < a_2 < \cdots < a_n$ are real numbers. Note that

$$a_1 + a_2 < a_1 + a_3 < \cdots < a_1 + a_n < a_2 + a_n < a_3 + a_n < \cdots < a_{n-1} + a_n,$$

and thus there are at least $(n-1) + (n-2) = 2n - 3$ distinct elements in A_S.

Now, let $T = \{1, 2, \ldots, n\}$. Note that sum of each two pairs of elements of S is at least $1 + 2 = 3$ and at most $(n-1) + n = 2n - 1$, which means the size of A_T is at most $2n - 1 - 2 = 2n - 3$. This example along with the above proof shows the answer to the problem is $2n - 3$. □

Example 13.2 (Putnam 1992, B2)**.** *For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that*

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$, with $\binom{a}{b} = 0$ otherwise.)

Scratch: Here is my first thoughts:

- Expanding that expression is a huge mess, obviously.
- We should try small cases first.
- The sum on the right hand side looks like a sum that we obtain from multiplying two polynomials. So, could we perhaps factor the expression that we are given?

At this point we realize we have a solution:

Solution.
$$(1+x+x^2+x^3)^n = (1+x)^n(1+x^2)^n = \left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)\left(\sum_{j=0}^{n}\binom{n}{j}x^{2j}\right)$$
Each term involving x^k is obtain by multiplying $\binom{n}{k-2j}x^{k-2j}$ and $\binom{n}{j}x^{2j}$, as j ranges from 0 to $\lfloor k/2 \rfloor$. Since when $j > k/2$, $\binom{n}{k-2j} = 0$, we can allow j to range from 0 to k. Thus, $Q(n,k) = \sum_{j=0}^{k}\binom{n}{j}\binom{n}{k-2j}$, as desired. □

Example 13.3 (IMC 2019, Problem 8)**.** *Let $x_1,\ldots,x_n$ be real numbers. For any set $I \subseteq \{1,2,\ldots,n\}$ let $s(I) = \sum_{i\in I} x_i$. Assume that the function $I \mapsto s(I)$ takes on at least 1.8^n values where I runs over all 2^n subsets of $\{1,2,\ldots,n\}$. Prove that the number of sets $I \subseteq \{1,2,\ldots,n\}$ for which $s(I) = 2019$ does not exceed 1.7^n.*

Scratch: Here are my first thoughts:

- Proof by contradiction seems the most reasonable approach since we want to show something does *not* happen.
- Why 1.8 and 1.7? Can we change these numbers and make the problem simpler?
- An obvious change is to change 1.8 to 2 and the problem becomes trivial! But no number seems nontrivial and at the same time easier than the given problem.
- We can probably get a contradiction by building too many sets, but $1.7^n + 1.8^n$ is less than 2^n. So, this doesn't seem a reasonable idea.
- Maybe we should take the pairs of sets? One from those with different sums and one from those with sums 2019? That gives us 3.06^n pairs. This seems good, but then we could have repeated elements. That gives us two copies of each element and thus $2n$ elements but that gives 2^{2n} subsets, so no hope there either... but wait! This is a multiset. Each element has *three* possibility, either is not in the set or appears exactly once or appears exactly twice. There we got a solution!

Solution. On the contrary suppose there are at least 1.7^n sets I with $s(I) = 2019$. Let $\{A_1,\ldots,A_m\}$ be such a set with $m \geq 1.7^n$. Suppose also that $\{B_1,\ldots,B_k\}$ is a set of size at least 1.8^n for which $s(B_j) \neq s(B_i)$ for all $i \neq j$. We will prove that two multisets $A_i \cup B_j$ and $A_r \cup B_s$ are the same if and only if $(i,j) = (r,s)$. Suppose $A_i \cup B_j = A_r \cup B_s$ as multisets. Thus, we must have $s(A_i) + s(B_j) = s(A_r) + s(B_s)$. The fact that $s(A_i) = s(A_r) = 2019$ implies $s(B_j) = s(B_s)$ and thus $j = s$. Since $A_i \cup B_j = A_r \cup B_s$ as multisets, $A_i = A_r$, or $i = r$, as desired. Since for every element $x \in \{1,2,\ldots,n\}$ the multiset $A_i \cup B_j$ either does not contain x or contains x exactly once or exactly twice, there must are at most 3^n of these multisets. However by assumption there are at least $1.7^n \cdot 1.8^n = 3.06^n$ of these multisets, which is a contradiction. □

Example 13.4 (IMC 2020, Problem 1)**.** *Let n be a positive integer. Compute the number of words w (finite sequences of letters) that satisfy all the following three properties:*

(1) w consists of n letters, all of them are from the alphabet $\{a,b,c,d\}$;

(2) w contains an even number of letters a;

(3) w contains an even number of letters b.

(For example, for $n=2$ there are 6 such words: aa, bb, cc, dd, cd and dc.)

Scratch: As usual, writing down a few examples may be helpful. For $n=1$, we have two possibilities: c, d.
For $n=2$, we have the six listed in the problem.
The number gets larger and listing a couple more cases you would see that it easily gets out of hand. It could be because the answer is exponential or something similar. We do see that to create words of length 3 we should take words of length 2 and add appropriate letters to them. For example if a word of length 2 has an even number of both a and b, we must add either c or d. If a word of length 2 has an odd number of a and b's we cannot create a word of length 3 with an even number of a's and b's. If a word of length 2 has an odd number of a's and an even number of b's we must add an a at the end to obtain a word of length 3 with an even number of a's and b's. So, this gives the idea of using recursions. However we see that we must also consider the number of words with an odd number of a's and/or b's.
Let x_n be the number of words of length n that have an even number of both a's and b's, y_n be the number of words of length n that have an even number of a's or b's and an odd number of the other one, and z_n be the number of words of length n that have an odd number of both a's and b's. Repeating the argument above we obtain the following recursions:

$$\begin{cases} x_{n+1} = 2x_n + y_n \\ y_{n+1} = 2y_n + z_n \\ z_{n+1} = 2x_n + 2y_2 + 2z_n \end{cases}$$

We can write this as a matrix equation of form:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}.$$

Using this repeatedly we obtain

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 2 & 2 \end{pmatrix}^n \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}.$$

To evaluate the power of this matrix we diagonalize it and that gives us

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 2 & 2 \end{pmatrix}^n \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4^n + 2^n \\ 4^n - 2^n \\ 2^{2n+1} \end{pmatrix}$$

Therefore, $x_n = 4^{n-1} + 2^{n-1}, y_n = 4^{n-1} - 2^{n-1}$, and $z_n = 2^{2n-1}$.

Now that we know what the answer is we could prove it by induction without having to go through the entire process of diagonalization of the matrix. That would save us some time when writing the solution.

Solution.(Video Solution) The answer is $\boxed{2^{n-1} + 4^{n-1}}$.

Throughout the solution we assume all words only use letters a, b, c and d.

Let X_n be the set of all words of length n that have an even number of both a's and b's, Y_n be the set of all words of length n that have an odd number of both a's and b's, and Z_n be the set of all words of length n that have an odd number of one of a's or b's and an even number of the other one. We will prove that:

Claim: $|X_n| = 4^{n-1} + 2^{n-1}, |Y_n| = 4^{n-1} - 2^{n-1}$, and $|Z_n| = 2^{2n-1}$. Set $x_n = |X_n|, y_n = |Y_n|$, and $z_n = |Z_n|$. By the discussion that we had earlier, we have

$$\begin{cases} x_{n+1} = 2x_n + y_n \\ y_{n+1} = 2y_n + z_n \\ z_{n+1} = 2x_n + 2y_2 + 2z_n \end{cases}$$

The claim can now be proved using induction on n. □

Example 13.5 (Putnam 2022, B3)**.** *Assign to each positive real number a color, either red or blue. Let D be the set of all distances $d > 0$ such that there are two points of the same color at distance d apart. Recolor the positive reals so that the numbers in D are red and the numbers not in D are blue. If we iterate this recoloring process, will we always end up with all the numbers red after a finite number of steps?*

Solution.(Video Solution) We will prove the answer is yes.

Let R_0, B_0 be the initial sets of red and blue numbers, respectively, and assume R_n, B_n are the sets of red and blue points after n iterations of this recoloring. We will prove that $R_2 = (0, \infty)$. First, we will prove the following claim:

Claim. If after n iterations, there are two points of distance x with different colors, then $x/2 \in R_{n+1}$.

Suppose after n iterations, $a, a+x$ have different colors. Then, since there are only two colors, the midpoint $a + x/2$ has the same color as either a or $a+x$. Thus $\frac{x}{2} \in R_{n+1}$.

Assume $x \in B_1$. By the above claim $\frac{x}{2} \in R_1$. Also, note that by assumption, in the initial coloring, the colors of the elements in the sequence $1, 1+x, 1+2x, 1+3x$ alternate, and hence $1, 1+3x$ have different colors. Thus, $\frac{3x}{2} \in R_1$.

Therefore, $\frac{3x}{2} - \frac{x}{2} \in R_2$, which means $x \in R_2$, as desired.

Now, assume $x \in R_1$, and assume $a, a+x$ have the same color in the original coloring for some $a > 0$. If $2x \in R_1$, then $2x - x \in R_2$, as desired. If $2x \in B_1$, then $a+x$ and $a+3x$ would have different colors. Since a and $a+x$ have the same color, a and $a+3x$ would have different colors. Therefore, by the claim above, $\frac{3x}{2} \in R_1$. This implies $\frac{3x}{2} - \frac{x}{2} \in R_2$, which means $x \in R_2$, as desired. □

Example 13.6 (IMC 2022, Problem 3)**.** *Let p be a prime number. A flea is staying at point 0 of the real line. At each minute, the flea has three possibilities: to stay at its position, or to move by 1 to the left or to the right. After $p-1$ minutes, it wants to be at 0 again. Denote by $f(p)$ the number of its strategies to do this (for example, $f(3) = 3$: it may either stay at 0 for the entire time, or go to the left and then to the right, or go to the right and then to the left). Find $f(p)$ modulo p.*

Scratch: In order for the flea to return to its original position, it needs to have the same number of moves to the right as to the left. If we represent each move to the right by $+1$, each move to the left by -1 and each stay by 0 we can evaluate $f(p)$ by counting the number of ways we can write 0 as a sum of a list of $p-1$ characters $+1$, -1 and 0. For example for $f(3) = 3$ can be seen using the following list:

$$0 = 0+0 = +1-1 = -1+1.$$

Using the language of generating functions, we can rewrite the above as

$$1 = x^0x^0 = x^1x^{-1} = x^{-1}x^1.$$

In other words, if we multiply two copies of $1+x+x^{-1}$, each of the terms above yields a constant term of 1. This can be restated as:

$$f(3) = \text{ The constant term of } (1+x+x^{-1})^2$$

Similarly, we can think of $f(p)$ as the constant term of $(1+x+x^{-1})^{p-1}$. After some algebra, we will obtain the following solution:

Solution. (Video Solution) The answer is

$$f(p) \pmod p = \begin{cases} 0 & \text{if } p = 3 \\ 1 & \text{if } p \equiv 1 \pmod 3 \\ p-1 & \text{if } p \equiv -1 \pmod 3 \end{cases}$$

Clearly, $f(3) = 3$ satisfies above. From now on assume $p \neq 3$. We note that

$$f(p) = \text{ The constant term of } (1+x+x^{-1})^{p-1}.$$

We will evaluate this constant in $\mathbb{Z}_p$, the field of integers modulo p. Using difference of cubes and the fact that in $\mathbb{Z}_p$, $(a+b)^p = a^p + b^p$ we obtain the following:

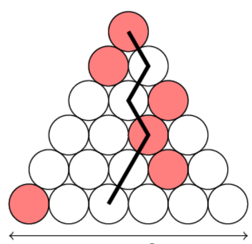
$$\frac{(1+x+x^2)^{p-1}}{x^{p-1}} = \frac{(1-x^3)^{p-1}}{x^{p-1}(1-x)^{p-1}} = \frac{(1-x^3)^p(1-x)}{x^{p-1}(1-x)^p(1-x^3)} = \frac{(1-x^{3p})(1-x)}{x^{p-1}(1-x^p)(1-x^3)}$$

Multiplying by x^{p-1} we need to find the coefficient of x^{p-1} in the following expression:

$$\frac{(1-x^{3p})(1-x)}{(1-x^p)(1-x^3)} = (1-x^{3p})(1-x)(\sum_{k=0}^{\infty} x^{pk})(\sum_{k=0}^{\infty} x^{3k})$$

If $p-1=3k$, then the only way we may obtain x^{p-1} is by multiplying x^{3k} from the last parenthesis above and 1 from the remaining ones. Which means $f(p) \equiv 1 \mod p$. If $p-1=3k+1$, the only way we may obtain x^{p-1} is by multiplying x^{3k} from the last parenthesis above, $-x$ from the second parenthesis, and 1 from the remaining ones. Thus, $f(p) \equiv -1 \mod p$, as desired. □

Example 13.7 (IMO 2023, Problem 5)**.** *Let n be a positive integer. A Japanese triangle consists of $1+2+\cdots+n$ circles arranged in an equilateral triangular shape such that for each $i=1,2,\ldots,n$, the i-th row contains exactly i circles, exactly one of which is coloured red. A ninja path in a Japanese triangle is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a Japanese triangle with $n=6$, along with a ninja path in that triangle containing two red circles. In terms of n, find the greatest k such that in each Japanese triangle there is a ninja path containing at least k red circles.*



Solution.(Video Solution) □

Example 13.8 (IMO 2021, Shortlisted Problem, C1)**.** *Let S be an infinite set of positive integers, such that there exist four pairwise distinct $a,b,c,d \in S$ with $\gcd(a,b) \neq \gcd(c,d)$. Prove that there exist three pairwise distinct $x,y,z \in S$ such that $\gcd(x,y) = \gcd(y,z) \neq \gcd(z,x)$.*

Solution.(Video Solution) Note that for every $s \in S$, the integer $\gcd(a,s)$ divides a. Therefore, there are finitely many possible values for $\gcd(a,s)$. Since S is infinite, the value $\gcd(a,s)$ must be the same for infinitely many elements of S. Let A be an infinite subset of $S-\{a,b,c,d\}$ for which $\gcd(a,x) = gcd(a,y)$ for all $x,y \in A$. Now, we will repeat the same argument replacing a by b, and S by A. There is an infinite subset B of A for which $\gcd(b,x) = gcd(b,y)$ for all $x,y \in B$. We will now repeat the same argument two more times, once by replacing b by c, and A by B to obtain an infinite subset C of B with the property that $\gcd(c,x) = gcd(c,y)$ for all $x,y \in C$, and once by replacing c by d and the set B by C to obtain an infinite subset D of C with the property that $\gcd(d,x) = gcd(d,y)$ for all $x,y \in D$. To summarize, we found an infinite subset D of $S-\{a,b,c,d\}$ for which

$$\forall\, x,y \in D \;\; \gcd(a,x) = \gcd(a,y);\; \gcd(b,x) = \gcd(b,y);\; \gcd(c,x) = \gcd(c,y);\; \gcd(d,x) = \gcd(d,y).$$

Now, if for two distinct $x,y \in D$ we have $\gcd(x,y) \neq \gcd(a,x)$, then we have found the desired triple a,x,y. So, let's assume $\gcd(x,y) = \gcd(a,x)$. Similarly, we may assume $\gcd(x,y) = \gcd(b,x)$. Therefore, $\gcd(a,x) = \gcd(b,x)$. If $\gcd(a,b) \neq \gcd(a,x)$, then the triple a,b,x would satisfy the desired condition. So, let's assume $\gcd(a,b) = \gcd(a,x)$. Thereofre, $\gcd(a,b) = \gcd(x,y)$. A similar argument shows $\gcd(c,d) = \gcd(x,y)$, whish implies $\gcd(a,b) = \gcd(a,b)$. This contradiction shows a triple with the desired condition must exist. □

Example 13.9 (IMC 2023, Problem 8). *Let T be a tree with n vertices; that is, a connected simple graph on n vertices that contains no cycle. For every pair u,v of vertices, let $d(u,v)$ denote the distance between u and v, that is, the number of edges in the shortest path in T that connects u with v.*

Consider the sums

$$W(T) = \sum_{\{u,v\} \subseteq V(T), u \neq v} d(u,v), \text{ and } H(T) = \sum_{\{u,v\} \subseteq V(T), u \neq v} \frac{1}{d(u,v)}.$$

Prove that

$$W(T)H(T) \geq \frac{(n-1)^3(n+2)}{4}.$$

Solution.(Video Solution)

□

Example 13.10 (Putnam 1985, A1). *How many ordered triples of sets (A,B,C) are there for which both of the following hold?*

$$A \cup B \cup C = \{1,2,3,4,5,6,7,8,9,10\} \text{ and } A \cap B \cap C = \emptyset$$

Solution.(Video Solution)

□

13.4 Exercises

Exercise 13.1 (VTRMC 1983). A finite set of roads connect n towns $T_1, T_2, \ldots, T_n$ where $n \geq 2$. We say that towns T_i and $T_j (i \neq j)$ are directly connected if there is a road segment connecting T_i and T_j which does not pass through any other town. Let $f(T_k)$ be the number of other towns directly connected to T_k. Prove that f is not one-to-one.

Exercise 13.2. Let $n \geq 2$ be an integer. A set $S = \{A_1, \ldots, A_k\}$ consisting of 2-subsets of $\{1,2,\ldots,n\}$ satisfies the following two properties:

- no two distinct A_i's share an element, and
- the sum of elements of A_i's are all different and do not exceed n.

Find the maximum size of S.

Exercise 13.3. Let $n \geq 2$ be an integer. We write n, 1's on aboard. In each step two of the numbers a and b on the board are erased and replaced by $(a+b)^4$. This is repeated until only one number is left. Prove that this number is at least $2^{\frac{4n^2-4}{3}}$.

Exercise 13.4 (VTRMC 1984)**.** Let the (x,y)-plane be divided into regions by n lines, any two of which may or may not intersect. Describe a procedure whereby these regions may be colored using only two colors so that regions with a common line segment as part of their boundaries have different colors.

Exercise 13.5 (VTRMC 1986)**.** Sets A and B are defined by $A = \{1, 2, \ldots, n\}$ and $B = \{1, 2, 3\}$. Determine the number of distinct functions from A onto B. (A function $f : A \to B$ is "onto" if for each $b \in B$ there exists $a \in A$ such that $f(a) = b$.)

Exercise 13.6 (VTRMC 1988)**.** Let $T(n)$ be the number of incongruent triangles with integral sides and perimeter $n \geq 6$. Prove that $T(n) = T(n-3)$ if n is even, or disprove by a counterexample. (Note: two triangles are congruent if there is a one-to-one correspondence between the sides of the two triangles such that corresponding sides have the same length.)

Exercise 13.7 (VTRMC 1989)**.** The integer sequence $\{a_0, a_1, \ldots, a_{n-1}\}$ is such that, for each i ($0 \leq i \leq n-1), a_i$ is the number of i 's in the sequence. (Thus for $n = 4$ we might have the sequence $\{1, 2, 1, 0\}$.)

(a) Prove that, if $n \geq 7$, such a sequence is a unique.

(b) Find such a sequence for $n = 7$.

Exercise 13.8 (VTRMC 1990)**.** Ten points in space, no three of which are collinear, are connected, each one to all the others, by a total of 45 line segments. The resulting framework F will be "disconnected" into two disjoint nonempty parts by the removal of one point from the interior of each of the 9 segments emanating from any one vertex of f. Prove that F cannot be similarly disconnected by the removal of only 8 points from the interiors of the 45 segments.

Exercise 13.9 (Putnam 1990, A6)**.** If X is a finite set, let $|X|$ denote the number of elements in X. Call an ordered pair (S, T) of subsets of $\{1, 2, \ldots, n\}$ *admissible* if $s > |T|$ for each $s \in S$, and $t > |S|$ for each $t \in T$. How many admissible ordered pairs of subsets of $\{1, 2, \ldots, 10\}$ are there? Prove your answer.

Exercise 13.10 (VTRMC 1991)**.** A and B play the following money game, where a_n and b_n denote the amount of holdings of A and B, respectively, after the nth round. At each round a player pays one-half his holdings to the bank, then receives one dollar from the bank if the other player had less than c dollars at the end of the previous round. If $a_0 = .5$ and $b_0 = 0$, describe the behavior of a_n and b_n when n is large, for (i) $c = 1.24$ and (ii) $c = 1.26$.

Exercise 13.11 (VTRMC 1991)**.** Mathematical National Park has a collection of trails. There are designated campsites along the trails, including a campsite at each intersection of trails. The rangers call each stretch of trail between adjacent campsites a "segment". The trails have been laid out so that it is possible to take a hike that starts at any campsite, covers each segment exactly once, and ends at the beginning campsite. Prove that it is possible to plan a collection $\mathscr{C}$ of hikes with all of the following properties:

(i) Each segment is covered exactly once in one hike $h \in \mathscr{C}$ and never in any of the other hikes of $\mathscr{C}$.

(ii) Each $h \in \mathscr{C}$ has a base campsite that is its beginning and end, but which is never passed in the middle of the hike. (Different hikes of $\mathscr{C}$ may have different base campsites.)

(iii) Except for its base campsite at beginning and end, no hike in $\mathscr{C}$ passes any campsite more than once.

Exercise 13.12 (Putnam 1991, A6)**.** Let $A(n)$ denote the number of sums of positive integers

$$a_1 + a_2 + \cdots + a_r$$

which add up to n with

$$a_1 > a_2 + a_3, a_2 > a_3 + a_4, \ldots,$$

$$a_{r-2} > a_{r-1} + a_r, a_{r-1} > a_r.$$

Let $B(n)$ denote the number of $b_1 + b_2 + \cdots + b_s$ which add up to n, with

1. $b_1 \geq b_2 \geq \cdots \geq b_s$,

2. each b_i is in the sequence $1, 2, 4, \ldots, g_j, \ldots$ defined by $g_1 = 1$, $g_2 = 2$, and $g_j = g_{j-1} + g_{j-2} + 1$, and

3. if $b_1 = g_k$ then every element in $\{1, 2, 4, \ldots, g_k\}$ appears at least once as a b_i.

Prove that $A(n) = B(n)$ for each $n \geq 1$.
(For example, $A(7) = 5$ because the relevant sums are $7, 6+1, 5+2, 4+3, 4+2+1$, and $B(7) = 5$ because the relevant sums are $4+2+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1$.)

Exercise 13.13 (Putnam 1991, B3)**.** Does there exist a real number L such that, if m and n are integers greater than L, then an $m \times n$ rectangle may be expressed as a union of 4×6 and 5×7 rectangles, any two of which intersect at most along their boundaries?

Exercise 13.14 (VTRMC 1992)**.** Some goblins, N in number, are standing in a row while "trick-or-treat"ing. Each goblin is at all times either 2' tall or 3' tall, but can change spontaneously from one of these two heights to the other at will. While lined up in such a row, a goblin is called a **Local Giant Goblin** (LGG) if he/she/it is not standing beside a taller goblin. Let $G(N)$ be the total of all occurrences of LGG's as the row of N goblins transmogrifies through all possible distinct configurations, where height is the only distinguishing characteristic. As an example, with $N = 2$, the distinct configurations are ˆ2 ˆ2, 2ˆ3, ˆ32, ˆ3 ˆ3, where a cap indicates an LGG. Thus $G(2) = 6$.

(i) Find $G(3)$ and $G(4)$.

(ii) Find, with proof, the general formula for $G(N), N = 1, 2, 3, \ldots$.

Exercise 13.15 (Putnam 1993, A3)**.** Let $\mathscr{P}_n$ be the set of subsets of $\{1, 2, \ldots, n\}$. Let $c(n,m)$ be the number of functions $f : \mathscr{P}_n \to \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n,m) = \sum_{j=1}^{m} j^n.$$

Exercise 13.16 (Putnam 1993, A4)**.** Let $x_1, x_2, \ldots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \ldots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.

Exercise 13.17 (Putnam 1994, A3)**.** Show that if the points of an isosceles right triangle of side length 1 are each colored with one of four colors, then there must be two points of the same color which are at least a distance $2-\sqrt{2}$ apart.

Exercise 13.18 (Putnam 1994, A6)**.** Let $f_1, \ldots, f_{10}$ be bijections of the set of integers such that for each integer n, there is some composition $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}$ of these functions (allowing repetitions) which maps 0 to n. Consider the set of 1024 functions

$$\mathscr{F} = \{f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}} \mid e_i = 0 \text{ or } 1 \text{ for } 1 \le i \le 10\},$$

where, f_i^0 is the identity function and $f_i^1 = f_i$. Show that if A is any nonempty finite set of integers, then at most 512 of the functions in $\mathscr{F}$ map A to itself.

Exercise 13.19 (Putnam 1995, A4)**.** Suppose we have a necklace of n beads. Each bead is labeled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, \ldots, x_n$ satisfy

$$\sum_{i=1}^{k} x_i \le k-1 \qquad \text{for} \quad k = 1, 2, \ldots, n.$$

Exercise 13.20 (Putnam 1995, B1)**.** For a partition π of $\{1,2,3,4,5,6,7,8,9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1,2,3,4,5,6,7,8,9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

Exercise 13.21 (VTRMC 1996)**.** There are $2n$ points in the plane such that no three points are on the same line. n points are red and the other n points are green. Show that there is at least one way to draw n line segments by connecting each point to a unique different colored point so that no two line segments intersect.

Exercise 13.22 (Putnam 1996, A3)**.** Suppose that each of 20 students has made a choice of anywhere from 0 to 6 courses from a total of 6 courses offered. Prove or disprove: there are 5 students and 2 courses such that all 5 have chosen both courses or all 5 have chosen neither course.

Exercise 13.23 (Putnam 1996, B1)**.** Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1,2,\ldots,n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

Exercise 13.24 (Putnam 1996, B5)**.** Given a finite string S of symbols X and O, we write $\Delta(S)$ for the number of X's in S minus the number of O's. For example, $\Delta(XOOXOOX) = -1$. We call a string S **balanced** if every substring T of (consecutive symbols of) S has $-2 \le \Delta(T) \le 2$. Thus, $XOOXOOX$ is not balanced, since it contains the substring $OOXOO$. Find, with proof, the number of balanced strings of length n.

Exercise 13.25 (VTRMC 1997)**.** Suppose that you are in charge of taking ice cream orders for a class of 100 students. If each student orders exactly one flavor from Vanilla, Strawberry, Chocolate and Pecan, how many different combinations of flavors are possible for the 100 orders you are taking. Here are some examples of possible combinations. You do not distinguish between individual students.

(a) $V = 30, S = 20, C = 40, P = 10$.

(b) $V = 80, S = 0, C = 20, P = 0$.

(c) $V = 0, S = 0, C = 0, P = 100$.

Exercise 13.26 (VTRMC 1997)**.** A business man works in New York and Los Angeles. If he is in New York, each day he has four options; to remain in New York, or to fly to Los Angeles by either the 8:00 a.m., 1:00 p.m. or 6:00 p.m. flight. On the other hand if he is in Los Angeles, he has only two options; to remain in Los Angeles, or to fly to New York by the 8:00 a.m. flight. In a 100 day period he has to be in New York both at the beginning of the first day of the period, and at the end of the last day of the period. How many different possible itineraries does the business man have for the 100 day period (for example if it was for a 2 day period rather than a 100 day period, the answer would be 4)?

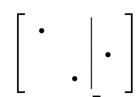
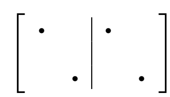
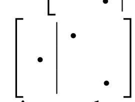
Exercise 13.27 (Putnam 1997, A5)**.** Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \ldots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

Exercise 13.28 (Putnam 1999, A3)**.** Consider the power series expansion

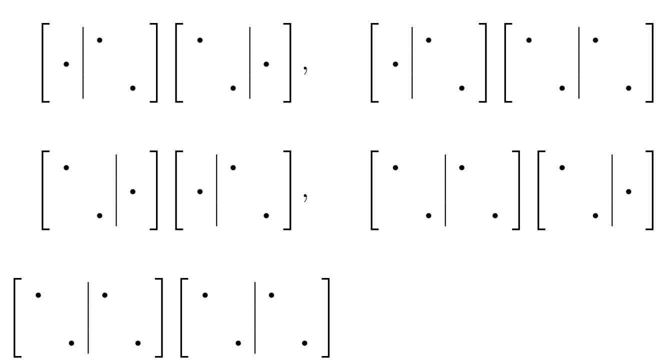
$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

Exercise 13.29 (VTRMC 2000)**.** Two types of domino-like rectangular tiles,  and , are available. The first type may be rotated end-to-end to produce a tile of type . Let $A(n)$ be the number of distinct chains of n tiles, placed end-to-end, that may be constructed if abutting ends are required to have the same number of dots.

Example $A(2) = 5$, since the following five chains of length two, and no others, are allowed.



(a) Find $A(3)$ and $A(4)$.

(b) Find, with proof, a three-term recurrence formula for $A(n+2)$ in terms of $A(n+1)$ and $A(n)$, for $n = 1, 2, \ldots$, and use it to find $A(10)$.

Exercise 13.30 (Putnam 2000, B1)**.** Let a_j, b_j, c_j be integers for $1 \le j \le N$. Assume for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \le j \le N$.

Exercise 13.31 (VTRMC 2001)**.** For each positive integer n, let S_n denote the total number of squares in an $n \times n$ square grid. Thus $S_1 = 1$ and $S_2 = 5$, because a 2×2 square grid has four 1×1 squares and one 2×2 square. Find a recurrence relation for S_n, and use it to calculate the total number of squares on a chess board (i.e. determine S_8).

Exercise 13.32 (Putnam 2001, B1)**.** Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is

$$(k-1)n+1, (k-1)n+2, \ldots, (k-1)n+n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

Exercise 13.33 (VTRMC 2002)**.** Let A and B be nonempty subsets of $S = \{1, 2, \ldots, 99\}$ (integers from 1 to 99 inclusive). Let a and b denote the number of elements in A and B respectively, and suppose $a + b = 100$. Prove that for each integer s in S, there are integers x in A and y in B such that $x + y = s$ or $s + 99$.

Exercise 13.34 (VTRMC 2002)**.** Let $\{1, 2, 3, 4\}$ be a set of abstract symbols on which the associative binary operation $*$ is defined by the following operation table (associative means $(a*b)*c = a*(b*c)$)

$*$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

If the operation $*$ is represented by juxtaposition, e.g., $2*3$ is written as 23 etc., then it is easy to see from the table that of the four possible "words" of length two that can be formed using only 2 and 3, i.e., 22, 23, 32 and 33 , exactly two, 22 and 33 , are equal to 1 . Find a formula for the number $A(n)$ of words of length n, formed by using only 2 and 3 , that equal 1 . From the table and the example just given for words of length two, it is clear that $A(1) = 0$ and $A(2) = 2$. Use the formula to find $A(12)$.

Exercise 13.35 (VTRMC 2002)**.** Let n be a positive integer. A bit string of length n is a sequence of n numbers consisting of 0 's and 1 's. Let $f(n)$ denote the number of bit strings of length n in which every 0 is surrounded by 1 's. (Thus for $n = 5$, 11101 is allowed, but 10011 and 10110 are not allowed, and we have $f(3) = 2$, $f(4) = 3$.) Prove that $f(n) < (1.7)^n$ for all n.

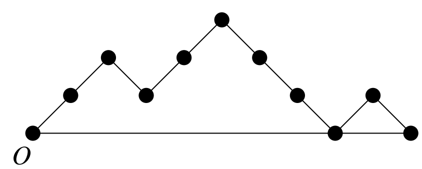
Exercise 13.36 (Putnam 2002, A3)**.** Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1,2,3,\ldots,n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

Exercise 13.37 (Putnam 2003, A1)**.** Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k,$$

with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

Exercise 13.38 (Putnam 2003, A5)**.** A Dyck n-path is a lattice path of n upsteps $(1,1)$ and n downsteps $(1,-1)$ that starts at the origin O and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n-1)$-paths.

Exercise 13.39 (Putnam 2003, A6)**.** For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?

Exercise 13.40 (VTRMC 2004)**.** A computer is programmed to randomly generate a string of six symbols using only the letters A, B, C. What is the probability that the string will not contain three consecutive A's?

Exercise 13.41 (VTRMC 2004)**.** An enormous party has an infinite number of people. Each two people either know or don't know each other. Given a positive integer n, prove there are n people in the party such that either they all know each other, or nobody knows each other (so the first possibility means that if A and B are any two of the n people, then A knows B, whereas the second possibility means that if A and B are any two of the n people, then A does not know B).

Exercise 13.42 (VTRMC 2004)**.** A 9×9 chess board has two squares from opposite corners and its central square removed (so 3 squares on the same diagonal are removed, leaving 78 squares). Is it possible to cover the remaining squares using dominoes, where each domino covers two adjacent squares? Justify your answer.

Exercise 13.43 (Putnam 2004, A1)**.** Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

Exercise 13.44 (Putnam 2004, A5)**.** An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, p and q, are in the same connected monochromatic region if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

Exercise 13.45 (VTRMC 2005)**.** We wish to tile a strip of n 1-inch by 1-inch squares. We can use dominos which are made up of two tiles which cover two adjacent squares, or 1-inch square tiles which cover one square. We may cover each square with one or two tiles and a tile can be above or below a domino on a square, but no part of a domino can be placed on any part of a different domino. We do not distinguish whether a domino is above or below a tile on a given square. Let $t(n)$ denote the number of ways the strip can be tiled according to the above rules. Thus for example, $t(1) = 2$ and $t(2) = 8$. Find a recurrence relation for $t(n)$, and use it to compute $t(6)$.

Exercise 13.46 (Putnam 2005, A2)**.** Let $\mathbf{S} = \{(a,b)|a = 1,2,\ldots,n, b = 1,2,3\}$. A *rook tour* of $\mathbf{S}$ is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that

(i) $p_i \in \mathbf{S}$,

(ii) p_i and p_{i+1} are a unit distance apart, for $1 \leq i < 3n$,

(iii) for each $p \in \mathbf{S}$ there is a unique i such that $p_i = p$.

How many rook tours are there that begin at $(1,1)$ and end at $(n,1)$?

Exercise 13.47 (Putnam 2005, B4)**.** For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m,n) = f(n,m)$.

Exercise 13.48 (VTRMC 2006)**.** Let $S(n)$ denote the number of sequences of length n formed by the three letters A,B,C with the restriction that the C's (if any) all occur in a single block immediately following the first B (if any). For example ABCCAA, AAABAA, and ABCCCC are counted in, but ACACCB and CAAAAA are not. Derive a simple formula for $S(n)$ and use it to calculate $S(10)$.

Exercise 13.49 (Putnam 2006, B2)**.** Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

Exercise 13.50 (Putnam 2007, A3)**.** Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

Exercise 13.51 (Putnam 2007, B6)**.** For each positive integer n, let $f(n)$ be the number of ways to make $n!$ cents using an unordered collection of coins, each worth $k!$ cents for some k, $1 \leq k \leq n$. Prove that for some constant C, independent of n,

$$n^{n^2/2 - Cn} e^{-n^2/4} \leq f(n) \leq n^{n^2/2 + Cn} e^{-n^2/4}.$$

Exercise 13.52 (Putnam 2008, B6)**.** Let n and k be positive integers. Say that a permutation σ of $\{1,2,\ldots,n\}$ is *k-limited* if $|\sigma(i)-i| \le k$ for all i. Prove that the number of k-limited permutations of $\{1,2,\ldots,n\}$ is odd if and only if $n \equiv 0$ or $1 \pmod{2k+1}$.

Exercise 13.53 (VTRMC 2008)**.** How many sequences of 1's and 3's sum to 16? (Examples of such sequences are $1,3,3,3,3,3$ and $1,3,1,3,1,3,1,3$.)

Exercise 13.54 (Putnam 2009, B3)**.** Call a subset S of $\{1,2,\ldots,n\}$ *mediocre* if it has the following property: whenever a and b are elements of S whose average is an integer, that average is also an element of S. Let $A(n)$ be the number of mediocre subsets of $\{1,2,\ldots,n\}$. [For instance, every subset of $\{1,2,3\}$ except $\{1,3\}$ is mediocre, so $A(3)=7$.] Find all positive integers n such that $A(n+2)-2A(n+1)+A(n)=1$.

Exercise 13.55 (Putnam 2010, A1)**.** Given a positive integer n, what is the largest k such that the numbers $1,2,\ldots,n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n=8$, the example $\{1,2,3,6\},\{4,8\},\{5,7\}$ shows that the largest k is *at least* 3.]

Exercise 13.56 (Putnam 2010, B3)**.** There are 2010 boxes labeled $B_1,B_2,\ldots,B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving *exactly* i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

Exercise 13.57 (Putnam 2012, B3)**.** A round-robin tournament of $2n$ teams lasted for $2n-1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

Exercise 13.58 (Putnam 2013, A1)**.** Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

Exercise 13.59 (Putnam 2013, A4)**.** A finite collection of digits 0 and 1 is written around a circle. An *arc* of length $L \ge 0$ consists of L consecutive digits around the circle. For each arc w, let $Z(w)$ and $N(w)$ denote the number of 0's in w and the number of 1's in w, respectively. Assume that $|Z(w)-Z(w')| \le 1$ for any two arcs w,w' of the same length. Suppose that some arcs $w_1,\ldots,w_k$ have the property that

$$Z = \frac{1}{k}\sum_{j=1}^{k} Z(w_j) \text{ and } N = \frac{1}{k}\sum_{j=1}^{k} N(w_j)$$

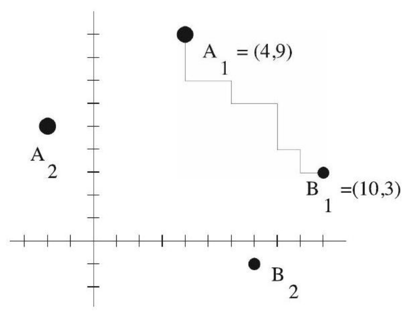
are both integers. Prove that there exists an arc w with $Z(w)=Z$ and $N(w)=N$.

Exercise 13.60 (Putnam 2013, B5)**.** Let $X=\{1,2,\ldots,n\}$, and let $k \in X$. Show that there are exactly $k \cdot n^{n-1}$ functions $f : X \to X$ such that for every $x \in X$ there is a $j \ge 0$ such that $f^{(j)}(x) \le k$. [Here $f^{(j)}$ denotes the j^{th} iterate of f, so that $f^{(0)}(x)=x$ and $f^{(j+1)}(x)=f(f^{(j)}(x))$.]

Exercise 13.61 (VTRMC 2014)**.** Suppose we are given a 19×19 chessboard (a table with 19^2 squares) and remove the central square. Is it possible to tile the remaining $19^2 - 1 = 360$ squares with 4×1 and 1×4 rectangles? (So each of the 360 squares is covered by exactly one rectangle.) Justify your answer.

Exercise 13.62 (VTRMC 2014)**.** Let A, B be two points in the plane with integer coordinates $A = (x_1, y_1)$ and $B = (x_2, y_2)$. (Thus $x_i, y_i \in \mathbb{Z}$, for $i = 1, 2$.) A path $\pi : A \to B$ is a sequence of down and right steps, where each step has an integer length, and the initial step starts from A, the last step ending at B. In the figure below, we indicated a path from $A_1 = (4,9)$ to $B_1 = (10,3)$. The distance $d(A,B)$ between A and B is the number of such paths. For example, the distance between $A = (0,2)$ and $B = (2,0)$ equals 6 . Consider now two pairs of points in the plane $A_i = (x_i, y_i)$ and $B_i = (u_i, z_i)$ for $i = 1, 2$, with integer coordinates, and in the configuration shown in the picture (but with arbitrary coordinates):

- $x_2 < x_1$ and $y_1 > y_2$, which means that A_1 is North-East of A_2; $u_2 < u_1$ and $z_1 > z_2$, which means that B_1 is North-East of B_2.
- Each of the points A_i is North-West of the points B_j, for $1 \leq i, j \leq 2$. In terms of inequalities, this means that $x_i < \min\{u_1, u_2\}$ and $y_i > \max\{z_1, z_2\}$ for $i = 1, 2$.



(a) Find the distance between two points A and B as before, as a function of the coordinates of A and B. Assume that A is North-West of B.

(b) Consider the 2×2 matrix $M = \begin{pmatrix} d(A_1, B_1) & d(A_1, B_2) \\ d(A_2, B_1) & d(A_2, B_2) \end{pmatrix}$. Prove that for any configuration of points A_1, A_2, B_1, B_2 as described before, $\det M > 0$.

Exercise 13.63 (Putnam 2015, B5)**.** Let P_n be the number of permutations π of $\{1, 2, \ldots, n\}$ such that

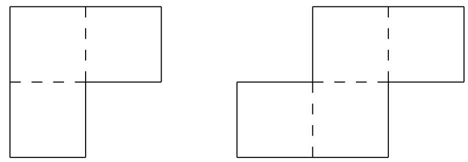
$$|i - j| = 1 \text{ implies } |\pi(i) - \pi(j)| \leq 2$$

for all i, j in $\{1, 2, \ldots, n\}$. Show that for $n \geq 2$, the quantity

$$P_{n+5} - P_{n+4} - P_{n+3} + P_n$$

does not depend on n, and find its value.

Exercise 13.64 (Putnam 2016, A4)**.** Consider a $(2m-1)\times(2n-1)$ rectangular region, where m and n are integers such that $m,n\geq 4$. This region is to be tiled using tiles of the two types shown:



(The dotted lines divide the tiles into 1×1 squares.) The tiles may be rotated and reflected, as long as their sides are parallel to the sides of the rectangular region. They must all fit within the region, and they must cover it completely without overlapping.
What is the minimum number of tiles required to tile the region?

Exercise 13.65 (Putnam 2017, B6)**.** Find the number of ordered 64-tuples $(x_0,x_1,\dots,x_{63})$ such that $x_0,x_1,\dots,x_{63}$ are distinct elements of $\{1,2,\dots,2017\}$ and

$$x_0+x_1+2x_2+3x_3+\cdots+63x_{63}$$

is divisible by 2017.

Exercise 13.66 (Putnam 2018, B1)**.** Let $\mathscr{P}$ be the set of vectors defined by

$$\mathscr{P}=\left\{\begin{pmatrix}a\\b\end{pmatrix}\middle|\, 0\leq a\leq 2, 0\leq b\leq 100, \text{ and } a,b\in\mathbb{Z}\right\}.$$

Find all $\mathbf{v}\in\mathscr{P}$ such that the set $\mathscr{P}\setminus\{\mathbf{v}\}$ obtained by omitting vector $\mathbf{v}$ from $\mathscr{P}$ can be partitioned into two sets of equal size and equal sum.

Exercise 13.67 (Putnam 2018, B6)**.** Let S be the set of sequences of length 2018 whose terms are in the set $\{1,2,3,4,5,6,10\}$ and sum to 3860. Prove that the cardinality of S is at most

$$2^{3860}\cdot\left(\frac{2018}{2048}\right)^{2018}.$$

Exercise 13.68 (Putnam 2020, A5)**.** Let a_n be the number of sets S of positive integers for which

$$\sum_{k\in S}F_k=n,$$

where the Fibonacci sequence $(F_k)_{k\geq 1}$ satisfies $F_{k+2}=F_{k+1}+F_k$ and begins $F_1=1,F_2=1,F_3=2,F_4=3$. Find the largest integer n such that $a_n=2020$.

Exercise 13.69 (Putnam 2020, B2)**.** Let k and n be integers with $1\leq k<n$. Alice and Bob play a game with k pegs in a line of n holes. At the beginning of the game, the pegs occupy the k leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Alice playing first. The game ends when the pegs are in the k rightmost holes, so whoever is next to play cannot move and therefore loses. For what values of n and k does Alice have a winning strategy?

Exercise 13.70 (Putnam 2021, A1)**.** A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?

Exercise 13.71 (Putnam 2022, A5)**.** Alice and Bob play a game on a board consisting of one row of 2022 consecutive squares. They take turns placing tiles that cover two adjacent squares, with Alice going first. By rule, a tile must not cover a square that is already covered by another tile. The game ends when no tile can be placed according to this rule. Alice's goal is to maximize the number of uncovered squares when the game ends; Bob's goal is to minimize it. What is the greatest number of uncovered squares that Alice can ensure at the end of the game, no matter how Bob plays?

Exercise 13.72 (Putnam 2022, B4)**.** Find all integers n with $n \geq 4$ for which there exists a sequence of distinct real numbers $x_1, \ldots, x_n$ such that each of the sets

$$\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \ldots,$$
$$\{x_{n-2}, x_{n-1}, x_n\}, \{x_{n-1}, x_n, x_1\}, \text{ and } \{x_n, x_1, x_2\}$$

forms a 3-term arithmetic progression when arranged in increasing order.

Exercise 13.73 (IMC 2020, Problem 1)**.** Let n be a positive integer. Compute the number of words w (finite sequences of letters) that satisfy all the following three properties:

1. w consists of n letters, all of them are from the alphabet $\{a, b, c, d\}$;
2. w contains an even number of letters a;
3. w contains an even number of letters b.

(For example, for $n = 2$ there are 6 such words: aa, bb, cc, dd, cd and dc.)

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1. w consists of n letters, all of them are from the alphabet $\{a, b, c, d\}$;
2. w contains an even number of letters a;
3. w contains an even number of letters b.

(For example, for $n = 2$ there are 6 such words: aa, bb, cc, dd, cd and dc.)

Exercise 13.74 (Putnam 2023, B1)**.** Consider an m-by-n grid of unit squares, indexed by (i, j) with $1 \leq i \leq m$ and $1 \leq j \leq n$. There are $(m-1)(n-1)$ coins, which are initially placed in the squares (i, j) with $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$. If a coin occupies the square (i, j) with $i \leq m-1$ and $j \leq n-1$ and the squares $(i+1, j)$, $(i, j+1)$, and $(i+1, j+1)$ are unoccupied, then a legal move is to slide the coin from (i, j) to $(i+1, j+1)$. How many distinct configurations of coins can be reached starting from the initial configuration by a (possibly empty) sequence of legal moves?

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Chapter 14

Probability and Miscellaneous

14.1 Basics

When dealing with questions on probability, we start with a sample space, typically denoted by S. There are two different types of probability questions you would face in Putnam or similar math competitions. The first type is discrete, i.e. those that typically involve counting, and the second type is called continuous, i.e. those that typically involve measuring lengths, areas, or volumes.

Definition 14.1. A **random variable** is a function $X : S \to \mathbb{R}$, from a sample space S to the set of real numbers. The image of X, i.e. the values that the random variable can take on, is called the **support** of X. A random variable X is said to be **continuous** if its support is an interval, and it is called **discrete** if its support is a finite or countable set,

Definition 14.2. The **probability density function** (pdf) of a continuous random variable X is a function f that satisfies:

$$P(a \leq X \leq b) = \int_a^b f(x)\ dx.$$

When X is a discrete random variable, i.e. the support of X is finite or a countable set $\{x_1, x_2, \ldots\}$, we have $f(x_i) = P(X = x_i)$ and $f(x) = 0$ for all other values of x. In this case, for every subset T of S, we have $P(X \in T) = \sum_{x \in T} f(x)$.

Note that by the Fundamental Theorem of Calculus, in the continuous case, $\frac{d}{dx}\left(P(a \leq X \leq x)\right) = f(x)$. This sometimes helps in finding the pdf.

Definition 14.3. The **expected value** of a continuous random variable X is given by $E[X] = \int_{-\infty}^{\infty} x\ f(x)\ dx$. When X is discrete with a support of $\{x_1, x_2, \ldots\}$, its expected value is given by $E[X] = \sum_{i=1}^{\infty} x_i P(X = x_i)$.

14.2 Important Theorems

Theorem 14.1 (Linearity of Expected Value)**.** *Given random variables $X_1, X_2, \ldots, X_n$ and constants $a_1, a_2, \ldots, a_n$ we have $E[\sum_{j=1}^{n} a_j X_j] = \sum_{j=1}^{n} a_j E[X_j]$.*

Theorem 14.2 (Monotone Convergence Theorem)**.** *Suppose $X_1, X_2, \ldots$ is a sequence of nonnegative random variables, then $E[\sum_{j=1}^{\infty} X_j] = \sum_{j=1}^{\infty} E[X_j]$.*

The above theorem holds if we replace the assumption that each X_n is nonnegtaive with the assumption that $E[\sum_{j=1}^{\infty} |X_j|]$ is finite.

14.3 Further Examples

Example 14.1 (Putnam 1992, A6)**.** *Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)*

Scratch: First, we may assume the sphere is the unit sphere centered at the origin.

We start with doing the problem on the plane. This means we are selecting three points on a circle and we want to determine the probability that the center of the circle is inside the triangle formed by these three points. One of the points can be assumed to be $(0,1)$. The other two points must be selected in a way that the triangle is acute. If $(1,\alpha)$ and $(1,\beta)$ with $\alpha \leq \beta$ are the polar coordinates of the two points, then we need $\alpha, \beta-\alpha, 2\pi-\beta < \pi$. This gives us a region in the plane that can be used to find the probability.

Can we try something similar in three dimensional space? For that we need to understand what it means for the center to be inside the tetrahedron. We don't really see an analogue to "being acute", so is there anyway we can understand "being acute" in the plane without really checking the angles? What we could do is to say B and C are on opposite sides of AO, and similar for the others. In other words, given A and B, I need to make sure C is inside the arc $A'B'$, where A' and B' are reflections of A and B about O. That means we are trying to find the average length of arc AB. This can be done in a similar way to what we did above, but we could also notice that the four arcs $AB, A'B, AB'$, and $A'B'$ all have the same average length. Thus the probability is 1/4. This seems to be a better approach that can be generalized. So, putting these together we can solve the problem.

Solution. The answer is $\boxed{\frac{1}{8}}$.

For simplicity assume the sphere is a unit sphere centered at the origin. Let A', B', C', D' be the reflections of A, B, C, D about the center of the sphere. For the center to be inside the tetrahedron we need vertex D to lie inside the spherical triangle $A'B'C'$. Thus, the question is reduced to evaluating the probability of the area of the spherical triangle $A'B'C'$ which is the same as the area of the spherical triangle ABC, where A, B, C are chosen randomly on the sphere. Since the sphere can be divided into eight spherical triangles $ABC, A'BC, AB'C, ABC', A'B'C, A'BC', AB'C', A'B'C'$ all with the same average area, the answer is 1/8. □

Example 14.2 (Putnam 2022, A4). *Suppose that $X_1, X_2, \dots$ are real numbers between 0 and 1 that are chosen independently and uniformly at random. Let $S = \sum_{i=1}^{k} X_i/2^i$, where k is the least positive integer such that $X_k < X_{k+1}$, or $k = \infty$ if there is no such integer. Find the expected value of S.*

Solution.(Video Solution) The answer is $\boxed{2\sqrt{e}-3}$.

For every $j \geq 1$ define the random variable Y_j by

$$Y_j = \begin{cases} X_j & \text{if } X_1 \geq X_2 \geq \cdots \geq X_j \\ 0 & \text{otherwise} \end{cases}$$

We note that $S = \sum\limits_{j=1}^{\infty} Y_j/2^j$. Let $t \in (0,1]$. For Y_j to be in $(0,t]$ we need two things:

(a) $X_j \leq t$, and

(b) $X_1 \geq X_2 \geq \cdots \geq X_j$.

In order for this to happen we need to do three things:

(i) Independently and uniformly select j number in $[0,1]$.

(ii) Make sure the smallest number that we selected does not exceed t.

(iii) Make sure the sequence $X_1, \dots, X_j$ is decreasing.

To find out the probability of (i) and (ii) we use complementary probability. The probability that all of these j numbers exceed t is $(1-t)^j$. Thus, the probability of (i) and (ii) happening is $1-(1-t)^j$. Since there are $j!$ different permutations of $X_1, X_2, \dots, X_j$, the probability that $X_1, X_2, \dots, X_j$ is decreasing is $1/j!$. Therefore, we have $P(Y_j \leq t) = \dfrac{1-(1-t)^j}{j!} + P(Y_j = 0)$. Thus, the probability density function for Y_j is $\dfrac{d}{dt}(P(Y_j \leq t)) = \dfrac{(1-t)^{j-1}}{(j-1)!}$. This implies that the expected value can be evaluated as follows:

$$E[Y_j] = \int_0^1 t\frac{(1-t)^{j-1}}{(j-1)!}\,dt = \int_0^1 \frac{(1-t)^{j-1}-(1-t)^j}{(j-1)!}\,dt = \frac{1}{j!} - \frac{1}{(j+1)(j-1)!} = \frac{1}{(j+1)!}.$$

Since every Y_j is nonnegative, we may apply the Monotone Convergence Theorem to obtain:

$$E[S] = \sum_{j=1}^{\infty} \frac{1}{2^j(j+1)!} = 2\frac{(1/2)^{j+1}}{(j+1)!} = 2(e^{1/2}-1-1/2) = 2\sqrt{e}-3.$$

□

Example 14.3 (IMO 2022, Shortlisted Problem, A2). *Let $k \geq 2$ be an integer. Find the smallest integer $n \geq k+1$ with the property that there exists a set of n distinct real numbers such that each of its elements can be written as a sum of k other distinct elements of the set.*

Solution.(Video Solution)

□

Example 14.4 (VTRMC 2012)**.** *Solve the equation in real numbers:* $3x - x^3 = \sqrt{x+2}$.

Solution.(Video Solution)

□

Example 14.5 (IMO 1961, Problem 3)**.** *Find all positive integers n and real numbers x for which* $\cos^n x - \sin^n x = 1$.

Solution.(Video Solution)

□

14.4 Exercises

Exercise 14.1 (VTRMC 1979)**.** Let S be a finite set of non-negative integers such that $|x-y| \in S$ whenever $x, y \in S$.

(a) Give an example of such a set which contains ten elements.

(b) If A is a subset of S containing more than two-thirds of the elements of S, prove or disprove that every element of S is the sum or difference of two elements from A.

Exercise 14.2 (VTRMC 1981)**.** Define $F(x)$ by $F(x) = \sum_{n=0}^{\infty} F_n x^n$ (wherever the series converges), where F_n is the nth Fibonacci number defined by $F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2}$, $n > 1$. Find an explicit closed form for $F(x)$.

Exercise 14.3 (VTRMC 1983)**.** Let $f(x) = 1/x$ and $g(x) = 1 - x$ for $x \in (0,1)$. List all distinct functions that can be written in the form $f \circ g \circ f \circ g \circ \cdots \circ f \circ g \circ f$ where $\circ$ represents composition. Write each function in the form $\frac{ax+b}{cx+d}$, and prove that your list is exhaustive.

Exercise 14.4 (Putnam 1989, A4)**.** If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both $\frac{1}{2}$. A game is finite if with probability 1 it must end in a finite number of moves.)

Exercise 14.5 (Putnam 1989, B6)**.** Let $(x_1, x_2, \ldots x_n)$ be a point chosen at random from the n-dimensional region defined by $0 < x_1 < x_2 < \cdots < x_n < 1$. Let f be a continuous function on $[0,1]$ with $f(1) = 0$. Set $x_0 = 0$ and $x_{n+1} = 1$. Show that the expected value of the Riemann sum

$$\sum_{i=0}^{n} (x_{i+1} - x_i) f(x_{i+1})$$

is $\int_0^1 f(t)P(t)\,dt$, where P is a polynomial of degree n, independent of f, with $0 \le P(t) \le 1$ for $0 \le t \le 1$.

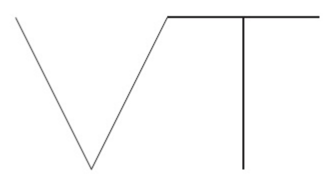
Exercise 14.6 (VTRMC 1990)**.** A person is engaged in working a jigsaw puzzle that contains 1000 pieces. It is found that it takes 3 minutes to put the first two pieces together and that when x pieces have been connected it takes $\frac{3(1000-x)}{1000+x}$ minutes to connect the next piece. Determine an accurate estimate of the time it takes to complete the puzzle. Give both a formula and an approximate numerical value in hours. (You may find useful the approximate value $\ln 2 = .69$.)

Exercise 14.7 (VTRMC 1993)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a surjective map with the property that if the points A, B and C are collinear, then so are $f(A), f(B)$ and $f(C)$. Prove that f is bijective.

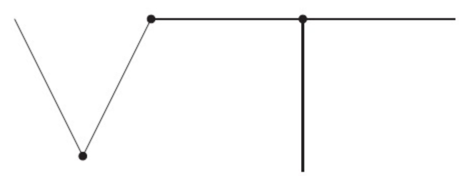
Exercise 14.8 (Putnam 1993, B2)**.** Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n+1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?

Exercise 14.9 (Putnam 1993, B3)**.** Two real numbers x and y are chosen at random in the interval (0,1) with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

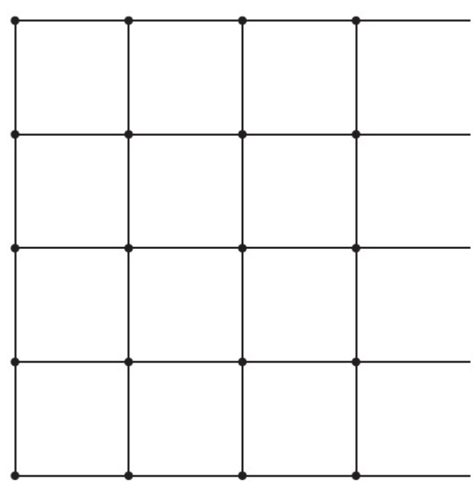
Exercise 14.10 (VTRMC 1993)**.** A popular Virginia Tech logo looks something like



Suppose that wire-frame copies of this logo are constructed of 5 equal pieces of wire welded at three places as shown:



If bending is allowed, but no re-welding, show clearly how to cut the maximum possible number of ready-made copies of such a logo from the piece of welded wire mesh shown. Also, prove that no larger number is possible.



Exercise 14.11 (VTRMC 1994)**.** Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ be a function which satisfies $f(0,0) = 1$ and

$$f(m,n) + f(m+1,n) + f(m,n+1) + f(m+1,n+1) = 0$$

for all $m,n \in \mathbb{Z}$ (where $\mathbb{Z}$ and $\mathbb{R}$ denote the set of all integers and all real numbers, respectively). Prove that $|f(m,n)| \geq 1/3$, for infinitely many pairs of integers (m,n).

Exercise 14.12 (Putnam 1994, A5)**.** Let $(r_n)_{n\geq 0}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} r_n = 0$. Let S be the set of numbers representable as a sum

$$r_{i_1} + r_{i_2} + \cdots + r_{i_{1994}},$$

with $i_1 < i_2 < \cdots < i_{1994}$. Show that every nonempty interval (a,b) contains a nonempty subinterval (c,d) that does not intersect S.

Exercise 14.13 (Putnam 1994, B5)**.** For any real number α, define the function $f_\alpha(x) = \lfloor \alpha x \rfloor$. Let n be a positive integer. Show that there exists an α such that for $1 \leq k \leq n$,

$$f_\alpha^k(n^2) = n^2 - k = f_{\alpha^k}(n^2).$$

Exercise 14.14 (VTRMC 1995)**.** Let $\mathbb{R}$ denote the real numbers, and let $\theta : \mathbb{R} \to \mathbb{R}$ be a map with the property that $x > y$ implies $(\theta(x))^3 > \theta(y)$. Prove that $\theta(x) > -1$ for all x, and that $0 \leq \theta(x) \leq 1$ for at most one value of x.

Exercise 14.15 (Putnam 1995, A6)**.** Suppose that each of n people writes down the numbers 1,2,3 in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums a,b,c of the resulting matrix be rearranged (if necessary) so that $a \leq b \leq c$. Show that for some $n \geq 1995$, it is at least four times as likely that both $b = a+1$ and $c = a+2$ occur as that $a = b = c$.

Exercise 14.16. Let $n \geq 3$ be a given integer. n points are randomly and uniformly selected on the circumference of a given circle C. What is the probability that the center of C lies inside the convex polygon formed by these n randomly selected points?

Exercise 14.17. Let n be an integer more than 1. Integers $0,1,\ldots,n-1$ are arranged around on a circle. A spider starts from 0 and randomly jumps to one of its neighboring numbers 1 or $n-1$. In each step the spider continues to randomly select one of the nearest neighboring numbers and jump to that number. For each i, $0 \leq i \leq n-1$ find the probability that the spider visits i for the first time after having visited all the other numbers first. (For example, the probability is 0 when $i = 0$.)

Exercise 14.18. Choose $X_1,\ldots,X_n$ randomly and uniformly from $[0,1]$. Let p_n be the probability that $X_i + X_{i+1} \leq 1$ for all $i = 1,\ldots,n-1$. Prove that $\lim_{n\to\infty} \sqrt[n]{p_n}$ exists and compute it.

Exercise 14.19. Suppose n and m are two positive integer for which $(n!)^m + n^m = (m!)^n + m^n$. Prove that $m = n$. Similarly show that the only solutions to $(n!)^m - n^m = (m!)^n - m^n$ are $m = n$ and $(m,n) = (1,2)$ or $(2,1)$.

Exercise 14.20 (Putnam 1996, A6)**.** Let $c > 0$ be a constant. Give a complete description, with proof, of the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = f(x^2 + c)$ for all $x \in \mathbb{R}$.

Exercise 14.21 (VTRMC 1998)**.** Ten cats are sitting on ten fence posts, numbered 1 through 10 in clockwise order and encircling a pumpkin patch. The cat on post #1 is white and the other nine cats are black. At 9:45 p.m. the cats begin a strange sort of dance. They jump from post to post according to the following two rules, applied in alternation at one second intervals. Rule 1 : each cat jumps clockwise to the next post. Rule 2 : all pairs of cats whose post numbers have a product that is 1 greater than a multiple of 11 exchange places. At 10 p.m., just as the Great Pumpkin rises out of the pumpkin patch, the dance stops abruptly and the cats look on in awe. If the first jump takes place according to Rule 1 at 9:45:01, and the last jump occurs at 10:00:00, on which post is the white cat sitting when the dance stops? (The first few jumps take the white cat from post 1 to posts $2,6,7,\ldots$.)

Exercise 14.22 (Putnam 1998, A5)**.** Let $\mathscr{F}$ be a finite collection of open discs in $\mathbb{R}^2$ whose union contains a set $E \subseteq \mathbb{R}^2$. Show that there is a pairwise disjoint subcollection $D_1,\ldots,D_n$ in $\mathscr{F}$ such that

$$E \subseteq \cup_{j=1}^{n} 3D_j.$$

Here, if D is the disc of radius r and center P, then $3D$ is the disc of radius $3r$ and center P.

Exercise 14.23 (VTRMC 2000)**.** Consider the initial value problem $y' = y^2 - t^2; y(0) = 0$ (where $y' = dy/dt$). Prove that $\lim_{t\to\infty} y'(t)$ exists, and determine its value.

Exercise 14.24 (Putnam 2000, A4)**.** Show that the improper integral

$$\lim_{B\to\infty} \int_0^B \sin(x)\sin(x^2)\,dx$$

converges.

Exercise 14.25 (Putnam 2001, A2)**.** You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

Exercise 14.26 (Putnam 2001, B4)**.** Let S denote the set of rational numbers different from $\{-1,0,1\}$. Define $f : S \to S$ by $f(x) = x - 1/x$. Prove or disprove that

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset,$$

where $f^{(n)}$ denotes f composed with itself n times.

Exercise 14.27 (Putnam 2002, B1)**.** Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

Exercise 14.28 (Putnam 2002, B2)**.** Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

> Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

Show that the player who signs first will always win by playing as well as possible.

Exercise 14.29 (Putnam 2002, B4)**.** An integer n, unknown to you, has been randomly chosen in the interval $[1,2002]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.

Exercise 14.30 (VTRMC 2003)**.** An investor buys stock worth $\$10,000$ and holds it for n business days. Each day he has an equal chance of either gaining 20% or losing 10%. However in the case he gains every day (i.e. n gains of 20%), he is deemed to have lost all his money, because he must have been involved with insider trading. Find a (simple) formula, with proof, of the amount of money he will have on average at the end of the n days.

Exercise 14.31 (VTRMC 2004)**.** Let $f(x)=\int_0^x \sin(t^2-t+x)\,\mathrm{d}t$. Compute $f''(x)+f(x)$ and deduce that $f^{(12)}(0)+f^{(10)}(0)=0$ ($f^{(10)}$ indicates 10-th derivative).

Exercise 14.32 (Putnam 2005, A6)**.** Let n be given, $n\geq 4$, and suppose that $P_1,P_2,\ldots,P_n$ are n randomly, independently and uniformly, chosen points on a circle. Consider the convex n-gon whose vertices are the P_i. What is the probability that at least one of the vertex angles of this polygon is acute?

Exercise 14.33 (Putnam 2005, B3)**.** Find all differentiable functions $f:(0,\infty)\to(0,\infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right)=\frac{x}{f(x)}$$

for all $x>0$.

Exercise 14.34 (Putnam 2006, A2)**.** Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n=17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

Exercise 14.35 (Putnam 2006, A4)**.** Let $S=\{1,2,\ldots,n\}$ for some integer $n>1$. Say a permutation π of S has a *local maximum* at $k\in S$ if

(i) $\pi(k)>\pi(k+1)$ for $k=1$;

(ii) $\pi(k-1)<\pi(k)$ and $\pi(k)>\pi(k+1)$ for $1<k<n$;

(iii) $\pi(k-1)<\pi(k)$ for $k=n$.

(For example, if $n=5$ and π takes values at $1,2,3,4,5$ of $2,1,4,5,3$, then π has a local maximum of 2 at $k=1$, and a local maximum of 5 at $k=4$.) What is the average number of local maxima of a permutation of S, averaging over all permutations of S?

Exercise 14.36 (Putnam 2006, A6)**.** Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.

Exercise 14.37 (VTRMC 2007)**.** Evaluate $\int_0^x \frac{d\theta}{2+\tan\theta}$, where $0 \le x \le \pi/2$. Use your result to show that $\int_0^{\pi/4} \frac{d\theta}{2+\tan\theta} = \frac{\pi+\ln(9/8)}{10}$

Exercise 14.38 (VTRMC 2007)**.** Find the third digit after the decimal point of

$$(2+\sqrt{5})^{100}\left((1+\sqrt{2})^{100}+(1+\sqrt{2})^{-100}\right)$$

For example, the third digit after the decimal point of $\pi = 3.14159\ldots$ is 1.

Exercise 14.39 (Putnam 2007, A5)**.** Let k be a positive integer. Prove that there exist polynomials $P_0(n), P_1(n), \ldots, P_{k-1}(n)$ (which may depend on k) such that for any integer n,

$$\left\lfloor \frac{n}{k} \right\rfloor^k = P_0(n) + P_1(n)\left\lfloor \frac{n}{k} \right\rfloor + \cdots + P_{k-1}(n)\left\lfloor \frac{n}{k} \right\rfloor^{k-1}.$$

($\lfloor a \rfloor$ means the largest integer $\le a$.)

Exercise 14.40 (VTRMC 2008)**.** Let $a_1, a_2, \ldots$ be a sequence of nonnegative real numbers and let π, ρ be permutations of the positive integers $\mathbb{N}$ (thus $\pi, \rho : \mathbb{N} \to \mathbb{N}$ are one-to-one and onto maps). Suppose that $\sum_{n=1}^{\infty} a_n = 1$ and ε is a real number such that

$$\sum_{n=1}^{\infty} |a_n - a_{\pi n}| + \sum_{n=1}^{\infty} |a_n - a_{\rho n}| < \varepsilon.$$

Prove that there exists a finite subset X of $\mathbb{N}$ such that $|X \cap \pi X|, |X \cap \rho X| > (1-\varepsilon)|X|$ (here $|X|$ indicates the number of elements in X; also the inequalities $<, >$ are strict).

Exercise 14.41 (Putnam 2009, A2)**.** Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$\begin{aligned} f' &= 2f^2gh + \frac{1}{gh}, \quad f(0) = 1, \\ g' &= fg^2h + \frac{4}{fh}, \quad g(0) = 1, \\ h' &= 3fgh^2 + \frac{1}{fg}, \quad h(0) = 1. \end{aligned}$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

Exercise 14.42 (Putnam 2009, B2)**.** A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, the cost of jumping from a to b is $b^3 - ab^2$. For what real numbers c can one travel from 0 to 1 in a finite number of jumps with total cost exactly c?

Exercise 14.43 (VTRMC 2011)**.** Let S be a set with an asymmetric relation $<$; this means that if $a, b \in S$ and $a < b$, then we do not have $b < a$. Prove that there exists a set T containing S with an asymmetric relation $\prec$ with the property that if $a, b \in S$, then $a < b$ if and only if $a \prec b$, and if $x, y \in T$ with $x \prec y$, then there exists $t \in T$ such that $x \prec t \prec y$ ($t \in T$ means "t is an element of T").

Exercise 14.44 (VTRMC 2012)**.** Evaluate

$$\int_0^{\pi/2} \frac{\cos^4 x + \sin x\,\cos^3 x + \sin^2 x\,\cos^2 x + \sin^3 x\,\cos x}{\sin^4 x + \cos^4 x + 2\sin x\,\cos^3 x + 2\sin^2 x\,\cos^2 x + 2\sin^3 x\,\cos x}\,\mathrm{d}x.$$

Exercise 14.45 (Putnam 2013, B6)**.** Let $n \geq 1$ be an odd integer. Alice and Bob play the following game, taking alternating turns, with Alice playing first. The playing area consists of n spaces, arranged in a line. Initially all spaces are empty. At each turn, a player either

- places a stone in an empty space, or
- removes a stone from a nonempty space s, places a stone in the nearest empty space to the left of s (if such a space exists), and places a stone in the nearest empty space to the right of s (if such a space exists).

Furthermore, a move is permitted only if the resulting position has not occurred previously in the game. A player loses if he or she is unable to move. Assuming that both players play optimally throughout the game, what moves may Alice make on her first turn?

Exercise 14.46 (Putnam 2014, A4)**.** Suppose X is a random variable that takes on only nonnegative integer values, with $E[X]=1$, $E\left[X^2\right]=2$, and $E\left[X^3\right]=5$. (Here $E[Y]$ denotes the expectation of the random variable Y.) Determine the smallest possible value of the probability of the event $X=0$.

Exercise 14.47 (VTRMC 2015)**.** Let n be a positive integer and let $x_1,\ldots,x_n$ be n nonzero points in $\mathbb{R}^2$. Suppose $\left\langle x_i,x_j\right\rangle$ (scalar or dot product) is a rational number for all $i,j(1 \leq i,j \leq n)$. Let S denote all points of $\mathbb{R}^2$ of the form $\sum_{i=1}^{i=n} a_i x_i$ where the a_i are integers. A closed disk of radius R and center P is the set of points at distance at most R from P (includes the points distance R from P). Prove that there exists a positive number R and closed disks $D_1, D_2, \ldots$ of radius R such that

(a) Each disk contains exactly two points of S

(b) Every point of S lies in at least one disk;

(c) Two distinct disks intersect in at most one point.

Exercise 14.48 (Putnam 2017, A5)**.** Each of the integers from 1 to n is written on a separate card, and then the cards are combined into a deck and shuffled. Three players, A, B, and C, take turns in the order $A,B,C,A,\ldots$ choosing one card at random from the deck. (Each card in the deck is equally likely to be chosen.) After a card is chosen, that card and all higher-numbered cards are removed from the deck, and the remaining cards are reshuffled before the next turn. Play continues until one of the three players wins the game by drawing the card numbered 1.
Show that for each of the three players, there are arbitrarily large values of n for which that player has the highest probability among the three players of winning the game.

Exercise 14.49 (IMC 2018, Problem 8)**.** Let $\Omega=\left\{(x,y,z)\in\mathbb{Z}^3 : y+1 \geq x \geq y \geq z \geq 0\right\}$. A frog moves along the points of Ω by jumps of length 1. For every positive integer n, determine the number of paths the frog can take to reach (n,n,n) starting from $(0,0,0)$ in exactly $3n$ jumps.

Exercise 14.50 (Putnam 2019, B6)**.** Let $\mathbb{Z}^n$ be the integer lattice in $\mathbb{R}^n$. Two points in $\mathbb{Z}^n$ are called *neighbors* if they differ by exactly 1 in one coordinate and are equal in all other coordinates. For which integers $n \geq 1$ does there exist a set of points $S \subset \mathbb{Z}^n$ satisfying the following two conditions?

(1) If p is in S, then none of the neighbors of p is in S.

(2) If $p \in \mathbb{Z}^n$ is not in S, then exactly one of the neighbors of p is in S.

Exercise 14.51 (IMC 2019, Problem 10)**.** 2019 points are chosen at random, independently, and distributed uniformly in the unit disc $\{(x,y) \in \mathbb{R}^2 \, : \, x^2+y^2 \leq 1\}$. Let C be the convex hull of the chosen points. Which probability is larger: that C is a polygon with three vertices, or a polygon with four vertices?

Exercise 14.52 (Putnam 2020, A4)**.** Consider a horizontal strip of $N+2$ squares in which the first and the last square are black and the remaining N squares are all white. Choose a white square uniformly at random, choose one of its two neighbors with equal probability, and color this neighboring square black if it is not already black. Repeat this process until all the remaining white squares have only black neighbors. Let $w(N)$ be the expected number of white squares remaining. Find

$$\lim_{N\to\infty} \frac{w(N)}{N}.$$

Exercise 14.53 (Putnam 2020, B3)**.** Let $x_0 = 1$, and let δ be some constant satisfying $0 < \delta < 1$. Iteratively, for $n = 0,1,2,\ldots$, a point x_{n+1} is chosen uniformly from the interval $[0,x_n]$. Let Z be the smallest value of n for which $x_n < \delta$. Find the expected value of Z, as a function of δ.

Exercise 14.54 (Putnam 2021, B1)**.** Suppose that the plane is tiled with an infinite checkerboard of unit squares. If another unit square is dropped on the plane at random with position and orientation independent of the checkerboard tiling, what is the probability that it does not cover any of the corners of the squares of the checkerboard?

Exercise 14.55 (Putnam 2021, B6)**.** Given an ordered list of $3N$ real numbers, we can *trim* it to form a list of N numbers as follows: We divide the list into N groups of 3 consecutive numbers, and within each group, discard the highest and lowest numbers, keeping only the median.

Consider generating a random number X by the following procedure: Start with a list of 3^{2021} numbers, drawn independently and uniformly at random between 0 and 1. Then trim this list as defined above, leaving a list of 3^{2020} numbers. Then trim again repeatedly until just one number remains; let X be this number. Let μ be the expected value of $|X-\frac{1}{2}|$. Show that

$$\mu \geq \frac{1}{4}\left(\frac{2}{3}\right)^{2021}.$$

Exercise 14.56 (Putnam 2022, A6)**.** Let n be a positive integer. Determine, in terms of n, the largest integer m with the following property: There exist real numbers $x_1,\ldots,x_{2n}$ with $-1 < x_1 < x_2 < \cdots < x_{2n} < 1$ such that the sum of the lengths of the n intervals

$$[x_1^{2k-1},x_2^{2k-1}],[x_3^{2k-1},x_4^{2k-1}],\ldots,[x_{2n-1}^{2k-1},x_{2n}^{2k-1}]$$

is equal to 1 for all integers k with $1 \leq k \leq m$.

Exercise 14.57 (Putnam 2022, B5)**.** For $0 \leq p \leq 1/2$, let $X_1, X_2, \ldots$ be independent random variables such that

$$X_i = \begin{cases} 1 & \text{with probability } p, \\ -1 & \text{with probability } p, \\ 0 & \text{with probability } 1-2p, \end{cases}$$

for all $i \geq 1$. Given a positive integer n and integers $b, a_1, \ldots, a_n$, let $P(b, a_1, \ldots, a_n)$ denote the probability that $a_1X_1 + \cdots + a_nX_n = b$. For which values of p is it the case that

$$P(0, a_1, \ldots, a_n) \geq P(b, a_1, \ldots, a_n)$$

for all positive integers n and all integers $b, a_1, \ldots, a_n$?

Exercise 14.58 (Putnam 2023, A6)**.** Alice and Bob play a game in which they take turns choosing integers from 1 to n. Before any integers are chosen, Bob selects a goal of "odd" or "even". On the first turn, Alice chooses one of the n integers. On the second turn, Bob chooses one of the remaining integers. They continue alternately choosing one of the integers that has not yet been chosen, until the nth turn, which is forced and ends the game. Bob wins if the parity of $\{k : \text{the number } k \text{ was chosen on the } k\text{th turn}\}$ matches his goal. For which values of n does Bob have a winning strategy?

Exercise 14.59 (Putnam 2023, B3)**.** A sequence $y_1, y_2, \ldots, y_k$ of real numbers is called *zigzag* if $k = 1$, or if $y_2 - y_1, y_3 - y_2, \ldots, y_k - y_{k-1}$ are nonzero and alternate in sign. Let $X_1, X_2, \ldots, X_n$ be chosen independently from the uniform distribution on $[0, 1]$. Let $a(X_1, X_2, \ldots, X_n)$ be the largest value of k for which there exists an increasing sequence of integers $i_1, i_2, \ldots, i_k$ such that $X_{i_1}, X_{i_2}, \ldots X_{i_k}$ is zigzag. Find the expected value of $a(X_1, X_2, \ldots, X_n)$ for $n \geq 2$.

Some Tips for the Day of the Competition

- Putnam problems in each session are generally ordered in level of difficulty. This means problems A1 and A6 are the easiest and the most difficult problems in the morning session, and problems B1 and B6 are the easiest and the most difficult problems in the afternoon session. So, it is generally not advisable to try A6 before having done A1.

- Remember the factorization of the year and perhaps a few primes near the year number. These frequently show up in the questions.

- The MAA logo, that appears on top of Putnam question sheets, shows a regular icosahedron. This object has appeared in Putnam problems enough that it is now expected that students are familiar with it. So, here is some basic information about it:

$$v = 12, e = 30, f = 20$$

- Beautiful problems are not necessarily the easiest ones. If anything, often times the exact opposite is true.

- Manage your time carefully. You are almost certainly not going to solve all of the problems! Read them all, but don't jump around trying to solve all of them. Try the ones that you think you have a chance on.

- You don't have to submit anything if you don't have anything valuable, but submit whatever you think could earn you points. Partial credit is awarded, even if it is done so sparingly.

- Don't put something down that you know it is mathematically false. Grader could lose patience after seeing an obviously false claim and not carefully read the rest of your solution. On the other hand, avoid writing things like "I don't know how to finish up the solution." There is no reason for you to advertise you didn't solve the problem.

- Put the final answer at the beginning. "Do not bury the lead!"

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